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Definable Types Over Banach Spaces

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Abstract We study connections between asymptotic structure in a Banach space and model theoretic properties of the space. We show that, in an asymptotic sense, a sequence (x_n) in a Banach space X generates copies of one of the classical sequence spaces ℓ_p or c_0 inside X (almost isometrically) if and only if the quantifier-free types approximated by (x_n) inside X are quantifier-free definable. More precisely, if (x_n) is a bounded sequence X such that no normalized sequence of blocks of (x_n) converges, then the following two conditions are equivalent. (1) There exists a sequence (y_n) of blocks of (x_n) such that for every finite dimensional subspace E of X, every quantifier-free type over $E + \overline{\operatorname{span}}\{y_n \mid n \in \mathbb{N}\}$ is quantifier-free definable. (2) One of the following two conditions holds: (a) there exists $1 \le p < \infty$ such that for every $\epsilon > 0$ and every finite dimensional subspace E of X there exists a sequence of blocks of (x_n) which is $(1+\epsilon)$-equivalent over E to the standard unit basis of ℓ_p; (b) for every $\epsilon > 0$ and every finite dimensional subspace E of X there exists a sequence of blocks of (x_n) which is $(1+\epsilon)$-equivalent over E to the standard unit basis of c_0. Several byproducts of the proof are analyzed.

1 Introduction

A central question in Banach space theory has been to identify the class of Banach spaces that contain the classical sequence spaces ℓ_p or c_0 almost isometrically. In this paper we show that there is a tight connection between this property and a model theoretic condition, namely, definability of types.

Our context is restricted to quantifier-free types. This allows us to present the arguments so that they are accessible not only to the reader versed in model theory, but also to the mathematician who does not have a background in mathematical logic. Nonetheless, throughout the paper we have included remarks indicating how the ideas at hand are related to ideas from model theory.

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In their famous paper [18], Krivine and Maurey introduced a concept of type for Banach spaces inspired by the concept of type studied in model theory. Since then, analysts have studied and applied the Krivine-Maurey concept of type in a variety of contexts (see, for example, Chaatit [5], Farmaki [7], Guerre [8], Haydon and Maurey [9], Maurey [21], Odell [22], Raynaud [24], [23], and [25], and Rosenthal [27] and [28]). This concept corresponds to a particular case of the model theoretic notion of quantifier-free type over a Banach space. Suppose that X is a Banach space and $a_1, \dots, a_n$ are elements of a Banach space ultrapower of X. From the perspective of model theory of Banach spaces, the quantifier-free type of $(a_1, \dots, a_n)$ over X is determined by the values of all the norms

$$\|\lambda_1 a_1 + \cdots + \lambda_n a_n + x\|,$$

where $\lambda_1, \dots, \lambda_n$ are scalars and $x \in X$. In particular, if a is an element of a Banach space ultrapower of X, the quantifier-free type of a over X is determined by the norms $\|a + x\|$, where $x \in X$; we may thus identify the quantifier-free type of a over X with the real-valued function

$$x \mapsto \|a + x\|, \qquad (x \in X).$$

Under this identification, the connection between types in the sense of Krivine-Maurey and (model theoretic) quantifier-free types becomes transparent. (See Section 2.)

Since all the types considered in this paper are quantifier-free, the word "type" will be used to refer to quantifier-free types, and $\mathrm{tp}(a_1, \dots, a_n/X)$ will denote the (quantifier-free) type of $(a_1, \dots, a_n)$ over X. Since $\mathrm{tp}(a_1, \dots, a_n/X)$ is determined by the family of types of the form $\mathrm{tp}(a/X)$, where $a \in \mathrm{span}\{ a_0, \dots, a_n \}$, for most purposes in this context it suffices to consider types of single elements rather than types of finite tuples.

Suppose that X is a Banach space and a is an element of a Banach space ultrapower of X. Following terminology from Banach space model theory, we will say that the type $\mathrm{tp}(a/X)$ is *quantifier-free definable* if for every ball B around a, the set $B \cap X$ can be approximated at will by finite Boolean combinations of balls in X. (For the precise kind of approximation required, see Definition 4.3.)

The goal of this paper is to expose the connection between quantifier-free definability and the following concept. Let X be a Banach space, let E be a subspace of X, and let ϵ be a positive number. If $1 \le p < \infty$, a sequence (x_n) is said to be $(1+\epsilon)$*-equivalent over E to the standard unit basis of ℓ_p* if whenever $x \in E$ and $\lambda_0, \dots, \lambda_n$ are scalars,

$$(1+\epsilon)^{-1} \left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\| \le \left\| x + \Big(\sum_{i=0}^{n} |\lambda_i|^p \Big)^{1/p} x_0 \right\| \le (1+\epsilon) \left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\|.$$

The sequence (x_n) is $(1+\epsilon)$*-equivalent over E to the standard unit basis of c_0* if whenever $x \in E$ and $\lambda_0, \dots, \lambda_n$ are scalars,

$$(1+\epsilon)^{-1} \left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\| \le \left\| x + \big(\max_{0 \le i \le n} |\lambda_i| \big) x_0 \right\| \le (1+\epsilon) \left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\|.$$

If (x_n) is a sequence in a Banach space, a sequence (y_n) is a *sequence of blocks* of (x_n) if there exist finite subsets $F_0, F_1, \dots$ of $\mathbb{N}$ such that $\max F_n < \min F_{n+1}$ and $y_n \in \mathrm{span}\{ x_k \mid k \in F_n \}$ for every $n \in \mathbb{N}$.

Theorem 1.1 *Let (x_n) be a bounded sequence in a Banach space X such that no normalized sequence of blocks of (x_n) converges. Then the following conditions are equivalent.*

1. *There exists a sequence (y_n) of blocks of (x_n) such that for every finite dimensional subspace E of X, every type over $E + \overline{\operatorname{span}}\{y_n \mid n \in \mathbb{N}\}$ is quantifier-free definable.*
2. *One of the following two conditions holds:*
   (a) *there exists $1 \leq p < \infty$ such that for every $\epsilon > 0$ and every finite dimensional subspace E of X there exists a sequence of blocks of (x_n) which is $(1+\epsilon)$-equivalent over E to the standard unit basis of ℓ_p;*
   (b) *for every $\epsilon > 0$ and every finite dimensional subspace E of X there exists a sequence of blocks of (x_n) which is $(1+\epsilon)$-equivalent over E to the standard unit basis of c_0.*

The argument combines the approach to type definability introduced in Iovino [14] with the ideas about Banach space stability and ℓ_p-types introduced by Krivine and Maurey in [18]. However, it does not suffice to invoke results from these papers, since we work with general Banach spaces rather than stable ones. We develop a "local" approach to stability, that is, we study stability of types rather than stability of spaces. For instance, the space c_0 is unstable, but it has a stable type, namely, the type determined by its standard unit basis.

The proof has several byproducts:

1. We prove that quantifier-free definable types determine a spreading model uniquely. (See Proposition 6.1.) Equivalently, a quantifier-free definable type τ can be extended uniquely to a strong type in the sense of Maurey [21] (and a convolution operation in the sense of Maurey [21] can be defined uniquely on the scalar multiples of τ).
2. We characterize stability of spreading models in terms of quantifier-free definability. We show that for this characterization, the definition of quantifier-free definable type can be strengthened in various ways, for example, by imposing restrictions on the geometry of the balls or on the form of the Boolean combinations used for the approximating sets. (See Corollary 8.8.)
3. The background material on spreading models of Banach spaces (Section 2) is presented so as to emphasize to the model theorist the connection between spreading models and important model theoretic concepts such as indiscernibility as well as Shelah's concepts of *semidefinability* and *average* of a sequence. (Section VII-4 of [29].)

The only general prerequisite is some familiarity with the definition of Banach space ultrapower, which can be found in virtually any Banach space theory textbook written after 1980, for example, Beauzamy [2]. More thorough expositions of applications of this concept can be found in Heinrich [10] and Sims [30]. A general concept of ultraproduct for structures of functional analysis and a study of the connections between the model theory of these structures and their ultraproducts is presented in Henson and Iovino [11]. We refer the reader to [11] for an introduction to the model theory of structures of functional analysis.

Aside from the concept of Banach space ultrapower, the background material, including a discussion of the connections among spreading models, types (in the sense of [18]), and ultrapowers (in the sense considered by functional analysts) is given in Section 2. To our knowledge these connections are not exposed elsewhere in the literature.

