Notre Dame Journal of Formal Logic
Volume VI, Number 2, April 1965

ON PROBABILITY LOGICS

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Our language contains the following symbols:

(1) the (individual) variables 'v_1', 'v_2', and so on;

(2) the sentential connectives '$\sim$' ('not'), '$\rightarrow$' ('only if'), '$\wedge$' ('and'), '$\vee$' ('or'), and '$\leftrightarrow$' ('if and only if');

(3) the variable binders 'ι' ('the'), '**P**' ('the probability that any - is a ...'), '**Q**' ('the probability that any - which is a ... is a ---'), '$\bigwedge$' ('for all'), and '$\bigvee$' ('for some');

(4) the individual constants '*0*', '*1*', 'c_3', 'c_4', and so on;

(5) the *1*-place operation symbols '$\dot{-}$' ('minus'), '0^1_2', '0^1_3', and so on;

(6) the 2-place operation symbols '+' ('plus'), '.' ('times'), '-' ('minus'), '/' ('divided by'), '$\urcorner$' ('to the power'), '$\ulcorner$' ('the -th non-negative root of'), '0^2_7', '0^2_8', and so on;

(7) the 3-place operation symbols '0^3_1', '0^3_2', and so on; and so on for any greater number of places;

(8) the *1*-place predicates '**R**' ('is a real number'), '**N**' ('is a positive integer'), 'P^1_3', 'P^1_4', and so on;

(9) the 2-place predicates '**I**' ('is identical with'), 'α' ('is less than'), '$\mathbf{P}^2_3$' '$\mathbf{P}^2_4$', and so on; and

(10) the 3-place predicates '$\mathbf{P}^3_1$', '$\mathbf{P}^3_2$', and so on; and so on for any greater number of places.

We use the symbols '<', '>' and '{', '}' in the metalanguage to mark the boundaries of non-empty finite sequences and sets respectively. The letter '*m*' will be used as a metalinguistic variable ranging over positive integers. Terms and formulas will be understood as follows:

(1) all variables and individual constants are terms;

(2) for any *m*-place operation symbol *o* and *m*-term sequence of terms *t*, $<ot>$ is a term;

(3) for any variable *v* and formulas *f* and *g*, $<$'ι' $vf>$, $<$'**P**' $vf>$, and $<$'**Q**' $vfg>$ are terms;

(4) for any *m*-place predicate *p* and *m*-term sequence of terms *t*, $<pt>$ is a formula;

(5) for any formulas *f* and *g*, $<$'$\sim$' $f>$, $<f$ '$\rightarrow$' $g>$, $<f$ '$\wedge$' $g>$, $<f$ '$\vee$' $g>$, and $<f$ '$\leftrightarrow$' $g>$ are formulas; and

Received April 22, 1962

(6) for any variable v and formula f, $\langle$'$\wedge$' $vf\rangle$ and $\langle$'$\vee$' $vf\rangle$ are formulas.

In the sequel, we omit superfluous sequence and quotation marks. Also, we write mentioned 2-place operation symbols and predicates between their arguments instead of in front of them. Given terms t and u and a term or formula f, we understand freedom and $\mathsf{PS}\,tuf$ (the result of properly substituting t for u in f) as follows:

(1) if $u = f$, then u is free in f and $\mathsf{PS}\,tuf = t$;

(2) if $u \neq f$, then

(a) if f is a variable or individual constant, then u is not free in f and $\mathsf{PS}\,tuf = f$;

(b) for any m-place operation symbol or predicate o and m-term sequence of terms v, if $f = \langle ov\rangle$, then u is free in f just in case u is free in some member of the range of v and $\mathsf{PS}\,tuf = \langle o$ the m-term sequence w such that $w(i) = \mathsf{PS}\,tuv(i)$ for any i in the domain of $w\rangle$;

(c) for any sentential connective c and formulas g and h,

(1) if $f = \langle cg\rangle$, then u is free in f just in case u is free in g and $\mathsf{PS}\,tuf = \langle c\,\mathsf{PS}\,tug\rangle$ and

(2) if $f = \langle gch\rangle$, then u is free in f just in case u is free in either g or h and $\mathsf{PS}\,tuf = \langle \mathsf{PS}\,tug\ c\ \mathsf{PS}\,tuh\rangle$; and

