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A NOTE ON CONTINUOUS GAMES, THE NOTION OF STRATEGY AND ZERMELO'S AXIOM*

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1. Introduction. The notion of a continuous game has appeared very naturally from a generalization of discrete (or sequential) games, and partly stems from the simple fact that a time scale is more often than not considered continuous. While for sequential games, finitist as they are, (see M. Claude Berge), the notion of strategy does not bring set-theorectical difficulties, the same is not the case for continuous games. In fact, the problem of existence of a single (pure) strategy for such games is strongly linked to Zermelo's Axiom of Choice for infinite sets.

A particular approach to continuous games has, in the recent years and because of its immediate practical applications, received increased attention. It was started by Rufus Isaacs in 1954 and 1955 in a series of Rand Corporation memoranda which were followed by, among others, papers by Scarf, Fleming and Berkovitz, and goes under the general heading of "Differential Games". The method is akin to and includes the widespread one in Control Theory, and is based on a set of differential equations:

$$\dot{x} = G(x : \phi; \psi; \ldots .)$$

where x is the state vector and $\phi; \psi; \ldots .$ are vector-valued functions of time chosen independently by the various players; associated with one or more optimization criteria (integral or terminal pay-offs) etc. .

Intrigued by the implications of the existence of pure strategy theorems for such games, the author has devised a very general set-theoretical description of a n-person continuous game with simultaneous moves (that is, a description of the game in extensive form) and attempted to determine the power of the sets involved. In behalf of clarity, we present here a two-person simplified game only. It seems that the same reasoning could be applied to the question of existence of single solutions to sets of differential equations without mentioning game theory, mutatis mutandis.

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2. Description of the Game. We shall take as our model for the game the set

(1) $\Gamma = \{ \mathfrak{M}_\kappa ; \mathfrak{F}_k \}$; $\kappa \varepsilon \mathfrak{K}$ $k = 1,2$

where

$\mathfrak{M}_\kappa$ = move corresponding to $\kappa \varepsilon \mathfrak{K}$
$\mathfrak{K}$ = an index set of special structure
$\mathfrak{F}_{1,2}$ = pay-off for player 1; 2

moreover:

(2) $\mathfrak{M}_\kappa = \{ \mathscr{A}_\kappa(\alpha_\kappa) | \alpha_\kappa \varepsilon \mathbf{A}_\kappa \}$ $\mathbf{A}_\kappa$ an index set

$\mathscr{A}_\kappa(\alpha_\kappa)$ are called "alternatives". We let also:

σ_κ = player's or chance choice on the move $\mathfrak{M}_\kappa$; $\sigma_\kappa \varepsilon \mathbf{A}_\kappa$

The final pay-off depending on all the choices made during the play:

(3) $\mathfrak{F} = \mathfrak{F}(\{\sigma_\kappa | \kappa \varepsilon \mathfrak{K}\})$

If $\mathfrak{M}_\kappa$ is a chance move we must know the probability distribution in $\mathbf{A}_\kappa$, that is, some $p(\alpha_\kappa)$ such that:

(4) $\int_{\mathbf{A}_\kappa} p(\alpha_\kappa)\, d\alpha_\kappa = 1$

the integral being taken in the Lebesgue-Stieltjes sense.

Most descriptions of games consider the moves as a finite sequence $S : \omega' \to U\mathfrak{M}$ where for some $q \varepsilon \omega$, $\omega' = \{p | p \varepsilon \omega ; p < q\}$, and $U\mathfrak{M}$ is the space of all possible moves. I suggest that a continuous play with simultaneous moves by several players could be described by a net $S : \mathfrak{K} \to U\mathfrak{M}$, the directed set $\mathfrak{K}$ having a peculiar structure, since the time scale is very important in the ordering of moves. This ordering in $\mathfrak{K}$ may be described, in our model, by the existence of an at most three-to-one map $h : \mathfrak{K} \to \mathfrak{K}'$, $\mathfrak{K}'$ being linearly ordered. We define:

(5) $[xy] : x, y \varepsilon \mathfrak{K} :\supset: x < y \,.\equiv.\, h(x) < h(y)$
$[xy] : x, y \varepsilon \mathfrak{K} :\supset: x < y \,.\vee.\, y < x \,.\vee.\, x \doteq y$

where $\doteq$ indicates "is at the same level as" or "is simultaneous to". The above is obviously a weakened linear ordering. Since the play must eventually start and finish, the set $\mathfrak{K}$ has to have at least one infimum κ and and one supremum κ^*.