In Section 3 we introduce the notion of *heir* of a type, and in Section 4 we introduce the central notion of the paper, that of quantifier-free definable type. These two concepts are then connected in Section 5: we prove that quantifier-free definable types are those which determine a unique heir. In Section 6 we observe that every quantifier-free definable type determines a spreading model (and hence a convolution) uniquely. In Section 7 we introduce a concept of stability for spreading models. The connection between quantifier-free definability and spreading model stability is then established in Section 8. (This is the most technical section of the paper; the proof of Proposition 8.3 is particularly involved.) The final section, Section 9, is devoted to the proof of the main theorem.

All Banach spaces considered in this paper are over the real field $\mathbb{R}$. If X is a Banach space and M is a nonnegative real number, the closed ball of radius M around 0 in X will be denoted $B(M)$, or $B_X(M)$, depending on whether the ambient space needs to be emphasized.

2 Background, Notation, and Terminology

Note to the logician. In this section we present a very brief introduction to quantifier-free types over Banach spaces and the connection between types and spreading models. The material is exposed so that logicians and specialists in nonstandard analysis can immediately see the connection with notions that are familiar to them. However, knowledge of logic is not required, not least because we deal only with the quantifier-free case, and moreover, with 1-types. Types are defined through Banach space ultrapowers.

The ultraproduct construction is widely used in functional analysis. We remind the logic-oriented reader, however, that an analytic ultrapower is not an ultrapower in the usual algebraic sense. From the point of view of model theory (or nonstandard analysis), a Banach space ultrapower is the result of performing two transformations in a regular ultrapower: first, excluding all the elements of infinite norm, and second, identifying any two elements which differ by an element of infinitesimal norm. (Thus, this is a particular case of Luxemburg's nonstandard hull construction [20].)

In first-order model theory, if one regards formulas as $\{0, 1\}$-valued functions, then the Stone topology on types can be naturally seen as a product topology (the product being taken over all finite disjunctions of formulas in the specified language and list of free variables). In analytic model theory the counterpart of the Stone topology can also be seen as a product topology, but the difference is that formulas should then be viewed as real-valued functions, rather than $\{0, 1\}$-valued functions. In the quantifier-free Banach space context (on which we focus here) this is easy to see, since the only quantifier-free formulas are norm estimates, so the real-valued functions corresponding to atomic formulas are translates of the norm. The non-quantifier-free case will not be needed here; we refer the reader to Iovino [13] or Iovino [15] and [16], and to [11].

From the point of view of model theory, a spreading model can be viewed naturally as follows. One starts with a quantifier-free type p over a Banach space X and constructs a sequence $(a_n)_{n<\omega}$ of realizations of p (in the monster model) such that $\mathrm{tp}(a_{n+1}/X + \mathrm{span}\{a_0, \dots, a_n\})$ extends $\mathrm{tp}(a_n/X + \mathrm{span}\{a_0, \dots, a_{n-1}\})$ and is finitely realized in X. The sequence (a_n) is quantifier-free indiscernible. The subspace of the monster model spanned by X and $\{ a_n \mid n < \omega \}$ is called a *spreading model* over X.

Logicians use partition theorems routinely to construct indiscernible sequences, so it is not surprising that a form of the basic Ramsey Theorem plays an important role in the ideas presented here. We will make heavy use of a simple but powerful lemma that in [28] Rosenthal called *the Ramsey principle for analysts*.

2.1 Ramsey's Theorem We will use a form of Ramsey's Theorem that was introduced by Brunel and Sucheston in [3].

Proposition 2.1 *Let*

$$(a_{m_1,m_2,\dots,m_d} \mid (m_1, m_2, \dots, m_d) \in \mathbb{N}^d)$$

be a family of real numbers such that the iterated limits

$$\lim_{m_1} \cdots \lim_{m_d} a_{m_1,m_2,\dots,m_d}$$

exist. Then there exist $k(0) < k(1) < \cdots$ such that

$$\lim_{i_1<i_2<\cdots<i_d} a_{k(i_1),k(i_2),\dots,k(i_d)} = \lim_{m_1} \cdots \lim_{m_d} a_{m_1,m_2,\dots,m_d}.$$

The proof is an exercise.

2.2 Types If X is a Banach space and $\hat{X}$ is a Banach space ultrapower of X, then $\hat{X}$ is finitely represented in X over any finite dimensional subspace of X, that is, for every finite dimensional subspace E of X, every $\epsilon > 0$, and every $a_1, \dots, a_n \in \hat{X}$ there exist $x_1, \dots, x_n \in X$ such that $E+\text{span}\{ a_1, \dots, a_n \}$ and $E+\text{span}\{ x_1, \dots, x_n \}$ are isomorphic via a $(1 + \epsilon)$-isomorphism that maps a_i to x_i and fixes E pointwise. Conversely, if Y is a Banach space which contains X and is finitely represented in X over any finite dimensional subspace of X, then there exists an ultrapower $\hat{X}$ of X and an embedding of Y into $\hat{X}$ which fixes X pointwise.

If $\hat{X}$ is an ultrapower of X and a is an element of $\hat{X}$, the *type of a over X* is the function $\text{tp}(a/X) : X \to \mathbb{R}$ defined by

$$\text{tp}(a/X)(x) = \|a + x\|.$$

We say that a function $\tau : X \to \mathbb{R}$ is a *type over X* if there exists an ultrapower $\hat{X}$ of X and $a \in \hat{X}$ such that $\tau = \text{tp}(a/X)$. In this case, we say that *a realizes τ*. The set of types over X is regarded as a topological space with the topology of pointwise convergence.

If a and a' are realizations of the same type over X (possibly in different ultrapowers), then the spaces $X + \text{span}\{a\}$ and $X + \text{span}\{a'\}$ are isometric via an isometry that maps a to a' and fixes X pointwise. Thus, the particular choice of ultrapower where we realize types is to a large extent irrelevant, and often we refer to a realization of a type without mentioning the extension where the realization lives. Also, it is easy to see that given any family of types $(\tau_i)_{i\in I}$ over X we can always take a single ultrapower of X where all the τ_is are realized.

The *norm* of a type τ over X is the norm of a realization of τ, that is, $\tau(0)$. We denote the norm of τ by $\|\tau\|$. If $M > 0$ and $(\tau_i)_{i\in I}$ is a family of types over X of norm less than or equal to M and a_i realizes τ_i in a space X_i, then $\lim_{i,\mathcal{U}} \tau_i = \text{tp}(a/X)$, where a is the element of the $\mathcal{U}$-ultraproduct of $(X_i)_{i\in I}$ represented by the family

$(a_i)_{i\in I}$. Hence, the set of types over X of norm less than or equal to any given constant is compact, so the space of types over X is σ-compact.

If λ, μ are nonnegative real numbers and $\epsilon > 0$, we write $\lambda \overset{1+\epsilon}{\sim} \mu$ if $(1+\epsilon)^{-1}\mu \le \lambda \le (1+\epsilon)\mu$. If τ is a type over X, then for every $\epsilon > 0$ and every finite dimensional subspace E of X there exists $x \in X$ such that $\mathrm{tp}(x/E) \overset{1+\epsilon}{\sim} \tau \restriction E$. Hence, if X is separable, there exists a sequence (x_k) in X such that $\lim_k \mathrm{tp}(x_k/X) = \tau$. Conversely (since the set of types of norm less than or equal to a given constant is compact), every bounded sequence (x_n) in X has a subsequence (x_{n_k}) such that $\lim_k \mathrm{tp}(x_{n_k}/X)$ exists. Thus, if the space X is separable, a function $\tau : X \to \mathbb{R}$ is a type if and only if there exists a bounded sequence (x_k) in X such that

$$(\dagger) \qquad \tau(x) = \lim_k \|x_k + x\|.$$

This is actually the original definition of "type" in functional analysis, introduced in [18] (where the context is restricted to separable spaces).

If ($\dagger$) holds, we say that (x_k) is an *approximating sequence* for τ or that (x_k) *approximates* τ.

