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THE RELATION OF WEAKLY DISCRETE TO SET AND EQUINUMEROSITY IN MEREOLOGY

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INTRODUCTION*

This paper deals with a formal system introduced by Leśniewski called mereology, in which, as the name implies, the concept of "party of the whole" is primitive. This system studies the properties of the collective class. Mereology is based on ontology, a formal system in which "is" is the primitive term. Ontology in turn is based on protothetic or on propositional calculus and quantification theory.

The collective class differs greatly from the distributive class. However, under the condition, "the a's are weakly discrete", which we introduce, the collective class of the a's and the distributive class of the a's become alike with respect to equinumerosity. We are thus able to prove the analogs of three important set-theoretic theorems under this condition. Two of these were previously known for the condition, "the a's are discrete", but the third is an entirely new theorem.

We then prove that for a certain class of statements dealing primarily with equinumerosity, discrete and weakly discrete are inferentially equivalent.

ONTOLOGICAL PRELIMINARIES

Ontology has the following very intuitive sole axiom.

01 $[Aa] \therefore A \varepsilon a . \equiv : [\exists B] . B \varepsilon a : [CD] : C \varepsilon A . D \varepsilon A . \supset . C \varepsilon D : [C] : C \varepsilon A . \supset . C \varepsilon a$

There is no rule which guides the use of capital and small letters. For easier understanding we shall use capital letters for names that are known to be proper and lower case letters otherwise. In this system two types of

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definitions are allowable, ontological definitions which are of the form

$$[Aabc\ldots]:A\varepsilon\alpha(abc\ldots).\equiv.A\varepsilon A.\beta(Aabc\ldots),$$

and protothetical definitions which are of the form

$$[abc\ldots]:\Sigma(abc\ldots).\equiv.\sigma(abc\ldots ABC\ldots\alpha\beta\gamma\ldots)$$

where $ABC\ldots$ occur bound in σ and $\alpha\beta\gamma\ldots$ are previous defined constants.

The construction of a constant of a certain semantical category automatically supplies the rule of extensionality for that category.

We shall now give a list of basic definitions together with a few scattered theorems. The basic properties of the terms defined are so obvious that we have not bothered to list them. Neither these definitions nor their properties will be specifically referred to in the body of the dissertation.

02 $[Aa]:A\varepsilon a.\supset.A\varepsilon A$

D01 $[a]:!\{a\}.\equiv.[\exists A].A\varepsilon a$

D02 $[a]\because\rightharpoonup\{a\}.\equiv:[AB]:A\varepsilon a.B\varepsilon a.\supset.A\varepsilon B$

03 $[A]:A\varepsilon A.\equiv.!\{A\}.\rightharpoonup\{A\}$

D03 $[AB]:A=B.\equiv.A\varepsilon B.B\varepsilon A$

04 $[AB]\because A=B.\equiv:A\varepsilon A.B\varepsilon B:[\sigma]:\sigma(A).\equiv.\sigma(B)$

D04 $[AB]:A\neq B.\equiv.A\varepsilon A.B\varepsilon B.\sim(A=B)$

D05 $[A]:A\varepsilon\vee.\equiv.A\varepsilon A$

D06 $[A]:A\varepsilon\wedge.\equiv.A\varepsilon A.\sim(A\varepsilon A)$

D07 $[Aa]:A\varepsilon\sim(a).\equiv.A\varepsilon A.\sim(A\varepsilon a)$

D08 $[ab]\therefore a\subset b.\equiv:[A]:A\varepsilon a.\supset.A\varepsilon b$

D09 $[ab]\because a\circ b.\equiv:[A]:A\varepsilon a.\equiv.A\varepsilon b$

05 $[ab]\because a\circ b.\equiv:[\sigma]:\sigma(a).\equiv.\sigma(b)$

D010 $[ab]:a\sqsubset b.\equiv.!\{a\}.a\subset b$

D011 $[Aab]:A\varepsilon a\cap b.\equiv.A\varepsilon a.A\varepsilon b$

D012 $[Aab]\because A\varepsilon a\cup b.\equiv:A\varepsilon A:A\varepsilon a.\vee.A\varepsilon b$

06 $[Aab]\because A\varepsilon a\cup b.\equiv:A\varepsilon a.\vee.A\varepsilon b$

D013 $[ab]:a\mathrel{\dot{\subset}}b.\equiv.a\subset b.\sim(b\subset a)$

D014 $[ab]:a\mathrel{\dot{\sqsubset}}b.\equiv.a\sqsubset b.\sim(b\sqsubset a)$

D015 $[ab\sigma]\because a\,{}^{\sigma}_{\infty}b.\equiv:[A]:A\varepsilon a.\supset.[\exists B].B\varepsilon b.B\varepsilon\sigma(A):[B]:B\varepsilon b$
$.\supset.[\exists A].A\varepsilon a.B\varepsilon\sigma(A):[ABC]:A\varepsilon a.C\varepsilon a.B\varepsilon b.B\varepsilon\sigma(A).B\varepsilon$
$\sigma(C).\equiv.A=C:[ABC]:A\varepsilon a.B\varepsilon b.C\varepsilon b.B\varepsilon\sigma(A).C\varepsilon\sigma(A).\equiv.B=C$

D016 $[ab]:a\infty b.\equiv.[\exists\sigma].a\,{}^{\sigma}_{\infty}b$[1]

D017 $[AB\sigma]:A\varepsilon\sigma^{-1}(B).\equiv.B\varepsilon\sigma(A).A\varepsilon A$

D018 $[ab]:a\,{}^{\propto}_{\infty}b.\equiv.[\exists c].c\subset b.a\infty c$

D019 $[ab]:a\propto b.\equiv.a\,{}^{\propto}_{\infty}b.\sim(a\infty b)$

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1. It can easily be shown that *D016* and the usual definition of equinumerosity are equivalent in the field of ontology.

MEREOLOGICAL PRELIMINARIES

In this paper we shall find it most convenient to use element **el**, instead of **pr** as the primitive term, so we shall give an axiom system for **el** closely related to the one given by Leśniewski for **pr**.

M1 $[A]:A\varepsilon A.\supset.A\varepsilon\mathbf{el}(A)$
M2 $[AB]:A\varepsilon\mathbf{el}(B).B\varepsilon\mathbf{el}(A).\supset.A=B$
M3 $[ABC]:A\varepsilon\mathbf{el}(B).B\varepsilon\mathbf{el}(C).\supset.A\varepsilon\mathbf{el}(C)$
M4 $[AB]:A\varepsilon\mathbf{el}(B).\supset.B\varepsilon B$
DM1 $[Aa]\therefore A\varepsilon\mathbf{Kl}(a).\equiv:A\varepsilon A:[D]:D\varepsilon a.\supset.D\varepsilon\mathbf{el}(A):[D]:D\varepsilon \mathbf{el}(A).\supset.[\exists EF].E\varepsilon a.F\varepsilon\mathbf{el}(D).F\varepsilon\mathbf{el}(E)$
M5 $[ABa]:A\varepsilon\mathbf{Kl}(a).B\varepsilon\mathbf{Kl}(a).\supset.A=B$
M6 $[Aa]:A\varepsilon a.\supset.[\exists B].B\varepsilon\mathbf{Kl}(a)$

There are many theorems that were proved by Leśniewski or Sobociński that will be needed in the proofs which follows. For the sake of convenient reference we shall give a complete list of all these below.