(d) for any variable binder b, variable v, and formulas g and h,

(1) if $f = \langle bvg\rangle$, then u is free in f just in case u is free in g and v is not free in u; also, if $f = \langle bvgh\rangle$, then u is free in f just in case u is free in either g or h and v is not free in u;

(2) if u is not free in f and either $f = \langle bvg\rangle$ or $f = \langle bvgh\rangle$, then $\mathsf{PS}\,tuf = f$;

(3) if u is free in f and v is not free in t, then $\mathsf{PS}\,tuf = \langle bv\,\mathsf{PS}\,tug\rangle$ if $f = \langle bvg\rangle$ and $\mathsf{PS}\,tuf = \langle bv\mathsf{PS}\,tug\ \mathsf{PS}\,tuh\rangle$ if $f = \langle bvgh\rangle$; and

(4) if u is free in f, v is free in t, and w is the first variable distinct from t not occurring in either f or t, then $\mathsf{PS}\,tuf = \langle bw\,\mathsf{PS}\,tu\,\mathsf{PS}\,wvg\rangle$ if $f = \langle bvg\rangle$ and $\mathsf{PS}\,tuf = \langle bw\,\mathsf{PS}\,tu\,\mathsf{PS}\,wvg\,\mathsf{PS}\,tu\mathsf{PS}\,wvh\rangle$ if $f = \langle bvgh\rangle$.

By an interpreter, we understand a function i of the following kind:

(1) the domain of i = the set of all variable binders, individual constants, operation symbols, predicates, and sentential connectives;

(2) there is a set s such that

(a) i(℩) = the function d such that the domain of d = the set of all subsets of s and, for any r in the domain of d, either there is just one object q in r and $d(r) = q$ or there is not just one object in r and $d(r) = i(0)$;

(b) $i(\mathsf{P})$ is a function p such that the domain of p = the set of all subsets of s and, for any r in the domain of p, $p(r)$ is in s;

(c) $i(\mathsf{Q})$ is a function q such that the domain of q = the set of all 2-term sequences whose ranges are included in the set of all subsets of s and, for any r in the domain of q, $q(r)$ is in s;

(d) $i(\wedge)$ = the function u such that the domain of u = the set of all subsets of s and, for any r in the domain of u, either $r = s$ and $u(r) = 1$ or $r \neq s$ and $u(r) = 0$;

(e) $i(\bigvee)$ = the function e such that the domain of e = the set of all subsets of s and, for any r in the domain of e, either r is not empty and $e(r) = 1$ or r is empty and $e(r) = 0$;

(f) for any individual constant c, $i(c)$ is in s;

(g) for any m-place operation symbol o, $i(o)$ is a function f such that the domain of f = the set of all m-term sequences whose ranges are included in s and, for any r in the domain of f, $f(r)$ is in s;

(h) for any m-place predicate p, $i(p)$ is included in the set of all m-term sequences whose ranges are included in s;

(i) $i(\mathrm{I})$ = the set of all 2-term sequences r such that, for some x in s, $r = \langle xx \rangle$;

(3) $i(\sim)$ = the function n whose domain is $\{0\ 1\}$ and such that, for any t in $\{0\ 1\}$, $n(t) = 1 - t$;

(4) for any sentential connective c, if $c \neq \sim$, then $i(c)$ is a function whose domain is the set of all 2-term sequences whose ranges are included in $\{0\ 1\}$; and

(5) for any t and u in $\{0\ 1\}$, $(i(\rightarrow))(\langle tu \rangle)$ = the smallest member of $\{1,\ (1-t)+u\}$, $(i(\wedge))(\langle tu \rangle)$ = the smallest member of $\{tu\}$, $(i(\vee))(\langle tu \rangle)$ = the greatest member of $\{tu\}$, and $(i(\leftrightarrow))(\langle tu \rangle) = (1$—the greatest member of $\{tu\})$ + the smallest member of $\{tu\}$.