2.3 Spreading models and convolutions Let (y_k) be a bounded sequence in a separable Banach space X and assume that (y_k) has no convergent subsequence (so (y_k) does not approximate a type realized in X). Inductively, we construct a sequence (a_n) (in some extension of X) and subsequences (y_k^n) of (y_k) as follows:

(i) (y_k^0) is a subsequence of (y_k);
(ii) (y_k^{n+1}) is a subsequence of (y_k^n);
(iii) (y_k^n) approximates $\mathrm{tp}(a_n/X + \mathrm{span}\{\, a_0, \dots, a_{n-1}\,\})$.

Let

$$\tau = \bigcup_n \mathrm{tp}(a_n/X + \mathrm{span}\{a_0, \dots, a_{n-1}\}).$$

Then τ is a type over $X + \mathrm{span}\{\, a_n \mid n \in \mathbb{N}\,\}$, approximated by the diagonal subsequence (x_k) of (y_k) defined by $x_i = y_i^i$. If $x \in X$ and $\lambda_0, \dots, \lambda_n$ are scalars, we have

$$\|x + \lambda_0 a_0 + \cdots + \lambda_n a_n\| = \lim_{k_n} \cdots \lim_{k_0} \|x + \lambda_0 x_{k_0} + \cdots + \lambda_n x_{k_n}\|.$$

By Ramsey's Theorem (Proposition 2.1) and further diagonalization, we can refine (x_k) so that

$$\|x + \lambda_0 a_0 + \cdots + \lambda_n a_n\| = \lim_{k_n < \cdots < k_0} \|x + \lambda_0 x_{k_0} + \cdots + \lambda_n x_{k_n}\|$$

for $x \in X$ and scalars $\lambda_0, \dots, \lambda_n$.

The space $X + \overline{\mathrm{span}}\{a_n \mid n \in \mathbb{N}\}$ is traditionally called the *spreading model* of the sequence (y_k). However, the spreading model is not uniquely determined by (y_k). It is uniquely determined by the sequence (x_k) (and hence by the type τ), so it would be more appropriate to call it the spreading model of the sequence (x_k) (or the spreading model of the type τ), although we will not use this terminology. The sequence (a_n) is called the *fundamental sequence* of the spreading model.[1] We will also say that (a_n) is a fundamental sequence for the type $\tau \restriction X$.

Notice that a fundamental sequence (a_n) for a type τ is 1-*subsymmetric over* X, that is, if $x \in X$ and $n_1 < \cdots < n_k$,

$$\|x + \lambda_0 a_0 + \cdots + \lambda_k a_k\| = \|x + \lambda_0 a_{n_0} + \cdots + \lambda_k a_{n_k}\|.$$

If τ is a type over a Banach space X realized by an element a (in some extension of X) and λ is a scalar, we define $\lambda\tau$ to be the type of λa over X.

Every fundamental sequence (a_n) for a type τ induces a function $*$ on finite tuples of scalar multiples of τ as follows. If $\lambda_0, \ldots, \lambda_k$ are scalars, we define

$$(\ddagger) \qquad \lambda_0\tau * \cdots * \lambda_k\tau(x) = \|x + \lambda_0 a_0 + \cdots + \lambda_k a_k\|.$$

We refer to the function $*$ as the *convolution* induced on the scalar multiples of τ by the fundamental sequence (a_n). The set of all types of the form $\lambda_0\tau * \cdots * \lambda_n\tau$, where $\lambda_0, \ldots, \lambda_n$ are scalars, will be denoted

$$\mathrm{span}(\tau, *).$$

If we are given convolution $*$ on the scalar multiples of τ and a sequence (a_n) (in some extension of X) that satisfies ($\ddagger$) above, we will say that (a_n) is a *fundamental sequence corresponding to* $\mathrm{span}(\tau, *)$.

The closure of $\mathrm{span}(\tau, *)$ with respect to the pointwise convergence topology will be denoted

$$\overline{\mathrm{span}}(\tau, *).$$

If $\sigma \in \overline{\mathrm{span}}(\tau, *)$, by the 1-subsymmetry of (a_n) there exists a sequence (b_n) of blocks of (a_n) such that $\sigma = \lim_n \mathrm{tp}(b_n/X)$. Furthermore, (b_n) can be taken so that if $\circ$ is a convolution on the scalar multiples of σ, then for $x \in X$ and scalars $\lambda_0, \ldots, \lambda_k$ we have

$$\lambda_0\sigma \circ \cdots \circ \lambda_k\sigma(x) = \lim_{n_k < \cdots < n_0} \|x + \lambda_0 b_{n_0} + \cdots + \lambda_k b_{n_k}\|,$$

and hence there exists a sequence (y_n) of blocks of (x_n) such that

$$\lambda_0\sigma \circ \cdots \circ \lambda_k\sigma(x) = \lim_{n_k < \cdots < n_0} \|x + \lambda_0 y_{n_0} + \cdots + \lambda_k y_{n_k}\|.$$

Let τ be a type and let $*$ be a convolution on the scalar multiples of τ. We will say that $\mathrm{span}(\tau, *)$ is 1-*unconditional* if given scalars $\lambda_1, \ldots, \lambda_n$, one has

$$\lambda_0\tau * \cdots * \lambda_n\tau(x) = \pm\lambda_0\tau * \cdots * \pm\lambda_n\tau,$$

that is, if any fundamental sequence corresponding to $\mathrm{span}(\tau, *)$ is 1-unconditional. Let us prove that 1-unconditional types exist. Suppose that τ is a type over a separable Banach space X, the sequence (x_n) is approximating sequence for τ, the sequence (a_n) is fundamental for τ, and $d_1, \ldots, d_k \in X + \overline{\mathrm{span}}\{a_n \mid n \in \mathbb{N}\}$. Then the Borsuk-Ulam antipodal map theorem ensures that for every finite set of indices $n_1 < \cdots < n_k$ there is an element s in the unit sphere of $\mathrm{span}\{x_{n_1}, \ldots, x_{n_k}\}$ such that $\|s - d_i\| = \|s + d_i\|$ for $i = 1, \ldots, k$. This in fact proves that there exist $\sigma \in \overline{\mathrm{span}}(\tau, *)$ and a convolution $\circ$ on the scalar multiples of σ such that $\mathrm{span}(\sigma, \circ)$ is 1-unconditional.

Remark 2.2 Using standard model theoretic ideas, it is possible to define a binary operation $*$ on all types over X such that for every type τ over X, the operation $*$ acts as a convolution on scalar multiples of τ. This is convenient in many settings, but will not be needed here. We refer the reader to Iovino [12] for the details.

2.4 ℓ_p- and c_0-types

Definition 2.3 Let X be a Banach space and let $\hat{X}$ be an extension of X. We say that a sequence (a_n) in $\hat{X}$ is *isometric over X to the standard unit basis of ℓ_p* if for $x \in X$ and scalars $\lambda_0, \ldots, \lambda_n$,

$$\Big\| x + \sum_{i=0}^{n} \lambda_i a_i \Big\| = \Big\| x + \big(\sum_{i=0}^{n} |\lambda_i|^p \big)^{1/p} a_0 \Big\| .$$

We say that (a_n) is *isometric over X to the standard unit basis of c_0* if for $x \in X$ and scalars $\lambda_0, \ldots, \lambda_n$,

$$\Big\| x + \sum_{i=0}^{n} \lambda_i a_i \Big\| = \Big\| x + \big(\max_{0 \le i \le n} |\lambda_i| \big) a_0 \Big\| .$$

Let τ be a type and let $*$ be a convolution on the scalar multiples of τ (induced by some fundamental sequence for τ). If $1 \le p < \infty$, we say that τ is an *ℓ_p-type with respect to* $*$ if for every pair of scalars μ, λ we have

$$\lambda\tau * \mu\tau = \big(|\lambda|^p + |\mu|^p \big)^{1/p} \tau.$$

Similarly, we say that τ is a *c_0-type with respect to* $*$ if for every pair of scalars μ, λ we have

$$\lambda\tau * \mu\tau = \max(|\lambda|, |\mu|)\tau.$$

Proposition 2.4 *Suppose that X is a separable Banach space, τ is a type over X, and $*$ is a convolution on the scalar multiples of τ such that* $\mathrm{span}(\tau, *)$ *is 1-unconditional. Then, if (a_n) is a fundamental sequence for* $\mathrm{span}(\tau, *)$ *and p is a positive real number, the following conditions are equivalent:*