3. State of Information. Consider the move $\mathfrak{M}_\kappa$. The player k_κ may or may not know the choices $\sigma_\lambda ; \lambda \varepsilon \mathfrak{K}$, $\lambda < \kappa$. One may attempt to describe the state of information of k_κ by a subset of $\mathfrak{K}$:

(6) $\Lambda_\kappa \subset \{ \lambda | \lambda \varepsilon \mathfrak{K} \,.\, \lambda < \kappa \} \subset \mathfrak{K}$.

and k_κ will know all σ_λ; $\lambda \varepsilon \Lambda_\kappa$.

(7) Def.: If $\lambda \varepsilon \Lambda_\kappa$ then λ is "preliminary" to κ.

(8) Def.: If $\lambda < \kappa$ then λ is "anterior" to κ.

Obviously, preliminarity $\rightarrow$ anteriority but not vice-versa. Moreover, preliminarity is not transitive.

(9) Def: If in Γ preliminarity $\leftrightarrow$ anteriority then Γ is a "perfect information game".

However Λ_κ is not enough to describe the situation at $\mathfrak{M}_\kappa$. That is, $\{\mathcal{A}_\kappa(\alpha_\kappa)\}$ may also depend on the previous choices. We must resort to a quite general description. Let:

(10) $\Phi_\kappa = \{h \mid h = h(\sigma_\lambda) \ ; \ \lambda \varepsilon \mathfrak{K} \ ; \ \lambda < \kappa\}$

of all sets h depending on the previous choices. Then $k_\kappa \varepsilon \Phi_\kappa$ and $\mathbf{A}_\kappa \varepsilon \Phi_\kappa$ etc..

4. The Notion of Strategy. Let Ω be the space of all "plays"[1], that is:

(11) $\Omega = \{S \mid S: \mathfrak{K} \rightarrow U\mathfrak{M}\}$

It is not enough, however, that $\mathfrak{K}$ would satisfy (5) and (6). The net S may send the elements of $\mathfrak{K}$ into $U\mathfrak{M}$ in an inordinate fashion and the result would be a chaotical description bearing little semblance to a game. This inclusion of "impossible" plays in Ω does not bring immediate difficulties. Consider then a certain "instant" $\lambda' \varepsilon \mathfrak{K}'$ and the elements:

(12) $\mathfrak{K}_{\lambda'} = \{\lambda \varepsilon \mathfrak{K} \mid h(\lambda) < \lambda'\}$

this set narrows Ω down to the subset $A_{\lambda'}$ of all possible plays that, up to λ', have the same course. That is, $A_{\lambda'}$ is the set of all nets S for which the corresponding $\mathfrak{K}$ agree in their $\mathfrak{K}'$ and $S[\mathfrak{K}_{\lambda'}]$ are the same. The $A_{\lambda'}$ are obviously disjoint. Consider:

(13) $\mathfrak{A}_{\lambda'} = \{A_{\lambda'} \mid \lambda' \varepsilon \mathfrak{K}'\}$

and $\mathfrak{A}_{\lambda'}$ is not a decomposition of Ω but, since it includes all possible plays, it divides a proper subset $\Omega^* \subset \Omega$ into equivalence classes.

Consider now $h^{-1}(\lambda')$. This may be none, one, two or at most three $\kappa \varepsilon \mathfrak{K}$. Obviously $h^{-1}(\lambda')$ is the same within each $A_{\lambda'}$ but may vary from one to another. In other words, in some cases the complete structure of $\mathfrak{K}$ is only known a posteriori.

From the point of view of player k, it may or may not be the case that $k_\kappa = k$ for some $\kappa \varepsilon h^{-1}(\lambda')$. In other words, k may or may not be called to make a decision at the instant λ'. Let's suppose he is and define:

(14) $B_\kappa(k) = \{A_{\lambda'} \mid k_\kappa = k \ ; \ \kappa \varepsilon h^{-1}(\lambda')\}$

of all plays in which it is k's turn to move at λ'. Then form:

(15) $\mathfrak{B}_{\lambda'} = \{B_\kappa(k) \mid k = 0,1,2\}$

Note that:

(16) $\bigcup_{\mathfrak{B}_{\lambda'}} B_\kappa(k) = \Omega^*$

1. In the sense of the French: "Partie" or Portuguese "Partida"

but the sets $B_\kappa(k)$ are not disjoint. Also, because of (13), (14), (15) and (16):

(17) $\mathfrak{A}_{\lambda'} \subset \mathfrak{B}_{\lambda'}$

Suppose now that for some $\kappa \varepsilon h^{-1}(\lambda')$ we have $k_\kappa = 0$, the player number zero representing chance moves. To $k_\kappa = 0$ is presented a set as (2) with probabilities as in (4). Note that this distribution is the same within each $A_{\lambda'}$.