M7 $[ABa]:A\varepsilon\mathbf{el}(B).B\varepsilon a.\supset.A\varepsilon\mathbf{el}(\mathbf{Kl}(a))$
M8 $[Aa]:A\varepsilon a.\supset.A\varepsilon\mathbf{el}(\mathbf{Kl}(a))$
M9 $[Aa]:A\varepsilon a.\supset.\mathbf{Kl}(a)\varepsilon\mathbf{Kl}(a)$
M10 $[Aa]:A\varepsilon\mathbf{Kl}(a).\supset.[\exists B].B\varepsilon a$
M11 $[Aa]:A\varepsilon\mathbf{Kl}(a).\supset.A=\mathbf{Kl}(a)$
M12 $[Aa]:A\varepsilon a.\supset.A=\mathbf{Kl}(A)$
M13 $[ABCab]:A\varepsilon\mathbf{Kl}(a).B\varepsilon\mathbf{Kl}(b).C\varepsilon\mathbf{Kl}(a\cup b).\supset.C\varepsilon\mathbf{Kl}(A\cup B)$
DM2 $[AB]:A\varepsilon\mathbf{pr}(B).\equiv.A\varepsilon\mathbf{el}(B).\sim(A=B)$
DM3 $[Aa]\therefore A\varepsilon\mathbf{st}(a).\equiv:A\varepsilon A:[D]:D\varepsilon\mathbf{el}(A).\supset.[\exists EF].E\varepsilon a.E\varepsilon\mathbf{el}(A).F\varepsilon\mathbf{el}(D).F\varepsilon\mathbf{el}(E)$
M14 $[Aa]:A\varepsilon a.\supset.A\varepsilon\mathbf{st}(a)$
M15 $[ab]:a\subset b.\supset.\mathbf{st}(a)\subset\mathbf{st}(b)$
M16 $[Aa]:A\varepsilon\mathbf{Kl}(a).\supset.A\varepsilon\mathbf{st}(a)$
M17 $[Aa]:A\varepsilon\mathbf{st}(a).a\subset\mathbf{el}(A).\equiv.A\varepsilon\mathbf{Kl}(a)$
M18 $[Aab]:A\varepsilon\mathbf{Kl}(a).a\subset b.\supset.A\varepsilon\mathbf{st}(b)$
M19 $[Aa]:A\varepsilon\mathbf{st}(a).\supset.[\exists b].A\varepsilon\mathbf{Kl}(b).b\subset a$
M20 $[Aa]:A\varepsilon\mathbf{st}(a).\supset.A\varepsilon\mathbf{el}(\mathbf{Kl}(a))$
M21 $[Aa]:a\subset\mathbf{el}(A).\equiv.\mathbf{Kl}(a)\subset\mathbf{el}(A)$
M22 $[ABab]:A\varepsilon\mathbf{Kl}(a).B\varepsilon\mathbf{Kl}(b).a\subset b.\supset.A\varepsilon\mathbf{el}(B)$
M23 $[Aab]:A\varepsilon\mathbf{Kl}(a).a\subset b.\supset.A\varepsilon\mathbf{el}(\mathbf{Kl}(b))$
M24 $[AB]\therefore A\varepsilon A:[C]:C\varepsilon\mathbf{el}(A).\supset.[\exists D].D\varepsilon\mathbf{el}(C).D\varepsilon\mathbf{el}(B):\supset.A\varepsilon\mathbf{el}(B)$
DM4 $[AB]\therefore A\varepsilon\mathbf{ex}(B).\equiv:A\varepsilon A.B\varepsilon B:[C]:C\varepsilon\mathbf{el}(A).\supset.\sim(C\varepsilon\mathbf{el}(B))$
M25 $[AB]:A\varepsilon\mathbf{ex}(B).\equiv.B\varepsilon\mathbf{ex}(A)$
M26 $[AB]:A\varepsilon\mathbf{el}(B).\supset.\sim(A\varepsilon\mathbf{ex}(B))$
M27 $[ABC]:A\varepsilon\mathbf{el}(B).B\varepsilon\mathbf{ex}(C).\supset.A\varepsilon\mathbf{ex}(C)$
M28 $[ABC]:A\varepsilon\mathbf{el}(B).C\varepsilon\mathbf{ex}(B).\supset.A\varepsilon\mathbf{ex}(C)$
DM5 $[A]:A\varepsilon\mathbf{Un}.\equiv.A\varepsilon\mathbf{Kl}(\vee)$
M29 $[A]:A\varepsilon A.\supset.A\varepsilon\mathbf{el}(\mathbf{Un})$
M30 $[A]:A\varepsilon A.[B].\sim(B\varepsilon\mathbf{ex}(A)).\equiv.A\varepsilon\mathbf{Un}$

DM6 $[a]::\mathbf{dscr}\{a\}.\equiv.\therefore[AB].\because.A\varepsilon a.B\varepsilon a.\supset:A=B.\mathrm{v}.A\varepsilon\mathbf{ex}(B)$[2]
DM7 $[ABC]:A\varepsilon\mathbf{Cm}(BC).\equiv.A\varepsilon\mathbf{Kl}(\mathbf{el}(C)\cap\mathbf{ex}(B)).B\varepsilon\bar{\mathbf{el}}(C)$
DM8 $[ABC]:A\varepsilon C\backslash B.\equiv.A\varepsilon\mathbf{Cm}(BC)$
M31 $[ABC]:A\varepsilon C\backslash B.\supset.A\varepsilon\mathbf{ex}(B)$
M32 $[ABC]:A\varepsilon C\backslash B.\supset.A\varepsilon\mathbf{el}(C)$
M33 $[AB]:A\varepsilon\mathbf{pr}(B).\equiv.B\backslash A\varepsilon B\backslash A$
M34 $[ABCD]:D\varepsilon C\backslash A.D\varepsilon C\backslash B.\supset.A=B$

1. THE TERMS WEAKLY OUTSIDE AND WEAKLY DISCRETE AND THEIR RELATION TO SET AND EQUINUMEROSITY

Leśniewski has proved that

$[a]:\mathbf{dscr}\{a\}.\sim(\rightharpoondown\{a\}).\supset.a\propto\mathbf{st}(a)$[3]

and Sobociński that

$[ab]:\mathbf{dscr}\{a\}.\mathbf{dscr}\{b\}.a\infty b.\supset.\mathbf{st}(a)\infty\mathbf{st}(b).$

The primary aim of this section is to prove that these theorems are still valid if the condition discrete is replaced by the condition weakly discrete (**w-dscr**) defined below. The proofs are based on the fact that **w-dscr** has the following characterization:

$[a].\therefore\mathbf{w\text{-}dscr}\{a\}.\equiv:[BCbc]:B\varepsilon\mathbf{Kl}(b).C\varepsilon\mathbf{Kl}(c).b\subset a.c\subset a.B=C.\supset.b\circ c,$

that is, **w-dscr** is precisely the property that insures that distinct ontological sets give rise to distinct mereological sets. Since the above theorems are valid for ontological sets without the additional hypothesis, we can infer that they are valid for mereological sets under the hypothesis of **w-dscr**. However, this argument carries us into a combination of meta-ontology and metamereology. We shall prove these theorems within the system of mereology.

From the above characterization it is clear that if the a's are discrete then the a's are weakly discrete.

In addition to the stronger versions of the above theorems we prove the new result

$[ab]:\mathbf{w\text{-}dscr}\{b\}.a\infty b.\supset.\mathbf{st}(a)\overset{\propto}{\infty}\mathbf{st}(b).$

We also introduce and investigate the term that two objects are weakly outside each other if neither is an element of the other. This is used in another characterization of weakly discrete.

2. The term **dscr** was introduced by Sobociński who proved that it is primitive.

3. Leśniewski did not formally use the term **dscr** $\{a\}$.

1.1 DEFINITION AND BASIC PROPERTIES OF WEAKLY OUTSIDE

D1 $[AB]:A\varepsilon\textbf{w-ex}(B).\equiv.A\varepsilon A.B\varepsilon B.\sim(A\varepsilon\textbf{el}(B)).\sim(B\varepsilon\textbf{el}(A))$

T1 $[A].\sim(A\varepsilon\textbf{w-ex}(A))$ [*D1*, *M1*]

T2 $[AB]:A\varepsilon\textbf{w-ex}(B).\equiv.B\varepsilon\textbf{w-ex}(A)$ [*D1*]

T3 $[AB]:A\varepsilon\textbf{w-ex}(B).\supset.A\neq B$ [*D1*, *T1*]

T4 $[AB]:A\varepsilon\textbf{el}(B).\supset.\sim(A\varepsilon\textbf{w-ex}(B))$ [*D1*]

T5 $[AB]:A\varepsilon\textbf{ex}(B).\supset.A\varepsilon\textbf{w-ex}(B)$

PF $[AB]:\mathrm{Hp}(1).\supset.$[4]

2) $\sim(A\varepsilon\textbf{el}(B)).$ [*M26*, 1]

3) $B\varepsilon\textbf{ex}(A).$ [*M25*, 1]

4) $\sim(B\varepsilon\textbf{el}(A)).$ [*M26*, 3]

$A\varepsilon\textbf{w-ex}(B)$ [*D1*, *DM4*, 1, 2, 4]

T6 $[ABC]:A\varepsilon\textbf{ex}(B).C\varepsilon\textbf{el}(B).\supset.C\varepsilon\textbf{w-ex}(A)$ [*M28*, *T5*]

T7 $[AB]\therefore A\varepsilon A.B\varepsilon B:[C]:C\varepsilon\textbf{el}(B).\supset.C\varepsilon\textbf{w-ex}(A):\supset.A\varepsilon\textbf{ex}(B)$

PF $[AB]\therefore\mathrm{Hp}(3):\supset:$

4) $[C]:C\varepsilon\textbf{el}(B).\supset.\sim(C\varepsilon\textbf{el}(A)):$ [*D1*, 3]

$A\varepsilon\textbf{ex}(B)$ [*DM4*, 1, 2, 4]

T8 $[AB]\therefore A\varepsilon\textbf{ex}(B).\equiv:A\varepsilon A.B\varepsilon B:[C]:C\varepsilon\textbf{el}(B).\supset.C\varepsilon\textbf{w-ex}(A)$ [*DM4*, *T6*, *T7*]

T9 $[AB]:A\varepsilon A.B\varepsilon B.\sim(A\varepsilon\textbf{el}(B)).\supset.[\exists C].C\varepsilon\textbf{el}(A).C\varepsilon\textbf{ex}(B)$[5]

PF $[AB]\therefore\mathrm{Hp}(3).\supset:$

$[\exists C]:$

4) $C\varepsilon\textbf{el}(A):$

5) $[D]:\sim(D\varepsilon\textbf{el}(C)).\mathrm{v}.\sim(D\varepsilon\textbf{el}(B)):$ } [*M24*, 3, 1]

6) $[D]:D\varepsilon\textbf{el}(C).\supset.\sim(D\varepsilon\textbf{el}(B)):$ [5]

7) $C\varepsilon\textbf{ex}(B).$ [*DM4*, 4, 2, 6]