1. *$1 \le p < \infty$ and τ is an ℓ_p-type with respect to $*$;*
2. *$1 \le p < \infty$ and (a_n) is isometric over X to the standard unit basis of ℓ_p;*
3. *for every $x \in X$ and every natural number k,*

$$\Big\| x + \sum_{i=0}^{m-1} \lambda_i a_i + (k+1)^{1/p} a_m + \sum_{i=m+1}^{n} \lambda_i a_i \Big\| = \Big\| x + \sum_{i=0}^{m-1} \lambda_i a_i + \sum_{i=m}^{m+k} a_i + \sum_{i=m+1}^{n} \lambda_i a_{i+k} \Big\| .$$

Proof (1) $\Rightarrow$ (2) We prove by induction on n that the first equality in Definition 2.3 holds. If $n \le 1$, the equality is immediate. Assume $n \ge 1$. Let (x_ν) be a sequence in X such that

$$\|x + \lambda_0 a_0 + \cdots + \lambda_n a_n\| = \lim_{\nu_n} \cdots \lim_{\nu_0} \|x + \lambda_0 x_{\nu_0} + \cdots + \lambda_n x_{\nu_n}\|,$$

for every choice of $x \in X$ and scalars $\lambda_0, \dots, \lambda_n$. Then,

$$\begin{aligned}
\Big\| x + \sum_{i=0}^{n} \lambda_i a_i \Big\| &= \lim_{\nu_n} \cdots \lim_{\nu_2} \Big\| x + \lambda_0 a_0 + \lambda_1 a_1 + \sum_{i=2}^{n} \lambda_i x_{\nu_i} \Big\| \\
&= \lim_{\nu_n} \cdots \lim_{\nu_2} \Big\| x + (|\lambda_0|^p + |\lambda_1|^p)^{1/p} a_0 + \sum_{i=2}^{n} \lambda_i x_{\nu_i} \Big\| \\
&= \Big\| x + (|\lambda_0|^p + |\lambda_1|^p)^{1/p} a_0 + \sum_{i=2}^{n} \lambda_i a_i \Big\| \\
&= \lim_{\nu_0} \Big\| x + (|\lambda_0|^p + |\lambda_1|^p)^{1/p} x_{\nu_0} + \sum_{i=2}^{n} \lambda_i a_i \Big\| \\
&= \lim_{\nu_0} \Big\| x + (|\lambda_0|^p + |\lambda_1|^p)^{1/p} x_{\nu_0} + \big(\sum_{i=2}^{n} |\lambda_i|^p \big)^{1/p} a_2 \Big\| \\
&= \Big\| x + (|\lambda_{n-1}|^p + |\lambda_n|^p)^{1/p} a_0 + \big(\sum_{i=2}^{n} |\lambda_i|^p \big)^{1/p} a_2 \Big\| \\
&= \Big\| x + \big(\sum_{i=0}^{n} |\lambda_i|^p \big)^{1/p} a_0 \Big\| .
\end{aligned}$$

(2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are immediate.

We prove (3) $\Rightarrow$ (2). Fix scalars $\lambda_0, \dots, \lambda_n$. Since $\mathrm{span}(\tau, *)$ is 1-unconditional, we can also assume $\lambda_0, \dots, \lambda_n \geq 0$. Furthermore, by density considerations, we may assume without loss of generality that λ_i^p is rational, for $i = 0, \dots, n$. We can therefore fix a positive integer M such that $M\lambda_i^p$ is an integer, for $i = 0, \dots, n$. Since (a_n) is subsymmetric over X, for every $x \in X$ we have

$$\begin{aligned}
\Big\| M^{1/p} x + \sum_{i=0}^{n} (M\lambda_i^p)^{1/p} a_0 \Big\| &= \Big\| M^{1/p} x + \sum_{i=0}^{n} \sum_{j=0}^{M\lambda_i^p - 1} a_{i+j} \Big\| \\
&= \Big\| M^{1/p} x + \big(\sum_{i=0}^{n} M\lambda_i^p \big)^{1/p} a_i \Big\| .
\end{aligned}$$

Dividing by $M^{1/p}$, we obtain the desired result. □

Proposition 2.5 *Suppose that X is a separable Banach space, τ is a type over X, and $*$ is a convolution on the scalar multiples of τ such that* $\mathrm{span}(\tau, *)$ *is 1-unconditional. Then, if (a_n) is a fundamental sequence for* $\mathrm{span}(\tau, *)$ *and p is a positive real number, the following conditions are equivalent:*

1. *τ is a c_0-type with respect to $*$;*
2. *(a_n) is isometric over X to the standard unit basis of c_0;*
3. *for every $x \in X$ and every natural number k,*

$$\Big\| x + \sum_{i=0}^{m-1} \lambda_i a_i + a_m + \sum_{i=m+1}^{n} \lambda_i a_i \Big\| = \Big\| x + \sum_{i=0}^{m-1} \lambda_i a_i + \sum_{i=m}^{m+k} a_i + \sum_{i=m+1}^{n} \lambda_i a_{i+k} \Big\| .$$

Proof Similar to the proof of Proposition 2.4. □

Remark 2.6 The equivalence (2) $\Leftrightarrow$ (3) in Propositions 2.4 and 2.5 holds for arbitrary (a_n). (The assumption that (a_n) is fundamental is not needed in the proof.)

2.5 Krivine's Theorem The following result was proved by Krivine in [17].

Theorem 2.7 *Let τ be a type over a Banach space X which is not realized in X and let $*$ be a convolution on the scalar multiples of τ such that* $\operatorname{span}(\tau, *)$ *is* 1*-unconditional. Then there exists a sequence (e_n) such that*

1. *(e_n) is isometric over X to the standard unit basis of c_0 or ℓ_p, for some p with $1 \leq p < \infty$;*
2. *there exists a sequence of types (σ_l) in* $\operatorname{span}(\tau, *)$ *such that for scalars $\lambda_0, \dots, \lambda_n$,*
$$\operatorname{tp}(\lambda_0 e_0 + \cdots + \lambda_n e_n / X) = \lim_{l} (\lambda_0 \sigma_l * \cdots * \lambda_n \sigma_l).$$

Krivine's Theorem is not usually stated in terms of types and convolutions but in terms of *block finite representability* of ℓ_p $(1 \leq p \leq \infty)$ in an arbitrary sequence (x_n) in Banach spaces. The original statement in [17] required a permutation of (x_n) in the c_0 case. Rosenthal [26] analyzed Krivine's proof and showed that the permutation in the c_0 case was unnecessary. The argument was further clarified by Lemberg [19].

3 Heirs of Types

Suppose that X is a Banach space and f is a real-valued function on X which is uniformly continuous on every bounded subset of X. If $\mathcal{U}$ is an ultrafilter on a set I, the $\mathcal{U}$*-ultrapower* of the structure (X, f) is defined as the pair $(\hat{X}, \hat{f})$, where

1. $\hat{X}$ is the $\mathcal{U}$-ultrapower of X;
2. $\hat{f}$ is the real-valued function defined on $\hat{X}$ as follows: If $x \in \hat{X}$ and $(x_i)_{i \in I}$ is a representative of x in X^I, we have
$$\hat{f}(x) = \lim_{i,\mathcal{U}} (f(x_i)).$$

The fact that f is uniformly continuous on every bounded subset of X ensures that $\hat{f}$ is well defined.

An ultrapower $(\hat{X}, \hat{f})$ of (X, f) has the property that it is *finitely represented in* (X, f); this means that whenever E is a finite dimensional subspace of $\hat{X}$ and $M, \epsilon > 0$, there exists a finite dimensional subspace F of X such that $(E, \hat{f} \restriction E)$ and $(F, f \restriction E)$ are $(1+\epsilon)$*-isomorphic* in the sense that there exists a $(1+\epsilon)$-isomorphism $\varphi \colon E \to F$ satisfying $|f(\varphi(x)) - \hat{f}(x)| \leq \epsilon$ for every element $x \in E$ of norm at most M.

