Notre Dame Journal of Formal Logic
Volume VI, Number 3, July 1965

NOTE ON A THEOREM OF W. SIERPIŃSKI

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As generalization of a theorem of Steiner-Riess [1], W. Sierpiński, using the axiom of choice, established in [3] the following

Theorem S. For every non-finite set E there exists a family F of triplets of elements of E such that any two distinct elements of E appear exactly in one triplet of F.

As shown by B. Sobociński in [4], theorem S is equivalent to the axiom of choice.

Here we prove the theorem S for the case of a denumerable E,[1] by establishing an effective construction of the family F. We simply suppose E to be the set of positive integers $1, 2, 3, \ldots$.

Consider the sequence $\{f_n\}$, $n = 1, 2, \ldots$ of functions, where

$$f_n(x) = x + 2 - n + \frac{(x-2)\,(x-1)}{2}$$

For the integer values of x, $f_n(x)$ takes integer values, and for $n \neq m$, $f_n(x) \neq f_m(x)$ for $x > \mathrm{Max}\,(m,n)$. Moreover $f_1(2) = 3$, $f_n(n+1) - f_1(n) = 1$, and for $x = n$, $f_{n-1}(x), f_{n-2}(x), \ldots, f_2(x)$ and $f_1(x)$ take as values all consecutive integers between $f_1(n-1)$ and $f_{n+1}(n)$ respectively (last two excluded). Therefore, for integer $x > n$ the functions $f_n(x)$ take as values all integers $\geqslant 3$, and every such integer appears in the double sequence $\{f_n(x)\}$, $n = 1, 2, \ldots, x > n$, exactly once. Also, if $n < x$ then $f_n(x) > x$.

In the following we suppose x to run only over positive integers.

Now construct the set F of triplets (p,q,r) of positive integers, $p < q < r$, in the form of a matrix as follows:

In the first row put successively all triplets

$$(1, x, f_1(x))$$

1. In [2], p. 57, Sierpiński notices that theorem S for the set of all natural numbers can be established without the aid of the axiom of choice.

Received October 13, 1964

for all $x > 1$ which are different from

$$(1) \qquad k + 2 - 1 + \frac{(k-2)\ (k-1)}{2} \qquad \textit{for } k = 2,\ 3,\ 4,\ \ldots$$

In the second row put successively all triplets

$$(2,\ x,\ f_2(x))$$

for all $x > 2$ which are different from

$$(2) \qquad k + 2 - 2 + \frac{(k-2)\ (k-1)}{2} \qquad \textit{for } k = 3,\ 4,\ 5,\ \ldots$$

Moreover, if the triplet $(1, 2, f_1(2))$ has appeared in the first row we eliminate $f_1(2)$ as the possible value of x.

Having constructed first n-1 rows, construct the n-th row as follows. Its elements are all triplets

$$(n,\ x,\ f_n(x))$$

for all $x > n$ which are different from

$$(n) \qquad k + 2 - n + \frac{(k-2)\ (k-1)}{2} \qquad \textit{for } k = n + 1,\ n + 2,\ \ldots$$

Moreover, if any of the triplets $(1,n,f_1(n))$, $(2,n,f_2(n))$, . . . , $(n-1,n,f_{n-1}(n))$ have appeared in the 1-st, resp. 2-d, ... , resp. (n-1)-th row, we eliminate all third coordinates $f_i(n)$ of such triplets as possible values of x.

We state: any two integers n,m such that $1 \leq n < m$ appear exactly in one of the triplets of F.

Namely, n,m will appear in the triplet $(n,m,f_n(m))$ if and only if the pair n,m has not appeared as the second and third coordinate respectively in any of the triplets $(k,\ n,\ f_k(n))$ for $k < n$. In such a case, as $m < f_n(m) < f_1(m)$, we need to consider at most first $f_1(m)$ columns in the first n-1 rows of the matrix F, and find the triplet containing n and m as the second and the third coordinate.

Therefore, constructing first n rows up to the $f_1(m)$-th column, we can effectively find the unique triplet containing n and m.

The following diagram gives some first elements of the matrix F. The empty places indicate the triplets which have been eliminated.

(1,2,3), , (1,4,8), (1,5,12), (1,6,17), (1,7,23), , (1,9,38), . . .
, (2,4,7), (2,5,11), (2,6,16), , (2,8,29), (2,9,37), . . .
(3,4,6), (3,5,10), , (3,7,21), (3,8,28), (3,9,36), . . .
(4,5,9) , , , , , (4,10,44)

If we imagine the matrix F as given by the above diagram, then to find the triplet with n and m it is enough to check all first n rows up to m-th column, (supposing that $n < m$), or more economically, only the n-th column and the m-th row till the place of $(n, m, f_n(m))$.

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