$[\exists C].C\varepsilon\textbf{el}(A).C\varepsilon\textbf{ex}(B)$ [4, 7]

T10 $[AB]:A\varepsilon\textbf{w-ex}(B).\supset.[\exists CD].C\varepsilon\textbf{el}(A).C\varepsilon\textbf{ex}(B).D\varepsilon\textbf{el}(B).D\varepsilon\textbf{ex}(A)$ [*D1*, *T9*]

T11 $[AB]\therefore[\exists C].C\varepsilon\textbf{el}(A).C\varepsilon\textbf{ex}(B):\supset.\sim(A\varepsilon\textbf{el}(B))$ [*M26*, *M3*]

T12 $[AB]:A\varepsilon\textbf{w-ex}(B).\equiv.[\exists CD].C\varepsilon\textbf{el}(A).C\varepsilon\textbf{ex}(B).D\varepsilon\textbf{el}(B).D\varepsilon\textbf{ex}(A)$
[*T10*, *D1*, *T11*, *M4*]

T13 $[AB]\therefore A\varepsilon A.B\varepsilon B.\sim(A\varepsilon\textbf{w-ex}(B)).\supset:[\exists ab]:A\varepsilon\textbf{Kl}(a).B\varepsilon\textbf{Kl}(b):a\subset b.\mathrm{v}.b\subset a$

PF $[AB]\therefore\mathrm{Hp}(3).\supset:$

4) $A\varepsilon\textbf{el}(A).$ [*M1*, 1]

5) $B\varepsilon\textbf{el}(B):$ [*M1*, 2]

6) $A\varepsilon\textbf{el}(B).\mathrm{v}.B\varepsilon\textbf{el}(A):$ [*D1*, 3, 1, 2]

7) $\textbf{Kl}(A\cup B)\varepsilon\textbf{el}(B).\mathrm{v}.\textbf{Kl}(A\cup B)\varepsilon\textbf{el}(A):$ [6, *M21*, 5, 4, *M9*, 1]

8) $A\varepsilon\textbf{Kl}(A).$ [*M12*, 1]

9) $B\varepsilon\textbf{Kl}(B):$ [*M12*, 2]

10) $A\varepsilon\textbf{Kl}(A\cup B).\mathrm{v}.B\varepsilon\textbf{Kl}(A\cup B):$ [7, *M2*, *M8*]

$[\exists ab].A\varepsilon\textbf{Kl}(a).B\varepsilon\textbf{Kl}(b):a\subset b.\mathrm{v}.b\subset a$ [10, 8, 9]

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4. 'Hp.' indicates the hypotheses, which are assumed to be numbered in the order of their appearance in the statement of the theorem.

5. This theorem was previously known.

$T14$ $[AB].\therefore.[\exists ab].A\varepsilon \mathsf{Kl}(a).B\varepsilon \mathsf{Kl}(b).a\subset b:\supset.\sim(A\varepsilon \mathsf{w\text{-}ex}(B))$ $[M22, T4]$

$T15$ $[AB].\therefore.A\varepsilon \mathsf{w\text{-}ex}(B).\equiv:A\varepsilon A.B\varepsilon B:[ab]:A\varepsilon \mathsf{Kl}(a).B\varepsilon \mathsf{Kl}(b).\supset.$
$\sim(a\subset b.\vee.b\subset a)$ $[T13, T14, D1, T2]$

$T16$ $[A]:A\varepsilon \mathsf{Un}.\supset.[B].\sim(B\varepsilon \mathsf{w\text{-}ex}(A))$

PF $[A].\therefore.\mathrm{Hp}(1).\supset:$
2) $[B]:B\varepsilon B.\supset.B\varepsilon \mathsf{el}(A):$ $[M29, 1]$
3) $[b]:B\varepsilon B.\supset.\sim(B\varepsilon \mathsf{w\text{-}ex}(A)):$ $[T4, 2]$
4) $[B]:\sim(B\varepsilon B).\supset.\sim(B\varepsilon \mathsf{w\text{-}ex}(A)):$ $[D1]$
$[B].\sim(B\varepsilon \mathsf{w\text{-}ex}(A))$ $[3, 4]$

$T17$ $[A]:A\varepsilon A.[B].\sim(B\varepsilon \mathsf{w\text{-}ex}(A)).\supset.A\varepsilon \mathsf{Un}$ $[M30, T5]$

$T18$ $[A]:A\varepsilon \mathsf{Un}.\equiv.A\varepsilon A.[B].\sim(B\varepsilon \mathsf{w\text{-}ex}(A))$ $[T16, T17]$

$T19$ $[AB]:A\varepsilon \mathsf{w\text{-}ex}(B).\equiv.A\varepsilon A.B\varepsilon B.\sim(A=B).\sim(A\varepsilon \mathsf{pr}(B)).\sim(B\varepsilon \mathsf{pr}(A))$
$[D1, DM2]$

1.2 DEFINITION AND BASIC PROPERTIES OF WEAKLY DISCRETE

$D2$ $[a].\therefore.\mathsf{w\text{-}dscr}\{a\}.\equiv:[Ab]:A\varepsilon a.b\subset a.A\varepsilon \mathsf{el}(\mathsf{Kl}(b)).\supset.A\varepsilon b$

$T20$ $[Aab]:\mathsf{dscr}\{a\}.A\varepsilon a.b\subset a.A\varepsilon \mathsf{el}(\mathsf{Kl}(b)).\supset.A\varepsilon b$

PF $[Aab]:\mathrm{Hp}(4).\supset.$
$[\exists EF].$
5) $E\varepsilon b.$
6) $F\varepsilon \mathsf{el}(A).$
7) $F\varepsilon \mathsf{el}(E).$ $[DM1, 4]$ (5–7)
8) $\sim(A\varepsilon \mathsf{ex}(E)).$ $[DM4, 6, 7]$
9) $E\varepsilon a.$ $[5, 3]$
10) $A=E.$ $[DM6, 1, 2, 9, 8]$
$A\varepsilon b$ $[5, 10]$

$T21$ $[a]:\mathsf{dscr}\{a\}.\supset.\mathsf{w\text{-}dscr}\{a\}$ $[T20, D2]$

$T22$ $[BCDacd].\therefore.[Ab]:A\varepsilon a.b\subset a.A\varepsilon \mathsf{el}(\mathsf{Kl}(b)).\supset.A\varepsilon b:C\varepsilon \mathsf{Kl}(c).D\varepsilon \mathsf{Kl}(d).c\subset a.$
$d\subset a.D=C.B\varepsilon c:\supset.B\varepsilon d$

PF $[BCDacd].\therefore.\mathrm{Hp}(7):\supset.$
8) $B\varepsilon a.$ $[7, 4]$
9) $B\varepsilon \mathsf{el}(C).$ $[M8, 7, 2, M11]$
10) $B\varepsilon \mathsf{el}(D).$ $[6, 9]$
$B\varepsilon d$ $[1, 8, 5, 10, 3, M11]$

$T23$ $[CDacd]:\mathsf{w\text{-}dscr}\{a\}.C\varepsilon \mathsf{Kl}(c).D\varepsilon \mathsf{Kl}(d).d\subset a,c\subset a.D=C.\supset.c\circ d$
$[D2, T22]$

$T24$ $[a].\therefore.\mathsf{w\text{-}dscr}\{a\}.\supset:[CDcd]:C\varepsilon \mathsf{Kl}(c).D\varepsilon \mathsf{Kl}(d).c\subset a.d\subset a.C=D.\supset.c\circ d$
$[T23]$

$T25$ $[Aa]:A\varepsilon a.A\varepsilon \mathsf{el}(\mathsf{Kl}(a\text{-}A)).\supset.[\exists CDcd].C\varepsilon \mathsf{Kl}(c).D\varepsilon \mathsf{Kl}(d).c\subset a.d\subset a.$
$C=D.\sim(c\circ d)$

PF $[Aa].\therefore.\mathrm{Hp}(2).\supset:$
3) $\mathsf{Kl}(a\text{-}A)\varepsilon \mathsf{Kl}(a\text{-}A).$ $[M4, 2]$
4) $\mathsf{Kl}(a\text{-}A)\varepsilon \mathsf{st}(a):$ $[M8, 3]$
5) $[D]:D\varepsilon a.\supset.D\varepsilon \mathsf{el}(\mathsf{Kl}(a\text{-}A)):$ $[M8, 2]$
6) $\mathsf{Kl}(a\text{-}A)=\mathsf{Kl}(a).$ $[M17, 4, 5, M11]$
7) $\sim(a\text{-}A\,\bar{\circ}\,a).$ $[1]$
$[\exists CDcd].C\varepsilon \mathsf{Kl}(c).D\varepsilon \mathsf{Kl}(d).c\subset a.d\subset a.C=D.\sim(c\circ d)$ $[6, 7]$

T26 $[Aa]{:}A\varepsilon a.\,\mathbf{Kl}(a\text{-}A)\varepsilon\mathbf{el}(A).\supset.[\exists CDcd].\,C\varepsilon\mathbf{Kl}(c).\,D\varepsilon\mathbf{Kl}(d).c\subset a.d\subset a.$
$C{=}D.\sim(c\circ d)$