Given an interpreter i, we understand by $\mathsf{U}i$ (the universe of i) the set s satisfying (2) above with respect to i. By an assigner for i, we mean a function whose domain is the set of all variables and which assigns to any variable a member of $\mathsf{U}i$. Given such an assigner for $i\ a$, variable v, and x in $\mathsf{U}i$, $a(^v_x)$ is the assigner for $i\ b$ such that b is a with the pair $v, a(v)$ removed and the pair v,x added in its place. Given an interpreter i and an assigner for $i\ a$, we understand the operation $\mathsf{Int}\ ia$ (the interpretation with respect to i and a of -) as follows:

(1) for any variable v, $\mathsf{Int}\ ia\,(v) = a(v)$;

(2) for any individual constant c, $\mathsf{Int}\ ia\,(c) = i(c)$;

(3) for any m-place operation symbol o and m-term sequence of terms t, $\mathsf{Int}\ ia\,(ot) = (i(o))$ (the m-term sequence u such that, for any j in its domain, $u(j) = \mathsf{Int}\ ia\,(t(j))$);

(4) for any variable v, any formulas f and g, and any b,

(a) if either $b = \mathsf{1}$ or $b = \mathsf{P}$, then $\mathsf{Int}\ ia\,(bvf) = (i(b))$ (the set of all x in $\mathsf{U}i$ such that $\mathsf{Int}\ ia(^v_x)\,(f) = 1$);

(b) if $b = \mathsf{Q}$, the $\mathsf{Int}\ ia\,(bvfg) = (i(b))$ ($\langle$the set of all x in $\mathsf{U}i$ such that $\mathsf{Int}\ ia(^v_x)\,(f) = 1$ the set of all x in $\mathsf{U}i$ such that $\mathsf{Int}\ ia(^v_x)(g) = 1\rangle$);

(5) for any m-place predicate p and m-term sequence of terms t, $\mathsf{Int}\ ia\,(pt)$ = the z such that either the m-term sequence u such that, for any j in its domain, $u(j) = \mathsf{Int}\ ia\,(t(j))$ is in $i(p)$ and $z = 1$ or is not and $z = 0$;

(6) for any formulas f and g and sentential connective c, either $c = \sim$ and $\mathsf{Int}\ ia\,(cf) = (i(c))(\mathsf{Int}\ ia\,(f))$ or $c \neq \sim$ and $\mathsf{Int}\ ia\,(fcg) = (i(c))(\langle \mathsf{Int}\ ia\,(f)\ \mathsf{Int}\ ia\,(g)\rangle)$; and

(7) for any variable v and formula f and any b, if either $b = \bigwedge$ or $b = \bigvee$, then $\mathsf{Int}\ ia\,(bvg) = (i(b))$ (the set of all x in $\mathsf{U}i$ such that $\mathsf{Int}\ ia(^v_x)\,(f) = 1$).

Given a formula f and an interpreter i, we say that f is true by i just in case, for any assigner for $i\,a$, $\mathsf{Int}\ ia\,(f) = 1$. We say that f is valid just in case, for any interpreter i, f is true by i. It follows that, for any variables v and w and formulas f, g, and h such that w is not free in g or h, $\bigwedge v \langle f \leftrightarrow g \rangle \rightarrow \mathrm{P}vf \;\mathrm{I}\; \mathrm{P}w\mathrm{PS}wvg \wedge \mathrm{Q}vfh \;\mathrm{I}\; \mathrm{Q}w\mathrm{PS}wvg\mathrm{PS}wvh \wedge \mathrm{Q}vhf \;\mathrm{I}\; \mathrm{Q}w\mathrm{PS}wvh\mathrm{PS}wvg$ is valid.

By an R-axiom, i.e., a special axiom of basic real number theory[1], we mean an f such that, for some distinct variables v, w, and x and for some formulas g and h in which v and w are free respectively, w and v are not free respectively, and x is not free, and some formula i in which v is free, f is one of the following:

(1) $\mathrm{R}v \wedge \mathrm{R}w \wedge \sim v\mathrm{I}w \rightarrow v\alpha w \vee w\alpha v$

(2) $\mathrm{R}v \wedge \mathrm{R}w \wedge v\alpha w \rightarrow \sim w\alpha v$

(3) $\mathrm{R}v \wedge \mathrm{R}w \wedge \mathrm{R}x \wedge v\alpha w \wedge w\alpha x \rightarrow v\alpha x$