Proposition 3.1 *Suppose that X is a Banach space and f is a real-valued function on X which is uniformly continuous on every bounded subset of X. Then for every ultrapower $\hat{X}$ of X there exists a real-valued function $\hat{f}$ on X such that $(\hat{X}, \hat{f})$ is an ultrapower of (X, f). Furthermore, if f is a type over X, then $\hat{f}$ is a type over $\hat{X}$.*

Proof We prove only the "furthermore" part of the statement, since the first one is immediate from the definitions. Suppose that f is a type over X. Then, for every $\epsilon > 0$ and any every subset A of X there exists an element $x_A \in X$ such that

$|\|x_A + x\| - f(x)| \leq \epsilon$ for every $x \in A$. Since $(\hat{X}, \hat{f})$ is finitely represented in (X, f), for every $\epsilon > 0$ and every finite subset B of $\hat{X}$ there exists an element $y_B \in \hat{X}$ such that $|\|y_B + x\| - \hat{f}(x)| \leq \epsilon$ for every $x \in B$. Thus, there exists an element a of an ultrapower of $\hat{X}$ such that $\mathrm{tp}(a/\hat{X}) = \hat{f}$. □

Definition 3.2 Let X be a Banach space and let Y be a superspace of X which is finitely represented in X. If τ is a type over X and σ is a type over Y extending τ, we will say that σ is an *heir* of τ if (Y, σ) is finitely represented in (X, τ).

Proposition 3.3 *Let X be a Banach space and let Y be a superspace of X which is finitely represented in X. If τ is a type over X and σ is a type over Y extending τ, then the following conditions are equivalent:*

1. *σ is an heir of τ;*
2. *there is an ultrapower $(\hat{X}, \hat{\tau})$ of (X, τ) and an isometric embedding $\varphi\colon Y \to \hat{X}$ such that φ fixes X pointwise, and $\varphi(\sigma)(y) = \hat{\tau}(\varphi(y))$ for every $y \in Y$.*

4 Quantifier-free Definable Functions

Note to the logician. In this section we introduce the concepts of *definable set* and *definable type*. We need only their quantifier-free version.

A quantifier-free definable set in a Banach space X is a finite Boolean combination of balls with centers in X. In Banach space model theory, the approach to definability is as follows. One generally does not talk about definable sets but definable *real-valued relations*. An m-ary real-valued relation on a Banach space structure X is a function $\mathcal{R} : X^m \to [-\infty, \infty]$ which is bounded and uniformly continuous on every bounded subset of X^m.

If X is a Banach space and $\mathcal{R} : X^m \to [-\infty, \infty]$ is an m-ary real-valued relation on X, we say that $\mathcal{R}$ is *definable* if the following condition holds. For every choice of $M, \epsilon > 0$ and every interval I there exists a formula $\theta(x_1, \dots, x_m)$ with parameters in X and $\delta > 0$ such that for every $\bar{x} = (x_1, \dots, x_m)$ in X^m with $\sup_i \|x_i\| \leq M$,

1. $\mathcal{R}(\bar{x}) \in I$ implies $\theta(\bar{x})$,
2. $\theta_\delta(\bar{x})$ implies $\mathcal{R}(\bar{x}) \in I + [-\epsilon, \epsilon]$,

where θ_δ is the result of relaxing all the norm estimates present in θ by a factor of $1 + \delta$ (see [13] for the precise definitions).

A real-valued relation is quantifier-free definable exactly when for every choice of M, ϵ, I, the formula θ above can be found so that it is quantifier-free. This is the "right" notion of definability in Banach space model theory, in the sense that it plays a role analogous to that played by the usual notion of definability in first-order model theory (for example, Beth's definability lemma holds, a complete theory is stable if and only if every type is definable and so on).

If X is a Banach space, the norm estimate $\|x\| \leq 1$ determines a subset of X, namely, the closed unit ball of X. If $x_0 \in X$, the norm estimate $\alpha \leq \|x - x_0\| \leq \beta$ also determines a subset of X, namely, an annulus around x_0. In general, we have the following concept.

Definition 4.1 Suppose that X is a Banach space, $x_1, \dots, x_m \in X$, and $I_1, \dots, I_n$ are closed intervals. A *quantifier-free expression C with parameters in X* is a

Boolean combination of syntactic norm estimates of the form

$$\|x + \lambda_1 x_1 + \cdots + \lambda_m x_m\| \in I_i, \qquad (i = 1, \ldots, n).$$

The elements $x_1, \ldots, x_m$ are called the *parameters* of C. We express the dependence of C on $x_1, \ldots, x_m$ and $I_1, \ldots, I_n$ by writing

$$C(x_1, \ldots, x_m; I_1, \ldots, I_n).$$

If $I_1, \ldots, I_n = I$, we write C as

$$C(x_1, \ldots, x_m; I).$$

If the Boolean combination is positive (that is, if it includes $\wedge$ and $\vee$, but not $\neg$), we say that C is a *positive quantifier-free expression*.

Note that if $C(x_1, \ldots, x_m; I_1, \ldots, I_n)$ is a quantifier-free expression with parameters in X, then $C(x_1, \ldots, x_m; I_1, \ldots, I_n)$ determines a subset of X, namely,

$$\left\{ x \in X \;\middle|\; C(x_1, \ldots, x_m; I_1, \ldots, I_n) \right\}.$$

We denote this set by

$$(*) \qquad [\, C(x_1, \ldots, x_m; I_1, \ldots, I_n)\,]_X.$$

We will call a subset of X *quantifier-free definable* if it is of the form $(*)$. When the ambient space X is clear from the context, we omit the subindex X in the preceding notation and write simply

$$[C(x_1, \ldots, x_m; I_1, \ldots, I_n)].$$

Remark 4.2 Suppose that X is a Banach space, $x_1, \ldots, x_m \in X$, and $I_1, \ldots, I_n$ are closed intervals.

1. The closure of $[C(x_1, \ldots, x_m; I_1, \ldots, I_n)]^c$ is quantifier-free definable (using the same parameters).
2. If δ is a positive real number, we have
   (a) $[C(x_1, \ldots, x_m; I_1, \ldots, I_n)]$
   $$\subseteq C(x_1, \ldots, x_m; J_1 + [-\delta, \delta], \ldots, J_n + [-\delta, \delta]);$$
   (b) $[C(x_1, \ldots, x_m; I_1, \ldots, I_n)]$
   $$= \bigcap_{\delta>0} C(x_1, \ldots, x_m; J_1 + [-\delta, \delta], \ldots, J_n + [-\delta, \delta]);$$
   (c) $[C(x_1, \ldots, x_m; I_1, \ldots, I_n)]^c$
   $$= \bigcup_{\delta>0} \overline{[C(x_1, \ldots, x_m; J_1 + [-\delta, \delta], \ldots, J_n + [-\delta, \delta])^c}.$$

Definition 4.3 Suppose that X is a Banach space and f is a real-valued function on X. We will say that f is *quantifier-free definable* if the following condition holds. For every choice of $M, \epsilon > 0$ and every interval I there exist a quantifier-free expression

$$C(x_1, \ldots, x_m; J_1, \ldots, J_n)$$

in X and $\delta > 0$ such that for every $x \in B(M)$,

1. $f(x) \in I$ implies $x \in [\, C(x_1, \ldots, x_m; J_1, \ldots, J_n)\,]$;
2. $x \in [C(x_1, \ldots, x_m; J_1 + [-\delta, \delta], \ldots, J_n + [-\delta, \delta])]$ implies $f(x) \in I + [-\epsilon, \epsilon]$.

We will express the fact that the implications (1) and (2) hold for every $x \in B(M)$ by saying that $[C(x_1,\ldots,x_m;\,J_1,\ldots,J_n)]$ *is* (ϵ,δ)*-equivalent to* $f^{-1}[I]$ *in the ball* $B(M)$.

Suppose that A is a subset of X. If for every choice of M, ϵ, and I, the expression $C(x_1,\ldots,x_m;\,J_1,\ldots,J_n)$ can be taken so that all of its parameters are in given subset A of X, we say that f is *quantifier-free definable over* A.

Remark 4.4 A real-valued function f on a Banach space X is quantifier-free definable if and only if it is quantifier-free definable over a dense subset of X.