PF $[Aa]{.}\therefore\,\mathrm{Hp}(2).\supset{:}$

3) $A\varepsilon\mathbf{el}(A){:}$ [*M1*, 1]
4) $[D]{:}\,D\varepsilon a\text{-}A.\supset.D\varepsilon\mathbf{el}(A){:}$ [*M3*, *M8*, 2]
5) $[D]{:}\,D\varepsilon a.\supset.D\varepsilon\mathbf{el}(A){:}$ [3, 4]
6) $A\varepsilon\mathbf{st}(a).$ [*M14*, 1]
7) $A{=}\mathbf{Kl}(a).$ [*M17*, 6, 5, *M11*]
8) $\mathbf{Kl}(a\text{-}A)\varepsilon\mathbf{Kl}(a\text{-}A).$ [2]
$[\exists B].$
9) $B\varepsilon a\text{-}A.$ [*M10*, 8]
10) $\sim(A\circ a).$ [9]
$[\exists CDcd].C\varepsilon\mathbf{Kl}(c).D\varepsilon\mathbf{Kl}(d).c\subset a.d\subset a.C{=}D.\sim(c\circ d)$ [7, *M12*, 1, 10]

T27 $[a]{.}\therefore[Cdcd]{:}\,C\varepsilon\mathbf{Kl}(c).D\varepsilon\mathbf{Kl}(d).c\subset a.d\subset a.C{=}D.\supset.c\circ d{:}\supset{:}[AB]{:}A\varepsilon a.$
$B\varepsilon\mathbf{Kl}(a\text{-}A).\supset.A\varepsilon\textbf{w-ex}(B)$ [*D1*, *T25*, *T26*]

T28 $[ABa]{:}A\varepsilon a.B\varepsilon\mathbf{st}(a\text{-}A).A\varepsilon\mathbf{el}(B).\supset.[\exists CD].\,C\varepsilon a.D\varepsilon\mathbf{Kl}(a\text{-}C).\sim(C\varepsilon\textbf{w-ex}(D))$

PF $[ABa]{:}\,\mathrm{Hp}(3).\supset.$

4) $B\varepsilon\mathbf{el}(\mathbf{Kl}(a\text{-}A)).$ [*M20*, 2]
5) $A\varepsilon\mathbf{el}(\mathbf{Kl}(a\text{-}A)).$ [*M3*, 3, 4]
6) $\sim(A\varepsilon\textbf{w-ex}(\mathbf{Kl}(a\text{-}A))).$ [*T4*, 5]
$[\exists CD].C\varepsilon a.D\varepsilon\mathbf{Kl}(a\text{-}C).\sim(C\varepsilon\textbf{w-ex}(D))$ [1, *M4*, 4, 6]

T29 $[ABa]{:}A\varepsilon a.B\varepsilon\mathbf{st}(a\text{-}A).B\varepsilon\mathbf{el}(A).\supset.[\exists CD].C\varepsilon a.D\varepsilon\mathbf{Kl}(a\text{-}C).\sim(C\varepsilon\textbf{w-ex}(D))$

PF $[ABa]{:}\,\mathrm{Hp}(3).\supset.$

4) $B\varepsilon\mathbf{el}(B).$ [*M1*, 2]
$[\exists C].$
5) $C\varepsilon a.$
6) $\sim(C\varepsilon A).$
7) $C\varepsilon\mathbf{el}(B).$ [*DM3*, 4, 2] (for 5–7)
8) $C\varepsilon\mathbf{el}(A).$ [*M3*, 7, 3]
9) $A\varepsilon a\text{-}C.$ [1, 6]
10) $A\varepsilon\mathbf{el}(\mathbf{Kl}(a\text{-}C)).$ [*M8*, 9]
11) $C\varepsilon\mathbf{el}(\mathbf{Kl}(a\text{-}C)).$ [*M3*, 8, 10]
12) $\sim(C\varepsilon\textbf{w-ex}(\mathbf{Kl}(a\text{-}C))).$ [*T4*, 11]
13) $\mathbf{Kl}(a\text{-}C)\varepsilon\mathbf{Kl}(a\text{-}C).$ [*M4*, 10]
$[\exists CD].C\varepsilon a.D\varepsilon\mathbf{Kl}(a\text{-}C).\sim(C\varepsilon\textbf{w-ex}(D))$ [5, 13, 12]

T30 $[a]{.}\therefore[CD]{:}C\varepsilon a.D\varepsilon\mathbf{Kl}(a\text{-}C).\supset.C\varepsilon\textbf{w-ex}(D){:}\supset{:}[AB]{:}A\varepsilon a$
$.B\varepsilon\mathbf{st}(a\text{-}A).\supset.A\varepsilon\textbf{w-ex}(D)$ [*D1*, *T28*, *T29*]

T31 $[Aab]{.}\therefore[BC]{:}B\varepsilon a.C\varepsilon\mathbf{st}(a\text{-}B).\supset.B\varepsilon\textbf{w-ex}(C){:}A\varepsilon a.b\subset a.$
$\sim(A\varepsilon b){:}\supset.\sim(A\varepsilon\mathbf{el}(\mathbf{Kl}(b)))$

PF $[Aab]{.}\therefore\mathrm{Hp}(4){:}\supset.$

5) $b\subset a\text{-}A.$ [3, 4]
6) $\mathbf{st}(b)\subset\mathbf{st}(a\text{-}A).$ [*M15*, 5]
7) $\mathbf{Kl}(b)\subset\mathbf{st}(a\text{-}A){:}$ [*M16*, 6]
8) $\mathbf{Kl}(b)\varepsilon\mathbf{Kl}(b).\supset.A\varepsilon\textbf{w-ex}(\mathbf{Kl}(b)){:}$ [1, 2, 7]
9) $\mathbf{Kl}(b)\varepsilon\mathbf{Kl}(b).\supset.\sim(A\varepsilon\mathbf{el}(\mathbf{Kl}(b))){:}$ [*T4*, 8]
$\sim(A\varepsilon\mathbf{el}(\mathbf{Kl}(b)))$ [9, *M4*]

T32 $[a]{.}\therefore[BC]{:}B\varepsilon a.C\varepsilon\mathbf{st}(a\text{-}B).\supset.B\varepsilon\textbf{w-ex}(C){:}\supset{:}[Ab]{:}A\varepsilon a$
$.b\subset a.A\varepsilon\mathbf{el}(\mathbf{Kl}(b)).\supset.A\varepsilon b.$ [*T31*]

T33 $[a].\because.\mathbf{w\text{-}dscr}\{a\}.\equiv:[CDcd]:C\varepsilon\mathbf{Kl}(c).D\varepsilon\mathbf{Kl}(d).c\subset a.d\subset a.C{=}D.\supset.c\circ d$ [*T24*; *T27*, *T30*, *T32*, *D2*]

T34 $[a].\because.\mathbf{w\text{-}dscr}\{a\}.\equiv:[AB]:A\varepsilon a.B\varepsilon\mathbf{Kl}(a\text{-}A).\supset.A\varepsilon\mathbf{w\text{-}ex}(B)$ [*T33*, *T27*, *T30*, *T32*, *D2*]

T35 $[a].\because.\mathbf{w\text{-}dscr}\{a\}.\equiv.[AB]:A\varepsilon a.B\varepsilon\mathbf{st}(a\text{-}A).\supset.A\varepsilon\mathbf{w\text{-}ex}(B)$ [*T34*, *T30*, *T32*, *D2*]

T36 $[AB]:A\neq B.\supset.B{=}\mathbf{Kl}((A\cup B)\text{-}A)$

PF $[AB]:\mathrm{Hp}(1).\supset.$

2) $B=(A\cup B)\text{-}A.$ [1]

$B=\mathbf{Kl}((A\cup B)\text{-}A)$ [*M12*, 2]

T37 $[AB]:A\varepsilon\mathbf{w\text{-}ex}(B).\supset.\mathbf{w\text{-}dscr}\{A\cup B\}$

PF $[AB].\because.\mathrm{Hp}(1).\supset.$

2) $A\neq B.$ [*T3*, 1]

3) $A\varepsilon\mathbf{w\text{-}ex}(\mathbf{Kl}((A\cup B)\text{-}A)).$ [*T36*, 2, 1]

4) $B\varepsilon\mathbf{w\text{-}ex}(\mathbf{Kl}((A\cup B)\text{-}B)):$ [*T36*, 2, 1, *T2*]

5) $[CD]:C\varepsilon A\cup B.D\varepsilon\mathbf{Kl}((A\cup B)\text{-}C).\supset.C\varepsilon\mathbf{w\text{-}ex}(D):$ [3, 4]

$\mathbf{w\text{-}dscr}\{A\cup B\}$ [*T34*, 5]

T38 $[ABa]:\mathbf{w\text{-}dscr}\{a\}.A\varepsilon a.B\varepsilon a.\sim(A{=}B).\supset.A\varepsilon\mathbf{w\text{-}ex}(B)$

PF $[ABa]:\mathrm{Hp}(4).\supset.$

5) $B\varepsilon a\text{-}A.$ [3, 4]

6) $B\varepsilon\mathbf{st}(a\text{-}A).$ [*M14*, 5]

$A\varepsilon\mathbf{w\text{-}ex}(B)$ [*T35*, 1, 2, 6]

T39 $[AB]:A\varepsilon\mathbf{w\text{-}ex}(B).\equiv.A\neq B.\mathbf{w\text{-}dscr}\{A\cup B\}$