(4) $\bigwedge v\langle g \rightarrow \mathrm{R}v\rangle \wedge \bigwedge w\langle h \rightarrow \mathrm{R}w\rangle \wedge \bigwedge v \bigwedge w\langle g \wedge h \rightarrow v\alpha w\rangle \rightarrow \bigvee x \langle \mathrm{R}x \wedge \bigwedge v \bigwedge w\langle g \wedge h \wedge \sim v\mathrm{I}x \wedge \sim w\mathrm{I}x \rightarrow v\alpha x \wedge x\alpha w\rangle\rangle$

(5) $\mathrm{R}v \wedge \mathrm{R}w \rightarrow \mathrm{R}v+w$

(6) $\mathrm{R}v \wedge \mathrm{R}w \rightarrow v+w\mathrm{I}w+v$

(7) $\mathrm{R}v \wedge \mathrm{R}w \wedge \mathrm{R}x \rightarrow v+\langle w+x\rangle \mathrm{I} \langle v+w\rangle + x$

(8) $\mathrm{R}v \wedge \mathrm{R}w \rightarrow \bigvee x\langle \mathrm{R}x \wedge v\mathrm{I}w+x\rangle$

(9) $\mathrm{R}v \wedge \mathrm{R}w \wedge \mathrm{R}x \wedge w\alpha x \rightarrow v+w\alpha v+x$

(10) $\mathrm{R}O$

(11) $\mathrm{R}v \rightarrow v+O\mathrm{I}v$

(12) $\mathrm{R}v \rightarrow \dot{-}v\mathrm{I}\iota x\langle \mathrm{R}x \wedge O\mathrm{I}v+x\rangle$

(13) $\mathrm{R}v \wedge \mathrm{R}w \rightarrow v-w\mathrm{I}v+\langle \dot{-}w\rangle$

(14) $\mathrm{R}v \wedge \mathrm{R}w \rightarrow \mathrm{R}v\cdot w$

(15) $\mathrm{R}v \wedge \mathrm{R}w \rightarrow v\cdot w\mathrm{I}w\cdot v$

(16) $\mathrm{R}v \wedge \mathrm{R}w \wedge \mathrm{R}x \rightarrow v\cdot\langle w\cdot x\rangle \mathrm{I}\langle v\cdot w\rangle \cdot x$

(17) $\mathrm{R}v \wedge \mathrm{R}w \wedge \sim w\mathrm{I}O \rightarrow \bigvee x\langle \mathrm{R}x \wedge v\mathrm{I}w\cdot x\rangle$

(18) $\mathrm{R}v \wedge \mathrm{R}w \wedge \mathrm{R}x \wedge O\alpha v \wedge w\alpha x \rightarrow v\cdot w\alpha v\cdot x$

(19) $\mathrm{R}v \wedge \mathrm{R}w \wedge \mathrm{R}x \rightarrow v\cdot\langle w+x\rangle \mathrm{I}\langle v\cdot w\rangle + \langle v\cdot x\rangle$

(20) $\mathrm{R}1$

(21) $\mathrm{R}v \rightarrow v\cdot 1\mathrm{I}v$

(22) $\sim O\mathrm{I}1$

(23) $\mathrm{R}v \wedge \mathrm{R}w \rightarrow v/w\mathrm{I}\iota x\langle\langle \sim w\mathrm{I}O \wedge \mathrm{R}x \wedge v\mathrm{I}w\cdot x\rangle \vee \langle w\mathrm{I}O \wedge x\mathrm{I}O\rangle\rangle$

(24) $\mathrm{N}v \leftrightarrow v\mathrm{I}1 \vee \bigvee w\langle \mathrm{N}w \wedge v\mathrm{I}w+1\rangle$

(25) $\mathrm{R}v \rightarrow v\urcorner O\mathrm{I}1$

(26) $\mathrm{R}v \wedge \mathrm{N}w \rightarrow v\urcorner w\mathrm{I}\langle v\urcorner\langle w-1\rangle\rangle \cdot v$

(27) $\mathrm{R}v \wedge \sim v\alpha O \rightarrow O\ulcorner v\mathrm{I}1$

(28) $\mathrm{R}v \wedge \sim v\alpha O \wedge \mathrm{N}w \rightarrow w\ulcorner v\mathrm{I}\iota x\langle \mathrm{R}x \wedge \sim x\alpha O \wedge x\urcorner w\mathrm{I}v\rangle$

(29) $\bigwedge v\langle i \rightarrow \mathrm{N}v\rangle \wedge \mathrm{PS}\,1vi \wedge \bigwedge v\langle \mathrm{N}v \wedge i \rightarrow \mathrm{PS}\langle v+1\rangle vi\rangle \rightarrow \bigwedge v\langle \mathrm{N}v \rightarrow i\rangle$.