Proposition 4.5 *Suppose that X is a Banach space and f is a real-valued function on X which is uniformly continuous on every bounded subset of X. If f is quantifier-free definable and $(\hat{X},\hat{f})$ is an ultrapower of (X,f), then $\hat{f}$ is quantifier-free definable. Furthermore, if $A \subseteq X$ and f is quantifier-free definable over A, so is every heir of $\hat{f}$.*

Proof Fix an interval I and real numbers $M',\epsilon',\epsilon'',\delta',\delta''$ such that

$$0 < M < M', \qquad 0 < \epsilon < \epsilon' < \epsilon'', \qquad 0 < \delta < \delta' < \delta''.$$

It is easy to verify that if

$$[\,C(x_1,\ldots,x_m;\,J_1,\ldots,J_n)\,]_X$$

is $(\epsilon'-\epsilon,\delta'')$-equivalent to $f^{-1}[I+[-\epsilon,\epsilon]]$ in the ball $B_X(M')$, then

$$[\,C(x_1,\ldots,x_m;\,J_1+[-\delta,\delta],\ldots,J_n+[-\delta,\delta])\,]_{\hat{X}}$$

is $(\epsilon'',\delta'-\delta)$-equivalent to $\hat{f}^{-1}[I]$ in the ball $B_{\hat{X}}(M)$. □

Corollary 4.6 *Let X be a Banach space and τ a type over X. Then, if τ is quantifier-free definable, every heir of τ is quantifier-free definable. Furthermore, if $A \subseteq X$ and τ is quantifier-free definable over A, so is every heir of τ.*

Proof By Propositions 3.3 and 4.5. □

5 Quantifier-free Definability and Uniqueness of Heirs

Note to the logician. In this section we prove a Banach space model theoretic analog of a well-known application of Beth's definability lemma, namely, if M is a model, a type $p \in S(M)$ is definable if and only if p has a unique heir over any given elementary extension of M, or equivalently, p has a unique heir over the monster model. We deal only with the quantifier-free version of this fact. (The full version can be found in [13].) Since we work in a quantifier-free context, any superstructure of a Banach space X contained in the monster model serves as an elementary extension of X. For many purposes it will suffice to consider extensions of the form $\operatorname{span}\{X\cup\{a\}\}$ where a is an element of the monster model.

Proposition 5.1 *Suppose that X is a Banach space and f is a real-valued function on X. The following conditions are equivalent.*

1. *f is quantifier-free definable.*
2. *If Y is a superspace of X which is finitely represented in X, then there exists a unique extension g of f to Y such that (Y,g) is finitely represented in (f,X).*
3. *If a is a realization of a type over X and $Y = X + \operatorname{span}\{a\}$, then there exists a unique extension g of f to Y such that (Y,g) is finitely represented in (f,X).*

Proof (1) $\Rightarrow$ (2) follows from Proposition 4.5.

We prove (3) $\Rightarrow$ (1). Suppose that f is not quantifier-free definable. Take $M, \epsilon > 0$ and an interval I such that there do not exist $[\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,]$ with $x_1, \dots, x_m \in X$ and $\delta > 0$ such that $[\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,]$ is (ϵ, δ)-equivalent to $f^{-1}[I]$ in the ball $B_X(M)$. Without loss of generality, we can assume that the interval I is bounded.

Let

$$\mathfrak{C} = \Big\{ [\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,] \;\Big|\; x_1, \dots, x_m \in X \text{ and } x \in B_X(M) \wedge f(x) \in I \text{ implies } x \in [\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,] \Big\}.$$

By our assumption, whenever $[\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,] \in \mathfrak{C}$ and $\delta > 0$, there exists $x \in B_X(M)$ such that

$$x \in [\, C(x_1, \dots, x_m; J_1 + [-\delta, \delta], \dots, J_n + [-\delta, \delta])\,] \text{ and } f(x) \notin I + [-\epsilon, \epsilon].$$

Since $\mathfrak{C}$ is closed under finite intersections, there exists an ultrapower $(\hat{X}, \hat{f})$ of (X, f) and an element $a \in B_{\hat{X}}(M)$ such that

$$a \in \bigcap_{[C(x_1,\dots,x_m; J_1,\dots,J_n)] \in \mathfrak{C}} [\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,] \text{ and } \hat{f}(a) \notin I + [-\epsilon/2, \epsilon/2].$$

Now notice that if $x_1, \dots, x_m \in X$, $J_1, \dots, J_n$ are closed intervals, and

$$a \in [\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,],$$

then for every $\delta > 0$ there exists $x \in B_X(M)$ such that

$$f(x) \in I \text{ and } x \in [\, C(x_1, \dots, x_m; J_1 + [-\delta, \delta], \dots, J_n + [-\delta, \delta])\,]$$

(otherwise, the closure of $[\, C(x_1, \dots, x_m; J_1 + [-\delta, \delta], \dots, J_n + [-\delta, \delta])\,]^c$ is in $\mathfrak{C}$ and then a is in the closure of $[\, C(x_1, \dots, x_m; J_1 + [-\delta, \delta], \dots, J_n + [-\delta, \delta])\,]^c$, which is impossible). Hence, there exists an ultrapowrer $(\hat{X}', \hat{f}')$ of (X, f) and an element $a' \in B_{\hat{X}'}(M)$ such that

$$\hat{f}'(a') \in I, \text{ and } a' \in \bigcap_{a \in C(x_1,\dots,x_m; J_1,\dots,J_n)\,]} [\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,].$$

Since

$$a' \in \bigcap_{a \in [\, C(x_1,\dots,x_m; J_1,\dots,J_n)\,]} [\, C(x_1, \dots, x_m; J_1, \dots, J_n)\,],$$

the spaces $X + \mathrm{span}\{a\}$ and $X + \mathrm{span}\{a'\}$ are isometric via an isometry that maps a to a' and fixes X pointwise. However, $\hat{f}(a) \notin I + [-\epsilon/2, \epsilon/2]$ and $\hat{f}'(a') \in I$. Thus, by taking $Y = X + \mathrm{span}\{a\}$, we contradict (3). □

Proposition 5.2 *Suppose that X is a Banach space and τ is a type over X. The following conditions are equivalent.*

1. *τ is quantifier-free definable.*
2. *If Y is a superspace of X which is finitely represented in X, then τ has a unique heir over Y.*
3. *If a is a realization of a type over X and $Y = X + \mathrm{span}\{a\}$, then τ has a unique heir over Y.*

6 Quantifier-free Definability and Uniqueness of Spreading Models

Proposition 6.1 *Let τ be a type over a separable Banach space X and suppose that τ is definable. Then, if (a_n) and (b_n) are fundamental sequences for τ, we have*

$$\|x + \lambda_0 a_0 + \cdots + \lambda_n a_n\| = \|x + \lambda_0 b_0 + \cdots + \lambda_n b_n\|,$$

for every choice of $x \in X$ and scalars $\lambda_0, \ldots, \lambda_n$.

Remark 6.2 By the definitions introduced in Section 2, the conclusion of Proposition 6.1 means that a convolution on the scalar multiples of τ can be defined uniquely.

Proof of Proposition 6.1 Suppose that (a_n) and (b_n) are fundamental sequences for τ and that there exists $x \in X$ and scalars $\lambda_0, \ldots, \lambda_{n-1}$ such that

$$\|x + a_0 + \cdots + \lambda_{n-1} a_{n-1} + \lambda_n a_n\| \neq \|x + b_0 + \cdots + \lambda_{n-1} b_{n-1} + \lambda_n b_n\|.$$

We may assume that n is minimal with this property. (Note that $n > 0$.) The types

$$\mathrm{tp}(a_0/X + \mathrm{span}\{a_1, \ldots, a_n\}),$$
$$\mathrm{tp}(b_0/X + \mathrm{span}\{b_1, \ldots, b_n\})$$

are heirs of $\mathrm{tp}(a_0/X) = \mathrm{tp}(b_0/X) = \tau$.

By the minimality assumption of n, for $k = 1, \ldots, n$ we have

$$\mathrm{tp}\left(a_k/X + \mathrm{span}\{a_i \mid 1 \leq i \leq k-1\}\right) = \mathrm{tp}\left(b_k/X + \mathrm{span}\{b_i \mid 1 \leq i \leq k-1\}\right).$$

Therefore there exists an isometry

$$\varphi\colon X + \mathrm{span}\{a_1, \ldots, a_n\} \longrightarrow X + \mathrm{span}\{b_1, \ldots, b_n\}$$

such that φ fixes X pointwise and $\varphi(a_k) = b_k$, for $k = 1, \ldots, n$.