T40 $[ab]:a\subset b.\mathbf{w\text{-}dscr}\{b\}.\supset.\mathbf{w\text{-}dscr}\{a\}$[6]

PF $[ab].\because.\mathrm{Hp}(1).\supset:$

3) $[A].a\text{-}A\subset b\text{-}A.$ [1]

4) $[A].\mathbf{st}(a\text{-}A)\subset\mathbf{st}(b\text{-}A):$ [*M15*, 3]

5) $[AB]:A\varepsilon a.B\varepsilon\mathbf{st}(a\text{-}A).\supset.A\varepsilon\mathbf{w\text{-}ex}(B):$ [*T35*, 1, 4]

$\mathbf{w\text{-}dscr}\{a\}$ [*T35*, 5]

1.3 THE RELATION OF WEAKLY DISCRETE TO SET

T41 $[ABa]:A\varepsilon a.B\varepsilon a.\sim(A{=}B).\supset.\sim(\mathbf{w\text{-}dscr}\{\mathbf{st}(a)\})$

PF $[ABa].\because.\mathrm{Hp}(3).\supset:$

4) $A\cup B\subset a.$ [1, 2]

5) $A\varepsilon A\cup B.$ [1]

6) $B\varepsilon A\cup B.$ [2]

7) $\mathbf{Kl}(A\cup B)\varepsilon\mathbf{st}(a).$ [*M9*, 5, *M18*, 4]

8) $A\varepsilon\mathbf{st}(a).$ [*M14*, 1]

9) $B\varepsilon\mathbf{st}(a):$ [*M14*, 2]

10) $\sim(A{=}\mathbf{Kl}(A\cup B)).\vee.\sim(B{=}\mathbf{Kl}(A\cup B)):$ [3]

11) $\sim(A\varepsilon\mathbf{w\text{-}ex}(\mathbf{Kl}(A\cup B))).$ [*M8*, 5, *T4*]

12) $\sim(B\varepsilon\mathbf{w\text{-}ex}(\mathbf{Kl}(A\cup B))).$ [*M8*, 6, *T4*]

$\sim(\mathbf{w\text{-}dscr}\{\mathbf{st}(a)\})$ [*T38*, 10, 7, 8, 11, 7, 9, 12]

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6. Proved for discrete by Sobociński.

T42 $[a]:\mathbf{w\text{-}dscr}\{\mathbf{st}(a)\}.\supset.\rightharpoonup\{a\}$[7] [*T41*]

T43 $[Aab]:\mathbf{w\text{-}dscr}\{a\}.\mathbf{st}(a)\circ\mathbf{st}(b).A\varepsilon a.\supset.A\varepsilon b$

PF $[Aab]\therefore \mathrm{Hp}(3).\supset.$

4) $A\varepsilon\mathbf{st}(a).$ [*M14*, 3]
5) $A\varepsilon\mathbf{st}(b).$ [4, 2]
6) $A\varepsilon\mathbf{el}(A):$ [*M1*, 3]
$[\exists E]:$
7) $E\varepsilon b.$ [*DM3*, 5, 6]
8) $E\varepsilon\mathbf{el}(A).$
9) $E\varepsilon\mathbf{st}(b).$ [*M14*, 7]
10) $E\varepsilon\mathbf{st}(a),$ [9, 2]
11) $E\varepsilon\mathbf{el}(E).$ [*M1*, 7]
$[\exists F].$
12) $F\varepsilon a.$ [*DM3*, 10, 11]
13) $F\varepsilon\mathbf{el}(E).$
14) $F\varepsilon\mathbf{el}(A).$ [*M3*, 8, 13]
15) $\sim(F\varepsilon\mathbf{w\text{-}ex}(A)).$ [*T4*, 14]
16) $F{=}A.$ [*T38*, 1, 3, 12, 15]
17) $A\varepsilon\mathbf{el}(E).$ [13, 16]
18) $A{=}E.$ [*M2*, 8, 17]
$A\varepsilon b$ [7, 18]

T44 $[ab]:\mathbf{w\text{-}dscr}\{a\}.\mathbf{st}(a)\circ\mathbf{st}(b).\supset.a\subset b$[8] [*T43*]

T45 $[ab]:\mathbf{w\text{-}dscr}\{a\}.\mathbf{w\text{-}dscr}\{b\}.\mathbf{st}(a)\circ\mathbf{st}(b).\supset.a\circ b$[8] [*T44*]

T46 $[a]:\mathbf{w\text{-}dscr}\{a\}.\sim(\rightharpoonup\{a\}).\supset.\sim(a\circ\mathbf{st}(a))$[8] [*T42*]

1.4 THE RELATION OF WEAKLY DISCRETE AND SET TO EQUINUMEROSITY. Our first objective in this section will be to prove the analog of the Great Theorem of Cantor.

D3 $[Aa\sigma]:A\varepsilon\Sigma_\alpha(a\sigma).\equiv.A\varepsilon a.[\exists B].B\varepsilon\mathbf{st}(a).B\varepsilon\sigma(A).\sim(A\varepsilon\mathbf{el}(B))$

T47 $[Aa\sigma]:\mathbf{w\text{-}dscr}\{a\}.a\overset{\sigma}{\underset{\infty}{\circ}}\mathbf{st}(a).A\varepsilon\mathbf{Kl}(\Sigma_\alpha(a\sigma)).\supset.[\exists B].B\varepsilon\Lambda$

PF $[Aa\sigma]\therefore \mathrm{Hp}(3).\supset.$

4) $A\varepsilon\mathbf{st}(a):$ [*M18*, 3, *D3*]
5) $[C]:C\varepsilon\mathbf{st}(a).\supset.[\exists D].D\varepsilon a.C\varepsilon\sigma(D):$ [2]
6) $[CDE]:C\varepsilon a.D\varepsilon\mathbf{st}(a).E\varepsilon\mathbf{st}(a).D\varepsilon\sigma(C).E\varepsilon\sigma(C).\supset.D{=}E:$ [2]
$[\exists B]:$
7) $B\varepsilon a.$ [4, 5]
8) $A\varepsilon\sigma(B):$
9) $B\varepsilon\mathbf{el}(A).\supset.B\varepsilon\Sigma_\alpha(a\sigma):$ [*D2*, 1, 7, *D3*, 3, *M12*]
10) $B\varepsilon\mathbf{el}(A).\supset.[C]:C\varepsilon\mathbf{st}(a).C\varepsilon\sigma(B).\supset.B\varepsilon\mathbf{el}(C):$ [6, 7, 4, 8]
11) $B\varepsilon\mathbf{el}(A).\supset.\sim(B\varepsilon\Sigma_\alpha(a\sigma)):$ [*D3*, 10]
12) $\sim(B\varepsilon\mathbf{el}(A)).$ [9, 11]
13) $B\varepsilon\Sigma_\alpha(a\sigma).$ [*D3*, 7, 4, 8, 12]

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7. Proved for dscr by Sobociński.

8. Proved for discrete by Sobociński.

14) $\sim(B\varepsilon\Sigma_\alpha(a\sigma))$: [*M8*, 12, 3, *M11*]

$[\exists B].B\varepsilon\Lambda$ [13, 14]

T48 $[a\sigma]:\mathbf{w\text{-}dscr}\{a\}.a\overset{\sigma}{\infty}\mathbf{st}(a).\supset.[A].\sim(A\varepsilon\mathbf{Kl}(\Sigma_\alpha(a\sigma)))$ [*T47*]

T49 $[Aa\sigma]:\mathbf{w\text{-}dscr}\{a\}.a\overset{\sigma}{\infty}\mathbf{st}(a).A\varepsilon a.\supset.A\varepsilon\sigma(A)$

PF $[Aa\sigma]\therefore\mathrm{Hp}(3).\supset:$

4) $A\varepsilon\mathbf{st}(a)$: [*M14*, 3]

5) $[C]:C\varepsilon\mathbf{st}(a).\supset.[\exists D].D\varepsilon a.C\varepsilon\sigma(D)$: [2]

$[\exists B].$

6) $B\varepsilon a.$ [5, 4]

7) $A\varepsilon\sigma(B).$

8) $B\varepsilon\mathbf{el}(A).$ [*T48*, 1, 2, *M9*, 6, 4, 7, *D3*]

9) $\sim(B\varepsilon\mathbf{w\text{-}ex}(A)).$ [*T4*, 8]

10) $A=B.$ [*T38*, 9, 1, 3, 6]

$A\varepsilon\sigma(A).$ [7, 10]

T50 $[Aa]:\mathbf{w\text{-}dscr}\{a\}.a\infty\mathbf{st}(a).A\varepsilon\mathbf{st}(a).\supset.A\varepsilon a$

PF $[Aa]::\mathrm{Hp}(3).\supset\therefore$

$[\exists\sigma]\therefore$

4) $a\overset{\sigma}{\infty}\mathbf{st}(a)$: [2]

5) $[C]:C\varepsilon\mathbf{st}(a).\supset.[\exists D].D\varepsilon a.C\varepsilon\sigma(D)$: [4]

6) $[CDE]:C\varepsilon a.D\varepsilon\mathbf{st}(a).E\varepsilon\mathbf{st}(a).D\varepsilon\sigma(C):E\varepsilon\sigma(C).\supset.D=E$: [4]

$[\exists B].$

7) $B\varepsilon a.$ [3, 5]

8) $A\varepsilon\sigma(B).$

9) $B\varepsilon\sigma(B).$ [*T49*, 1, 4, 7]

10) $B\varepsilon\mathbf{st}(a).$ [*M14*, 7]

11) $A=B\therefore$ [6, 7, 3, 10, 8, 9]

$A\varepsilon a$ [7, 11]

T51 $[a]:\mathbf{w\text{-}dscr}\{a\}.a\infty\mathbf{st}(a).\supset.\rightharpoonup\{a\}$ [*T50*, *M14*, *T46*]

T52 $[a]:\mathbf{w\text{-}dscr}\{a\}.\sim(\rightharpoonup\{a\}).\supset.a\propto\mathbf{st}(a)$ [*T51*, *M14*]

We have thus proved the analog of the Great Theorem of Cantor.