If $k_\kappa \neq 0$ we have a so-called personnal move. The player k_κ does not necessarily possess the full state of information described by $A_{\lambda'}$, but is aware of some Φ_κ as in (10). Φ_κ then divide $B_\kappa(k)$ into disjoint sets $D_\kappa \varepsilon \mathfrak{D}_\kappa(k)$ and $UD_\kappa = B_\kappa(k)$. In other words, k_κ is not aware of $A_{\lambda'}$ but of D_κ only. For a chance move we may assume $D_\kappa = A_{\lambda'}$.

At λ' all players $k_\kappa = k$; $\kappa \varepsilon h^{-1}(\lambda')$ make their decisions (there are at most three). These will divide $A_{\lambda'}$ into three subsets which correspond to the restrictions imposed by the choices σ_κ, since in $A_{\lambda'}$ these choices will induce a disjoint cover with as many sets as the elements of:

(18) $\bigtimes_{\mathfrak{K} \varepsilon h^{-1}(\lambda')} \mathsf{A}_\kappa$

From the point of view of k_κ, the set D_κ is such that certainly $A_{\lambda'} \subset D_\kappa$ (unless he is unrepairably deluded as the fool in Wiener's considerations). So that his choice σ_κ will divide D_κ into subsets C_κ disjoint and their family $\mathfrak{C}_\kappa$ covers D_κ.

Note that for:

(19) $[\lambda'] \therefore \lambda' \varepsilon \mathfrak{K}' . \lambda' < \kappa_*' :\supset: \mathfrak{A}_{\lambda'} = \{\Omega\}$

and

(20) $[\lambda'] \therefore \lambda' \varepsilon \mathfrak{K}' . \lambda' > \kappa^{*'} :\supset: \mathfrak{A}_{\lambda'} = \Omega$

Here the distinction between $\{\Omega\}$ and Ω itself is crucial. Obviously, after the play is finished each $A_{\lambda'}$ corresponds to one $\pi \varepsilon \Omega$ (or one $S \varepsilon \Omega$).

(21) *Def.: A strategy of the player k is a function $\mathbf{E}_k$ on $\mathfrak{K} \times \mathfrak{D}_\kappa(k)$ into $\mathfrak{C}_\kappa(k)$ for all $\kappa \varepsilon \mathfrak{K}$ and $\kappa \varepsilon h^{-1}(\lambda')$ and $k_\kappa = k$ and $k = 1,2$ such that:*

$$\mathbf{E}_k < \kappa; D_\kappa > = C_\kappa \qquad C_\kappa \varepsilon \mathfrak{C}_\kappa(k); \quad C_\kappa \subset D_\kappa$$

(22) *Def.: An Umpire's Choice is a function $\mathbf{E}_0$ on $\mathfrak{K} \times \mathfrak{A}_{\lambda'}$ into $\mathfrak{C}_\kappa(0)$ such that, for all $\lambda' \varepsilon \mathfrak{K}'$:*

$$\mathbf{E}_0 < \lambda' ; A_{\lambda'} > = C_\kappa \qquad C_\kappa \varepsilon \mathfrak{C}\ (0)$$

In the space Σ_0 of the functions $\mathbf{E}_0$ we have an induced probability density. If for all $\lambda' \varepsilon \mathfrak{K}'$ such that $k_\kappa = 0$ for $k \varepsilon h^{-1}(\lambda')$ we have a set of choices $\mathcal{A}_\kappa(\alpha_\kappa)$ for $\alpha_\kappa \varepsilon \mathsf{A}_\kappa$ as in (2), we see that $p(\alpha_\kappa) = p(C_\kappa)$. The event $\mathbf{E}_0$ would then be a conjunction of all particular C_κ for all $\lambda' \varepsilon \mathfrak{K}'$ as above and the probability density:

(23) $p(\mathbf{E}_0) = \prod_{\lambda' \varepsilon \mathfrak{K}';\ \kappa \varepsilon h^{-1}(\lambda');\ k_\kappa = 0} p(c_\kappa)$

where we consider the events for each λ' as independent.

Certainly:

(24) $\int_{\Sigma} p(E_0)dE_0 = 1$

5. The Power of These Sets. Throughout this analysis we shall accept a restricted continuum hypothesis for sets with power up to $\mathfrak{g} = 2^{\mathfrak{f}}$.