We have that $\mathrm{tp}(\varphi(a_0)/X + \mathrm{span}\{b_1, \ldots, b_n\})$ and $\mathrm{tp}(b_0/X + \mathrm{span}\{b_1, \ldots, b_n\})$ are both heirs of τ. However,

$$\|x + \varphi(a_0) + \sum_{i=1}^{n} \lambda_i b_i\| = \|x + a_0 + \sum_{i=1}^{n} \lambda_i a_i\| \neq \|x + b_0 + \sum_{i=1}^{n} \lambda_i b_i\|,$$

so these two heirs are distinct. Hence, τ is not quantifier-free definable, by Proposition 5.2. $\square$

7 Stable Types

Note to the logician. In this section we introduce spreading model analogs of the concept of stable type from model theory. Definition 7.2 corresponds to the failure of the order property. Proposition 7.3 is a counterpart of the well-known characterization of stability in terms of the algebraic properties of forking (symmetry, finite character and so on).

Definition 7.1 Let τ be a type over a separable Banach space X and let $*$ be a convolution on the scalar multiples of τ. We say that a sequence (x_k) in X *approximates* $\mathrm{span}(\tau, *)$, or is an *approximating sequence* for $\mathrm{span}(\tau, *)$, if for arbitrary $x \in X$ and scalars $\lambda_0, \ldots, \lambda_n$ we have

$$\lambda_0\tau * \cdots * \lambda_n\tau = \lim_{k_n < \cdots < k_0} \|x + \lambda_0 x_{k_0} + \cdots + \lambda_n x_{k_n}\|.$$

Definition 7.2 Let X be a separable Banach space, let τ be a type over X which is not realized in X, and let $*$ be a convolution on the scalar multiples of τ. We will say that $\mathrm{span}(\tau, *)$ is *stable* if the convolution $*$ has an extension to $\overline{\mathrm{span}}(\tau, *)$ which is commutative and separately continuous.

Proposition 7.3 *Let X be a separable Banach space, let τ be a type over X which is not realized in X, and let $*$ be a convolution on the scalar multiples of τ. Then the following conditions are equivalent.*

1. $\mathrm{span}(\tau, *)$ *is stable.*
2. *Given nonprincipal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, the convolution $*$ on* $\mathrm{span}(\tau, *)$ *can be extended to* $\overline{\mathrm{span}}(\tau, *)$ *by defining, for every pair of sequences* $(\sigma_m), (\rho_n)$ *in* $\mathrm{span}(\tau, *)$ *with* $\lim_m \sigma_m = \sigma$ *and* $\lim_n \rho_n = \rho$,
$$\sigma * \rho = \lim_{\mathcal{U},m} \lim_{\mathcal{V},n} (\sigma_m * \rho_n),$$
and this extension is commutative.
3. *There do not exist sequences (σ_m) and (ρ_n) in* $\mathrm{span}(\tau, *)$ *and $x \in X$ such that*
$$\sup_{m<n} (\sigma_m * \rho_n)(x) < \inf_{n<m} (\rho_n * \sigma_m)(x).$$
4. *If (x_ν) is an approximating sequence for* $\mathrm{span}(\tau, *)$ *in X, there do not exist double sequences $(y_{m,l})_{m,l}$ and $(z_{n,l})_{n,l}$ in X satisfying the following properties:*
   (a) *for each m and n, $(y_{m,k})_k$ and $(z_{n,l})_l$ are sequences of blocks of (x_ν) which approximate types in* $\mathrm{span}(\tau, *)$,
   (b) *for some $x \in X$,*
$$\sup_{m<n} \left(\lim_{k<l} \|y_{m,k} + z_{n,l} + x\| \right) < \inf_{n<m} \left(\lim_{l<k} \|y_{m,k} + z_{n,l} + x\| \right).$$

Proof The implication (1) $\Rightarrow$ (2) is trivial and the equivalence (3) $\Leftrightarrow$ (4) follows from the definitions.

(2) $\Rightarrow$ (1) First we note that, in (2), the commutativity assumption allows us to permute the order of the limits; this will also be used in the proof of (2) $\Rightarrow$ (1), so, although it is easy, we record it below:
$$\begin{aligned}\lim_{\mathcal{U},m} \lim_{\mathcal{V},n} (\sigma_m * \rho_n) &= \sigma * \rho \\ &= \rho * \sigma \\ &= \lim_{\mathcal{V},n} \lim_{\mathcal{U},m} (\rho_n * \sigma_m) \\ &= \lim_{\mathcal{V},n} \lim_{\mathcal{U},m} (\sigma_m * \rho_n).\end{aligned}$$

Now we just have to show that the extension of $*$ given by (2) is separately continuous. To this effect, fix $\sigma, \rho \in \overline{\mathrm{span}}(\tau, *)$ and a sequence (ρ_n) in $\overline{\mathrm{span}}(\tau, *)$ such that $\lim_n \rho_n = \rho$. Take sequences $(\sigma_k), (\rho_{n,l})$ in $\mathrm{span}(\tau, *)$ such that $\lim_k \sigma_k = \sigma$ and $\lim_l \rho_{n,l} = \rho_n$ for every n. By Ramsey's Theorem (Proposition 2.1), we can assume that $\lim_{n<l} \rho_{n<l} = \rho$. Hence, after replacing (σ_n) by a subsequence and $(\rho_{n,l})$ by an

$(\omega \times \omega)$-submatrix if necessary, we have

$$\begin{aligned}\sigma * \rho &= \lim_k \lim_{n<l} (\sigma_k * \rho_{n,l}) \\ &= \lim_{n<l} \lim_k (\sigma_k * \rho_{n,l}) \\ &= \lim_n \lim_l \lim_k (\sigma_k * \rho_{n,l}) \\ &= \lim_n \lim_k \lim_l (\sigma_k * \rho_{n,l}) \\ &= \lim_n (\sigma * \rho_n).\end{aligned}$$

(2) $\Rightarrow$ (3) Suppose that (2) is true and (3) is false, and fix sequences (σ_m), (ρ_n) in $\text{span}(\tau, *)$ and element $x \in X$ satisfying the inequality given in (3). Since, by (2), $*$ is commutative on $\text{span}(\tau, *)$, we have

$$\sup_{m<n} (\sigma_m * \rho_n)(x) < \inf_{n<m} (\sigma_m * \rho_n)(x).$$

But this contradicts the fact that the order of the limits in (2) can be exchanged (see the proof of (2) $\Rightarrow$ (1) above).

(3) $\Rightarrow$ (2) Suppose that (2) does not hold. By Ramsey's Theorem, the operation defined in (2) is continuous on the first coordinate. Commutativity of the extension would imply continuity on the second coordinate. Thus, since we are assuming that (2) is not true, the operation defined in (2) cannot be commutative. Take $\sigma, \rho \in \overline{\text{span}}(\tau, *)$ and $x \in X$ such that $(\sigma * \rho)(x) \neq (\rho * \sigma)(x)$. Without loss of generality, assume

$$(\sigma * \rho)(x) < (\rho * \sigma)(x).$$

Take sequences (σ_m), (ρ_n) in $\text{span}(\tau, *)$ such that $\lim_m \sigma_m = \sigma$ and $\lim_n \rho_n = \rho$. Then,

$$\lim_{\mathcal{U},m} \lim_{\mathcal{V},n} (\sigma_m * \rho_n)(x) < \lim_{\mathcal{V},n} \lim_{\mathcal{U},m} (\rho_n * \sigma_m)(x),$$

which contradicts (3), by Ramsey's Theorem. □

Definition 7.4 We will refer to the extension of $*$ given by (2) of Proposition 7.3 as the *Krivine-Maurey convolution* on $\overline{\text{span}}(\tau, *)$, corresponding to the ultrafilters $\mathcal{U}$ and $\mathcal{V}$.

Example 7.5 For every type τ over the Banach space ℓ_p $(1 \leq p < \infty)$ and every convolution on τ, we have that $\text{span}(\tau, *)$ is stable. To prove this, fix a type τ over ℓ_p and let (x_n) be an approximating sequence for τ. Since (x_n) is bounded, it is coordinatewise bounded, so there exists $\bar{x} \in \ell_p$ such that $x_n \to \bar{x}$ coordinatewise. Let $z_n = x_n - \bar{x}$. Then,

$$\tau(x) = \lim_n \|\bar{x} + x + z_n\|$$

and $z_n \to 0$ coordinatewise. By further extracting a subsequence, we can assume that there exists a real number λ such that $\|z_n\| \to \lambda$ as $n \to \infty$. It is an easy exercise to show that

$$\tau(x) = \left(\|\bar{x} + x\|^p + \lambda^p\right)^{1/p}.$$

Therefore, $\bar{x}$ and λ are uniquely determined by τ.