T53 $[ab]:\mathbf{w\text{-}dscr}\{a\}.b\subset a.b\infty\mathbf{st}(a).\supset.\rightharpoonup\{a\}$[9]

PF $[ab]:\mathrm{Hp}(2).\supset.$

4) $b\overset{\propto}{\infty}a.$ [2]

5) $a\infty\mathbf{st}(a).$ [3, 4, *M14*]

$\rightharpoonup\{a\}$ [*T51*, 1, 5]

The definition *D3* used here is a simplification of the definition used in the proof of *T52* under discrete. In that proof *T53* was proved first and then *T52* followed almost immediately. The proofs of *T52* and *T53* under discrete required seven theorems and 105 points of proof as opposed to seven theorems and 44 points of proof under weakly discrete.

We now proceed to the second theorem of our primary aim.

$$[ab]:\mathbf{w\text{-}dscr}\{a\}.\mathbf{w\text{-}dscr}\{b\}.a\infty b.\supset.\mathbf{st}(a)\infty\mathbf{st}(b).$$

9. Proved by Leśniewski for discrete.

We actually shall prove a somewhat more specialized theorem, namely, if σ is a one-to-one relation from a onto b, then σ can be extended to a one-to-one relation Σ, from $\mathbf{st}(a)$ onto $\mathbf{st}(b)$. There seem to be two natural methods of procedure, each of which yields a solution.

The first method. Consider an arbitrary non-empty subname, c, of a. Let d be the name of the images of c under σ. Now form the mereological classes of c and d, which are non-empty subnames of $\mathbf{st}(a)$ and $\mathbf{st}(b)$ respectively. Now define Σ so that $\Sigma(\mathbf{Kl}(c)) = \mathbf{Kl}(d)$.

The second method. Consider any non-empty subname A of $\mathbf{st}(a)$. Let c be the name of all those a's which are mereological elements of A, and d be the name of all the images of c. Now form B, the mereological class of d, and define Σ so that $\Sigma(A) = B$.

At first glance these two methods seem to be equivalent, and indeed, if both the a's and the b's are weakly discrete, the same Σ is defined. The distinction lies in the order in which the concepts of subname and mereological class are used. In the first method, first subname is used and then mereological class, while in the second method the order is reversed. Were both concepts distributive, the order should be immaterial, but we shall see below that if weakly discrete is removed from the hypotheses the two relations defined are quite distinct.

Consider the first method. If b is not weakly discrete, then Σ is many-to-one. If a is not weakly discrete, then Σ is one-to-many. If neither a nor b is weakly discrete, the Σ is many-to-many. In every case, Σ is onto $\mathbf{st}(b)$.

In the second method, if b is not weakly discrete, then Σ is many-to-one and onto $\mathbf{st}(b)$. If a is not weakly discrete, then Σ is one-to-one, but strictly into $\mathbf{st}(b)$. If neither is weakly discrete, then Σ is many-to-one and strictly into $\mathbf{st}(b)$.

The second method can be refined as follows. Once B has been defined, let f be the name of all those b which are mereological elements of B and e be the name of all the pre-images of f under σ. Let C be the mereological class of e. C is, in fact, the largest (in the mereological sense) of all those $\mathbf{st}(a)$ which are related to B by Σ in the second method. Now define Σ so that $\Sigma(C) = B$. This relation has the advantage of always being one-to-one. If b is not weakly discrete, then it fails to have all of $\mathbf{st}(a)$ as its domain. If a is not weakly discrete, it is strictly into $\mathbf{st}(b)$.

The theorem

$$[ab]\colon \mathbf{w\text{-}dscr}\{b\}.\, a \infty b. \supset .\, \mathbf{st}(a) \overset{\propto}{\underset{\infty}{}} \mathbf{st}(b)$$

is also of importance. Since the first method fails to prove this theorem, we shall employ the second method in this dissertation. The refinement of the second would be useful in studying equinumerosity if neither a nor b is weakly discrete. Some of the definitions used here were employed by B. Sobociński in his proof using discrete.

T54 $[BCa] .\!\cdot\, B \varepsilon \mathbf{st}(a) . C \varepsilon \mathbf{st}(a) \colon [A] \colon A \varepsilon a . A \varepsilon \mathbf{el}(B) . \supset . A \varepsilon \mathbf{el}(C) \colon \supset . B \varepsilon \mathbf{el}(C)$[10]

PF $[BCa] .\!\cdot\, \mathrm{Hp}(3) \colon \supset \colon$

$[\exists b] \colon$

10. This theorem was previously established by Sobociński.

4) $b\subset a$.
5) $B\varepsilon\mathsf{Kl}(b)$: } [*M19*, 1]
6) $[A]:A\varepsilon b.\supset .A\varepsilon a.A\varepsilon\mathsf{el}(B)$: [4, *M8*, 5, *M11*]
7) $[A]:A\varepsilon b.\supset .A\varepsilon\mathsf{el}(C)$: [6, 3]
8) $\mathsf{Kl}(b)\subset\mathsf{el}(C)$. [*M21*, 7]
$B\varepsilon\mathsf{el}(C)$ [5, 8]

D4 $[ABab\sigma]:B\varepsilon\Phi_\alpha(Aab\sigma).\equiv .B\varepsilon b.[\exists C].C\varepsilon a.C\varepsilon\mathsf{el}(A).B\varepsilon\sigma(C)^2$

D5 $[Bab\sigma]:B\varepsilon\Sigma(ab\sigma).\equiv .B\varepsilon\mathsf{st}(b)\text{-}b.[\exists A].A\varepsilon\mathsf{st}(a)\text{-}a.B\varepsilon\mathsf{Kl}(\Phi_\alpha(Aab\sigma))$

D6 $[ABab\sigma]:B\varepsilon\Sigma\{ab\sigma\}(A).\equiv .B\varepsilon\mathsf{Kl}(\Phi_\alpha(Aab\sigma))$

T55 $[ABC]:B\varepsilon\Sigma\{ab\sigma\}(A).C\varepsilon\Sigma\{ab\sigma\}(A).\supset .B=C$ [*D6*, *M5*]

This proves that $\Sigma\{ab\sigma\}$ is a many-to-one relation without the use of the condition of weakly discrete.

T56 $[ACDab\sigma]:a\,{}^{\sigma}_{\infty}\,b.\Phi_\alpha(Aab\sigma)\,o\,\Phi_\alpha(Cab\sigma).D\varepsilon a.D\varepsilon\mathsf{el}(A).\supset .D\varepsilon\mathsf{el}(C)$

PF $[ACDab\sigma].\!\cdot.\,\mathrm{Hp}(4).\supset :$
5) $[E]:E\varepsilon a.\supset .[\exists F].F\varepsilon b.F\varepsilon\sigma(E)$: [1]
6) $[EFG]:E\varepsilon a.F\varepsilon a.G\varepsilon b.G\varepsilon\sigma(E).G\varepsilon\sigma(F).\supset .E=F$: [1]
$[\exists B]$:
7) $B\varepsilon b$.
8) $B\varepsilon\sigma(D)$. } [5, 3]
9) $B\varepsilon\Phi_\alpha(Aab\sigma)$. [*D4*, 7, 3, 4, 8]
10) $B\varepsilon\Phi_\alpha(Cab\sigma)$. [9, 2]
$[\exists E]$.
11) $E\varepsilon a$.
12) $E\varepsilon\mathsf{el}(C)$.
13) $B\varepsilon\sigma(E)$. } [*D4*, 10]
14) $D=E$. [6, 3, 11, 8, 13]
$D\varepsilon\mathsf{el}(c)$ [14, 12]

T57 $[ABCab\sigma]:\mathsf{w\text{-}dscr}\{b\}.a\,{}^{\sigma}_{\infty}\,b.A\varepsilon\mathsf{st}(a)\text{-}a.C\varepsilon\mathsf{st}(a)\text{-}a.B\varepsilon\Sigma\{ab\sigma\}(A).$
$B\varepsilon\Sigma\{ab\sigma\}(C).\supset .A=C$

PF $[ABCab\sigma].\!\cdot.\,\mathrm{Hp}(6).\supset :$
7) $B\varepsilon\mathsf{Kl}(\Phi_\alpha(Aab\sigma))$. [*D6*, 5]
8) $B\varepsilon\mathsf{Kl}(\Phi_\alpha(Cab\sigma))$. [*D6*, 6]
9) $\Phi_\alpha(Aab\sigma)\,o\,\Phi_\alpha(Cab\sigma)$: [*T33*, 1, 7, 8, *D4*]
10) $[D]:D\varepsilon a.D\varepsilon\mathsf{el}(A).\supset .D\varepsilon\mathsf{el}(C)$: [*T56*, 2, 9]
11) $A\varepsilon\mathsf{el}(C)$: [*T54*, 3, 4, 10]
12) $[D]:D\varepsilon a.D\varepsilon\mathsf{el}(C).\supset .D\varepsilon\mathsf{el}(A)$: [*T56*, 2, 9]
13) $C\varepsilon\mathsf{el}(A)$. [*T54*, 4, 3, 12]
$A=C$ [*M2*, 9, 11]

T57 together with *T55* proves that under the condition that b is weakly discrete, $\Sigma\{ab\sigma\}$ is one-to-one.