By an R-interpreter, we mean an interpreter i such that every R-axiom is true by i. By a P-axiom, i.e., a special axiom of basic probability logic, we mean an e such that, for some variables v and w and formulas f, g, and h such that w is not free in h, e is one of the following:

(1) $\bigwedge v\langle g \leftrightarrow h\rangle \rightarrow \mathrm{P}vg\,\mathrm{I}\,\mathrm{P}w\mathrm{PS}wvh$

(2) $\mathrm{RP}vf$

(3) $\mathrm{P}vf\,\mathrm{I}\,O \vee O\alpha\mathrm{P}vf$

(4) $\bigwedge vf \leftrightarrow \mathsf{P}vf \,\mathbf{I}\, 1$
(5) $\sim \bigvee v < f \wedge g > \rightarrow \mathsf{P}v\, f \vee g \,\mathbf{I}\, \mathsf{P}vf + \mathsf{P}vg$
(6) $\mathsf{Q}vfg \,\mathbf{I}\, Pv\, f \wedge g / \mathsf{P}vf$.

By a **P**-interpreter, we mean an **R**-interpreter i such that every **P**-axiom is true by i. In the language of probability theory, this means both a bit less and a bit more than that $<\mathsf{U}i$ the set of all subsets of $\mathsf{U}i\ i(\mathsf{P})>$ is a probability field where the real number system involved is the one whose components are the i-values of **R**, **N**, α, and so on[2].

We say that a formula is probability valid just in case it is true by any **P**-interpreter. The construction of an appropriately sound and complete probability logic is now a trivial matter; we simply add to any appropriate sound and complete ordinary logic (in which both identity and definite descriptions are dealt with) both the **R**-axioms and the **P**-axioms. One probability logic[3] is the logic whose inference rules are modus ponens and universal generalization and whose axioms are the e such that, for some variables v and w, terms t and u, and formulas f and g, v is not free in g and e is one of the following:

(1) a tautology
(2) $\bigwedge v < g \rightarrow f > \wedge g \rightarrow \bigwedge vf$
(3) $\bigwedge vf \rightarrow \mathsf{PS}tvf$
(4) $\bigvee vf \leftrightarrow \sim \bigwedge v \sim f$
(5) $t \mathbf{I} t$
(6) $t \mathbf{I} u \wedge \mathsf{PS}\, tuf \rightarrow f$
(7) $\bigvee v\ \bigwedge w < g \leftrightarrow w \mathbf{I} v > \rightarrow \mathsf{PS} < \iota wg > wg$
(8) $\sim \bigvee v \bigwedge w < g \leftrightarrow w \mathbf{I} v > \rightarrow \iota wg \mathbf{I} O$
(9) an **R**-axiom
(10) a **P**-axiom.

This logic can be called basic probability logic. It could also be called basic inductive logic; however, such a procedure would be misleading since the concerned logic is entirely deductive.

Notice that a probability logic is not one, but two steps removed from its associated system of ordinary deductive logic (the first step being that of the real number axioms). This is a sense in which deductive logic is far more fundamental than probability logic.

By methods analogous to those used above, probability logics containing probability variable binders which bind *2*, *3*, and so on variables can also be interpreted and formalized. Such probability logics will not be discussed here.

NOTES

1. This axiom set is essentially the second one given by A. Tarski in his *Introduction to logic and to the methodology of the deductive sciences*, 2nd ed. (London and New York, 1946). It is, of course, slightly redundant.

2. The reader is referred to A. Kolmogorov's *Foundations of the theory of probability* (New York, 1950).

3. The description logic on which this probability logic is based is almost the same as one proven to be sound and complete by R. Montague and D. Kalish in 'Remarks on descriptions and natural deduction' (*Archiv für Mathematische Logik und Grundlagenforschung*, Vol. 3, 1957).

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