Consider the set $\mathfrak{K}'$. This set may be considered of the power:

(25) $\overline{\overline{\mathfrak{K}'}} = \mathfrak{x}$

since our time scale is continuous. $\mathfrak{K}$ is related to $\mathfrak{K}'$ by h. Since the number of players is only two we have at most:

(26) $\overline{\overline{\mathfrak{K}}} = 3\,\overline{\overline{\mathfrak{K}'}} = 3 \cdot \mathfrak{x} = \mathfrak{x}$

then, since there is a one-one correspondence $\mathfrak{M}_{\kappa} \rightarrow \kappa$:

(27) $\overline{\overline{U\mathfrak{M}}} = \overline{\overline{\mathfrak{K}}} = \mathfrak{x}$

Let in (2):

(28) $\overline{\overline{\mathbf{A}}}_{\kappa} = \mathfrak{x}$ All $\kappa\, \varepsilon\, \mathfrak{K}$ $\therefore$ $\overline{\overline{\mathfrak{M}}}_{\kappa} = \mathfrak{x}$

according to (11) and (27) then:

(29) $\overline{\overline{\Omega}} = \mathfrak{x}^{\mathfrak{x}} = 2^{\mathfrak{x}} = \mathfrak{f} = \overline{\overline{R^R}}$

and the space of all games has the same power as the space of all real valued functions of a real variable. This does not mean that $\overline{\overline{\Omega}}{}^* = \mathfrak{f}$ because the definition of "possible" game could make $\overline{\overline{\Omega}}{}^* = \mathfrak{x}$ or even $\overline{\overline{\Omega}}{}^* = \aleph_0$.

Now let's consider (13) and (19). We see that before the game starts, that is for $\lambda' < \kappa_*'$, we have:

(30) $\mathfrak{A}_{\lambda'} = \{\Omega\}$

as in (19). Hence:

$\overline{\overline{\mathfrak{A}}}_{\lambda'} = 1$

At $\lambda' = \kappa_*'$ the players make their first choice. This will divide the set $\mathfrak{A}_{\lambda'}$ into as many different subsets as the set:

$$\mathop{\times}_{\kappa \varepsilon h^{-1}(\lambda')} \mathbf{A}_{\kappa}$$

as in (18). And:

(31) $\overline{\overline{\mathfrak{A}}}_{\lambda'} = \overline{\overline{\times\, \mathbf{A}_{\kappa}}} = 3 \cdot \overline{\overline{\mathbf{A}}}_{\kappa} = 3 \cdot \mathfrak{x} = \mathfrak{x}$

this process repeating itself $\mathfrak{x}$ times, that is, after any non-zero interval in $\mathfrak{K}'$ has elapsed we have:

(32) $\overline{\overline{\mathfrak{A}}}_{\lambda'} = \mathfrak{x}^{\mathfrak{x}} = \mathfrak{f}$

and so, for $\lambda' > \kappa^{*\prime}$ as in (20):

$\overline{\overline{\mathfrak{A}}}_{\lambda'} = \overline{\overline{\Omega}} = \mathfrak{f}$

which agrees with the result of (29) above.

Now consider the player k. At $\lambda' < \kappa_*'$ he is aware of the whole $\mathfrak{A}_{\lambda'}$. At $\lambda' = \kappa^{*\prime}$ he is then required to choose one α_κ out of $\mathbf{A}_\kappa$. For simplification of the reasoning, we may consider perfect information games as in (9) without loss of generality. The process of decision repeats itself for all $\lambda' \varepsilon \mathfrak{K}'$. In other words, he must make $\mathfrak{c}$ choices out of sets, each of which has the power of the continuum.

Now consider the definition of strategy of (21). With $A_{\lambda'} | D_\kappa$ we have:

(33) $\mathbf{E}_k : (\mathfrak{K} \times \mathfrak{A}_{\lambda'}) \to C_\kappa$

and the power of the set of these functions:

(34) $\overline{\overline{\mathbf{E}}}_k = \overline{\overline{\mathfrak{S}(\kappa)}}^{\overline{\overline{\mathfrak{K}\times\mathfrak{A}_{\lambda'}}}} = \mathfrak{c}^{\mathfrak{c}\cdot\mathfrak{f}} = \mathfrak{c}^{\mathfrak{f}} = 2^{\mathfrak{f}}$

In other words, the choice of a single strategy for the player k amounts to, out of a set of power $2^{\mathfrak{f}}$, to choose one element.

In the other hand, the existence of one strategy for the player k amounts to the following: out of each set of a family:

$\mathfrak{K} \times \mathfrak{A}_{\lambda'}$

which, as it was seen, has the power $\mathfrak{f}$, k must be able to choose one element of each set, each set having the power $\mathfrak{c}$.

The possibility of choosing one strategy out of an infinite set is also strongly related to the notion of self-information. For the probability of a single point in an infinite set for which the probability distribution does not have impulses is zero (the singleton has zero Lebesgue measure) and hence the self-information is not finite. Hence, even if k could choose one strategy Σ_k it seems that he would never be able to announce his choice, since this would imply the transmission of infinitely many bits. Samuel Karlin has advanced a finitist idea to obviate these troubles. In essence, his K-strategies are simply the result of discretizing the time scale. Whether or not his ε-optimal solutions will, heuristically, yield truly continuous optimal solutions is a point considered by many as an unnecessary subtlety.

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