Next we shall show that $\Sigma\{ab\sigma\}$ maps the $\mathsf{st}(a)\text{-}a$ into the $\mathsf{st}(b)\text{-}b$.

D7 $[ABab\sigma]:A\varepsilon\Phi_\alpha(Bba\sigma).\equiv .A\varepsilon a.[\exists C].C\varepsilon b.C\varepsilon\mathsf{el}(B).C\varepsilon\sigma(A)$

T58 $[Aab\sigma]:\mathsf{w\text{-}dscr}\{b\}.a\,{}^{\sigma}_{\infty}\,b.A\varepsilon\mathsf{st}(a)\text{-}a.\supset .[\exists B].B\varepsilon\mathsf{st}(b)\text{-}b.$
$B\varepsilon\mathsf{Kl}(\Phi_\alpha(Aab\sigma))$.

PF $[Aab\sigma]:\cdot:\mathrm{Hp}(3).\supset ::$
4) $[C]:C\varepsilon a.\supset .[\exists B].B\varepsilon b.B\varepsilon\sigma(C)$: [2]

5) $[CDE]:C\varepsilon b.D\varepsilon a.E\varepsilon a.C\varepsilon\sigma(D).C\varepsilon\sigma(E).\supset.D=E::$ [2]

$[\exists E]::$

6) $E\varepsilon a.$

7) $E\varepsilon\mathbf{el}(A).$ [DM3, M1, 3] (6–7)

8) $\sim(A=E).$ [3, 6]

9) $\sim(A\varepsilon\mathbf{el}(E)).$ [M2, 8, 7]

10) $E\varepsilon\mathbf{st}(a).\because$ [M14, 6]

$[\exists FG].\because$

11) $G\varepsilon a.$

12) $G\varepsilon\mathbf{el}(A).$

13) $\sim(G\varepsilon\mathbf{el}(E)).$ [T54, 9, 3, 10] (11–13)

14) $\sim(G=E)$ [M1, 13]

15) $F\varepsilon b.$

16) $F\varepsilon\sigma(E).$ [4, 6] (15–16)

17) $F\varepsilon\Phi_\alpha(Aab\sigma):$ [D4, 15, 6, 7, 16]

$[\exists BH]:$

18) $B\varepsilon\mathbf{Kl}(\Phi_\alpha(Aab\sigma)).$ [M6, 17]

19) $F\varepsilon\mathbf{el}(B).$ [M9, 17, M11, 18]

20) $\sim(F\varepsilon\mathbf{w\text{-}ex}(B)).$ [T4, 19]

21) $H\varepsilon b.$

22) $H\varepsilon\sigma(G).$ [4, 11] (21–22)

23) $H\varepsilon\Phi_\alpha(Aab\sigma):$ [D4, 21, 11, 12, 22]

24) $F=H.\supset.G=E:$ [5, 15, 6, 11, 16, 22]

25) $\sim(F=H).$ [24, 14]

26) $H\varepsilon\mathbf{el}(B).$ [M8, 23, M11, 18]

27) $\sim(H\varepsilon\mathbf{w\text{-}ex}(B)):$ [T4, 24]

28) $\sim(B=F).\vee.\sim(B=H):$ [25]

29) $\sim(B\varepsilon b).$ [28, T38, 20, 1, 15, T38, 27, 1, 21]

30) $B\varepsilon\mathbf{st}(b)\text{-}b::$ [M18, 18, D4, 29]

$[\exists B].B\varepsilon\mathbf{st}(b)\text{-}b.B\varepsilon\mathbf{Kl}(\Phi_\alpha(Aab\sigma)).$ [30, 18]

T59 $[Aab\sigma]:\mathbf{w\text{-}dscr}\{b\}.a\overset{\sigma}{\underset{\infty}{}}b.A\varepsilon\mathbf{st}(a)\text{-}a.\supset.[\exists B].B\varepsilon\Sigma(ab\sigma)$
$.B\varepsilon\Sigma\{ab\sigma\}(A)$ [T58, D5, D6]

T60 $[Bab\sigma]:B\varepsilon\Sigma(ab\sigma).\supset.[\exists A].A\varepsilon\mathbf{st}(a)\text{-}a.B\varepsilon\Sigma\{ab\sigma\}(A)$ [D5, D6]

T61 $[ab\sigma]:\mathbf{w\text{-}dscr}\{b\}.a\overset{\sigma}{\underset{\infty}{}}b.\supset.\mathbf{st}(a)\text{-}a\overset{\Sigma\{ab\sigma\}}{\underset{\infty}{\propto}}\Sigma(ab\sigma)$

PF $[ab\sigma].\because.\mathrm{Hp}(2).\supset:$

3) $[A]:A\varepsilon\mathbf{st}(a)\text{-}a.\supset.[\exists B].B\varepsilon\Sigma(ab\sigma).B\varepsilon\Sigma\{ab\sigma\}(A):$ [T59, 1, 2]

4) $[ABC]:A\varepsilon\mathbf{st}(a)\text{-}a.C\varepsilon\mathbf{st}(a)\text{-}a.B\varepsilon\Sigma(ab\sigma).B\varepsilon\Sigma\{ab\sigma\}(A).B\varepsilon$
$\Sigma\{ab\sigma\}(C).\supset.A=C:$ [T57, 1, 2, D5]

$\mathbf{st}(a)\text{-}a\overset{\Sigma\{ab\sigma\}}{\underset{\infty}{\propto}}\Sigma\{ab\sigma\}$ [3, 4, T60, T55]

T62 $[ab]:\mathbf{w\text{-}dscr}\{b\}.a\infty b.\supset.\mathbf{st}(a)\text{-}a\underset{\infty}{\propto}\mathbf{st}(b)\text{-}b$ [T61, D5]

T63 $[ab]:\mathbf{w\text{-}dscr}\{b\}.a\infty b.\supset.\mathbf{st}(a)\underset{\infty}{\propto}\mathbf{st}(b)$ [T62]

The validity of *T63* depends upon the ontological fact that b and $\mathbf{st}(b)$ have no objects in common.

We now invoke the ontological analog at the Schroeder-Bernstein Theorem to arrive at

T64 $[ab]\colon \mathbf{w\text{-}dscr}\{a\}.\mathbf{w\text{-}dscr}\{b\}.a \infty b.\supset.\mathbf{st}(a) \infty \mathbf{st}(b)$[11] [*T63*]

2. THE EQUIVALENCE OF WEAKLY DISCRETE AND DISCRETE WITH RESPECT TO EQUINUMEROSITY

THEOREM. Let $\Sigma(a,b,\ldots,\propto,\infty,\ \mathbf{st}(a),\ \mathbf{st}(b),\ldots,)$ be a mereological proposition involving, only the mereological terms indicated, then the statements

(i) $[ab\ldots]\colon \mathbf{dscr}\{a\}.\mathbf{dscr}\{b\}\ldots\ldots\supset.\sigma$
(ii) $[ab\ldots]\colon \mathbf{w\text{-}dscr}\{a\}.\mathbf{w\text{-}dscr}\{b\}\ldots\ldots\supset.\sigma$

are inferentially equivalent. That (ii) implies (i) is an immediate consequence of *T21*. The converse depends upon a theorem which is proved below.

T65 $[ABCD]\colon A\varepsilon\mathbf{el}(\mathbf{Kl}(B\cup C)).\rightharpoonup\{C\}.A\varepsilon\mathbf{ex}(B).D\varepsilon\mathbf{el}(A).\supset.[\exists E].E\varepsilon$ $\mathbf{el}(D).E\varepsilon\mathbf{el}(C)$

PF $[ABCD]\colon$ Hp(4). $\supset$.

5) $D\varepsilon\mathbf{el}(\mathbf{Kl}(B\cup C))$. [*M3*, 4, 1]
6) $D\varepsilon\mathbf{ex}(B)$. [*M27*, 4, 3]
7) $\mathbf{Kl}(B\cup C)\varepsilon\mathbf{Kl}(B\cup C)$. [*M4*, 5]
8) $B\varepsilon B$. [*DM4*, 6]
$[\exists EF]$.
9) $E\varepsilon B\cup C$. } [*DM1*, 7, 5]
10) $F\varepsilon\mathbf{el}(D)$. } [*DM1*, 7, 5]
11) $F\varepsilon\mathbf{el}(E)$. } [*DM1*, 7, 5]
12) $\sim(E\varepsilon B)$. [9, 8, 11, 10, *DM4*, 6]
13) $E{=}C$. [9, 12, 2]
14) $F\varepsilon\mathbf{el}(C)$. [11, 13]
$[\exists E].E\varepsilon\mathbf{el}(D).E\varepsilon\mathbf{el}(C)$ [10, 14]

T66 $[ABC]\colon A\varepsilon\mathbf{el}(\mathbf{Kl}(B\cup C)).\rightharpoonup\{C\}.A\varepsilon\mathbf{ex}(B).\supset.A\varepsilon\mathbf{el}(C)$ [*M24*, *T65*]

D8 $[Ba]\colon B\varepsilon\delta(a).\equiv.[\exists A].B\varepsilon\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}A)$

T67 $[ABa]\colon A\varepsilon a.B\varepsilon\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}A).\supset.B\varepsilon\mathbf{el}(A)$

PF $[ABa]\colon$ Hp(2). $\supset$.

3) $B\varepsilon\mathbf{ex}(\mathbf{Kl}(a\text{-}A))$. [*M31*, 2]
4) $B\varepsilon\mathbf{el}(\mathbf{Kl}(a))$. [*M32*, 2]
5) $(a\text{-}A)\cup A \circ a$. [1]
6) $B\varepsilon\mathbf{el}(\mathbf{Kl}((a\text{-}A)\cup A))$. [4, 5]
7) $B\varepsilon\mathbf{el}(\mathbf{Kl}(\mathbf{Kl}(a\text{-}A)\cup A))$. [*M13*, 6, *M12*, 1]
$B\varepsilon\mathbf{el}(A)$ [*T66*, 7, 1, 3]

T68 $[ABa]\colon A\varepsilon\delta(a).B\varepsilon\delta(a).\sim(A{=}B).\supset.A\varepsilon\mathbf{ex}(B)$

PF $[ABa]\colon$ Hp(3). $\supset$.

$[\exists CD]$.
4) $C\varepsilon a$. } [*D8*, 1]
5) $A\varepsilon\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}C)$. } [*D8*, 1]

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11. To prove *T64* without the Schroeder-Bernstein result would require another three pages of print.

6) $D\varepsilon a.$ } [$D8$, 2]
7) $B\varepsilon \mathbf{Kl}(a)\setminus \mathbf{Kl}(a\text{-}D).$
8) $\sim(\mathbf{Kl}(a\text{-}C) \circ \mathbf{Kl}(a\text{-}D)).$ [3, 5, 7, $DM8$, $DM7$, $M5$]
9) $\sim(C \circ D).$ [8]
10) $C\varepsilon a\text{-}D.$ [4, 6, 9]
11) $A\varepsilon \mathbf{el}(C).$ [$T67$, 5, 7]
12) $A\varepsilon \mathbf{el}(\mathbf{Kl}(a\text{-}D)).$ [$M7$, 11, 10]
13) $B\varepsilon \mathbf{ex}(\mathbf{Kl}(a\text{-}D)).$ [$M31$, 7]
$A\varepsilon \mathbf{ex}(B)$ [$M28$, 12, 13]

$T69$ $[a].\mathbf{dscr}\{\delta(a)\}$ [$DM6$, $T68$]

$D9$ $[ABa]:B\varepsilon\delta\langle a\rangle(A). \equiv .A\varepsilon a.B\varepsilon \mathbf{Kl}(a)\setminus \mathbf{Kl}(a\text{-}A)$

$T70$ $[Ba]:B\varepsilon\delta(a).\supset.[\exists A].A\varepsilon a.B\varepsilon\delta\langle a\rangle(A)$ [$D8$, $D9$]

$T71$ $[Aa]:\mathbf{w\text{-}dscr}\{a\}.A\varepsilon a.\sim(\rightharpoonup\{a\}).\supset.[\exists B].B\varepsilon\delta(a).B\varepsilon\delta\langle a\rangle(A)$

PF $[Aa]$: Hp(3). $\supset$.
4) $!\{a\text{-}A\}.$ [2, 3]
5) $\mathbf{Kl}(a\text{-}A)\varepsilon\mathbf{Kl}(a\text{-}A).$ [$M9$, 4]
6) $\mathbf{Kl}(a\text{-}A)\varepsilon\mathbf{el}(\mathbf{Kl}(a)).$ [$M23$, 5]
7) $A\varepsilon\mathbf{w\text{-}ex}(\mathbf{Kl}(a\text{-}A)).$ [$T34$, 1, 2, 5]
8) $\sim(A\varepsilon\mathbf{el}(\mathbf{Kl}(a\text{-}A))).$ [$T4$, 7]
9) $A\varepsilon\mathbf{el}(\mathbf{Kl}(a)).$ [$M8$, 2]
10) $\sim(\mathbf{Kl}(a\text{-}A)=\mathbf{Kl}(a)).$ [8, 9]
11) $\mathbf{Kl}(a\text{-}A)\varepsilon\mathbf{pr}(\mathbf{Kl}(a)).$ [$DM2$, 6, 10]
12) $\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}A)\varepsilon\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}A).$ [$M33$, 11]
13) $\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}A)\varepsilon\delta(a).$ [$D8$, 2, 12]
14) $\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}A)\varepsilon\delta\langle a\rangle(A).$ [$D9$, 2, 12]
$[\exists B].B\varepsilon\delta(a).B\varepsilon\delta\langle a\rangle(A)$ [13, 14]

$T72$ $[ABCa]:B\varepsilon\delta\langle a\rangle(A).C\varepsilon\delta\langle a\rangle(A).\supset.B=C$ [$M5$, $D9$]

$T73$ $[ABCa]:C\varepsilon\delta\langle a\rangle(A).C\varepsilon\delta\langle a\rangle(B).\mathbf{w\text{-}dscr}\{a\}.\sim(A=B).\supset.C\varepsilon\wedge$

PF $[ABCa]$:: Hp(4). $\supset$.˙.
5) $C\varepsilon\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}A).$ } [$D9$, 1]
6) $A\varepsilon a.$
7) $C\varepsilon\mathbf{Kl}(a)\setminus\mathbf{Kl}(a\text{-}B).$ } [$D9$, 2]
8) $B\varepsilon a.$
9) $B\varepsilon a\text{-}A.$ [6, 8, 4]
10) $B\varepsilon\mathbf{el}(\mathbf{Kl}(a\text{-}A)).$ [$M9$, 9]
11) $\mathbf{Kl}(a\text{-}A)=\mathbf{Kl}(a\text{-}B).$ [$M34$, 5, 7]
12) $B\varepsilon\mathbf{el}(\mathbf{Kl}(a\text{-}B)).$ [10, 11]
13) $\sim(B\varepsilon\mathbf{w\text{-}ex}(\mathbf{Kl}(a\text{-}B))).$ [$T4$, 12]
14) $\sim(\mathbf{w\text{-}dscr}\{a\}).$ [$T34$, 8, 11, 13]
15) $C\varepsilon\wedge$ [3, 14]

$T74$ $[ABCa]:\mathbf{w\text{-}dscr}\{a\}.C\varepsilon\delta\langle a\rangle(A).C\varepsilon\delta\langle a\rangle(B).\supset.A=B$ [$T73$]

$T75$ $[Aa]:\mathbf{w\text{-}dscr}\{a\}.\sim(\rightharpoonup\{a\}).\supset.[\exists b].a\infty b.\mathbf{dscr}\{b\}$

PF $[a]$.˙. Hp(2). $\supset$:
3) $[A]:A\varepsilon a.\supset.[\exists B].B\varepsilon\delta(a).B\varepsilon\delta\langle a\rangle(A):$ [$T71$, 1, 2]
4) $[ABC]:C\varepsilon\delta\langle a\rangle(A).C\varepsilon\delta\langle a\rangle(B).\supset.A=B:$ [$T74$, 1]
5) $a\infty\delta(a).$ [3, $T70$, 4, $T74$]
$[\exists b].a\infty b.\mathbf{dscr}\{b\}$ [5, $T69$]

$T76$ $[a]:\mathbf{w\text{-}dscr}\{a\}.\supset.[\exists b].a\infty b.\mathbf{dscr}\{b\}$ [$T75$, $DM6$]

(i) implies (ii) now follows immediately from *T76*, *T21*, and *T64*.

We see therefore, that the main theorems proved in section 1.4 are only apparently stronger than the previously known theorems. Since **w-dscr** is precisely the condition that insures that distinct ontological names give rise to distinct mereological sets, the proofs using it are not only shorter, but also more natural.

REFERENCES

[1] S. Leśniewski, O podstawach matematyki (On the foundations of mathematics), *Przegląd Filozoficzny* (Philosophical Reviews), V.30 (1927), pp. 164-206; V.31 (1928), pp. 261-291; V.32 (1929), pp. 60-101; V.33 (1930), pp. 75-105; V.34 (1931), pp. 142-170.

[2] B. Sobociński: Studies in Leśniewski's mereology. *Yearbook for 1954-55 of Polish Society of Arts and Sciences Abroad*, vol. V. London, 1955, pp. 34-48.

[3] R. E. Clay. *Contributions to mereology*. Ph.D. thesis, University of Notre Dame, August 1961. Notre Dame, Indiana.

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