If σ is another type over ℓ_p (not necessarily in $\overline{\text{span}}(\tau, *)$) and (y_n) is an approximating sequence for σ, we have

$$\sigma(x) = \left(\|\bar{y} + x\|^p + \mu^p\right)^{1/p},$$

where $y_n \to \bar{y}$ coordinatewise and $\|y_n - \bar{y}\| \to \mu$. Therefore,

$$\lim_m \lim_n \|x_m + y_n + x\| = \lim_m \left(\|x_m + \bar{y} + x\|^p + \mu^p\right)^{1/p}$$
$$= \left(\|\bar{x} + \bar{y} + x\|^p + \lambda^p + \mu^p\right)^{1/p},$$

and the same value is obtained if the limits are taken in the opposite order.

Example 7.6 For every type τ over c_0 which is approximable over a weakly compact subset of c_0 and every convolution on the scalar multiples of τ, $\operatorname{span}(\tau, *)$ is stable. To prove this, let K be a weakly compact subset of c_0 and let τ be a type approximated by a sequence (x_n) in K. Fix $\bar{x} \in c_0$ such that $x_n \to \bar{x}$ weakly, let $z_n = x_n - \bar{x}$, and by taking a subsequence if necessary, assume $\|z_n\| \to \lambda$. Then,

$$\tau(x) = \max(\|\bar{x} + x\|, \lambda).$$

Arguing as in Example 7.5, we show that if (y_n) is another sequence in K,

$$\lim_m \lim_n \|x_m + y_n + x\| = \lim_n \lim_m \|x_m + y_n + x\|.$$

Definition 7.7 A type τ is called *symmetric* if $\tau = -\tau$.

Remark 7.8 Notice that if $\operatorname{span}(\tau, *)$ is stable, then $\sigma = \tau * (-\tau)$ is symmetric and stable, and hence $\operatorname{span}(\sigma, *)$ is 1-unconditional.

Definition 7.9 Let τ be a type over X which is not realized in X and let $*$ be a convolution on the scalar multiples of τ such that $\operatorname{span}(\tau, *)$ is 1-unconditional. Given $p \in [1, \infty]$, we will say that ℓ_p is *block represented in* $\operatorname{span}(\tau, *)$ if there exists a sequence (e_n) satisfying the following two conditions:

1. (e_n) is isometric over X to the standard unit basis of ℓ_p, if $1 \leq p < \infty$, and to the standard unit basis of c_0, if $p = \infty$;
2. there exists a sequence of types (σ_l) in $\operatorname{span}(\tau, *)$ such that for any scalars $\lambda_0, \dots, \lambda_n$,

$$\operatorname{tp}(\lambda_0 e_0 + \cdots + \lambda_n e_n / X) = \lim_l (\lambda_0 \sigma_l * \cdots * \lambda_n \sigma_l).$$

For a type τ, we define

$$\mathfrak{p}[\operatorname{span}(\tau, *)] = \{\, p \in [1, \infty] \mid \ell_p \text{ is block represented in } \operatorname{span}(\tau, *) \,\}.$$

Theorem 2.7 says exactly that for every Banach space X, every type τ over X which is not realized in X, and every convolution on the scalar multiples of τ such that $\operatorname{span}(\tau, *)$ is 1-unconditional, the set $\mathfrak{p}[\operatorname{span}(\tau, *)]$ is nonempty.

Lemma 7.10 *Suppose that τ is a type over X which is not realized in X and that $\operatorname{span}(\tau, *)$ is 1-unconditional and stable. Then, if $\tau' \in \overline{\operatorname{span}}(\tau, *)$, we have*

$$\mathfrak{p}[\operatorname{span}(\tau', *)] \subseteq \mathfrak{p}[\operatorname{span}(\tau, *)].$$

Proof Suppose that $p \in \mathfrak{p}[\operatorname{span}(\tau', *)]$ and take (e_n), and (σ_l) corresponding to p and τ' as in Definition 7.9. Since $\sigma_l \in \operatorname{span}(\tau', *)$, we can write

$$\sigma_l = \mu_0^l \tau' * \cdots * \mu_{j(l)}^l \tau',$$

where $\mu_0^l, \dots, \mu_{j(l)}^l$ are scalars. Also, since $\tau' \in \overline{\operatorname{span}}(\tau, *)$, there exists a sequence (ρ_m) in $\operatorname{span}(\tau, *)$ such that $\tau' = \lim_m \rho_m$. Then for any scalars $\lambda_1, \dots, \lambda_n$ we have the following equalities. The last one follows from the separate continuity of the Krivine-Maurey convolution on $\overline{\mathrm{span}}(\tau, *)$ (Proposition 7.3).

$$\begin{aligned}
&\mathrm{tp}(\lambda_0 e_0 + \cdots + \lambda_k e_k / X) = \\
&\lim_l \left[\lambda_0(\mu_0^l \tau' * \cdots * \mu_{j(l)}^l \tau') * \cdots * \lambda_k(\mu_0^l \tau' * \cdots * \mu_{j(l)}^l \tau') \right] = \\
&\lim_l \left[\lambda_0\big(\mu_0^l \lim_m \rho_m * \cdots * \mu_{j(l)}^l \lim_m \rho_m\big) * \cdots * \lambda_k\big(\mu_0^l \lim_m \rho_m * \cdots * \mu_{j(l)}^l \lim_m \rho_m\big) \right] = \\
&\lim_l \left[\lambda_0 \lim_{m_0} \cdots \lim_{m_{j(l)}} (\mu_0^l \rho_{m_0} * \cdots * \mu_{j(l)}^l \rho_{m_{j(l)}}) * \cdots * \lambda_k \lim_{m_0} \cdots \lim_{m_{j(l)}} (\mu_0^l \rho_{m_0} * \cdots * \mu_{j(l)}^l \rho_{m_{j(l)}}) \right].
\end{aligned}$$

Now Ramsey's Theorem allows us to replace each of the iterated limits inside the square brackets by the same single limit. We conclude that $p \in \mathfrak{p}[\mathrm{span}(\tau, *)]$. □

Proposition 7.11 *Suppose that τ is a symmetric type over a separable Banach space X and* $\mathrm{span}(\tau, *)$ *is stable. Then there exists a type τ' such that*

1. $\tau' \in \overline{\mathrm{span}}(\tau, *)$;
2. $\|\tau'\| = 1$;
3. $\mathfrak{p}[\mathrm{span}(\rho, *)] = \mathfrak{p}[\mathrm{span}(\tau', *)]$ *for every type* $\rho \in \overline{\mathrm{span}}(\tau', *)$.

Proof Suppose that the conclusion of the proposition is false. We construct, inductively, a sequence $(\tau_i)_{i<(2^{\aleph_0})^+}$ of types over X such that

1. $\tau_0 = \tau$;
2. $\|\tau_i\| = 1$;
3. $\tau_i \in \overline{\mathrm{span}}(\tau_j, *)$ for $i > j$;
4. $\mathfrak{p}[\mathrm{span}(\tau_i, *)] \subsetneq \mathfrak{p}[\mathrm{span}(\tau_j, *)]$ for $i > j$.

This is clearly impossible. We construct τ_i by induction on i. The case when i is a successor ordinal is given by assumption. Suppose that i is a limit ordinal. Fix a nonprincipal ultrafilter $\mathcal{U}$ on i. By compactness, there exists a type $\sigma \in \overline{\mathrm{span}}(\tau, *)$ such that $\lim_{j<i,\mathcal{U}} \tau_j = \sigma$. Conditions (1)–(3) are then satisfied by letting $\tau_i = \sigma$. □

8 Quantifier-free Definability and Stability

Note to the logician. In this section we prove an analog of Shelah's well-known theorem that every stable type is definable. More precisely, we will prove an analog of the fact that every quantifier-free type is quantifier-free definable. The proof is a refinement of the main lemma of [14].

Definition 8.1 Let X be a Banach space and let τ be a type over X. We will say that τ is *strongly quantifier-free definable* if τ is quantifier-free definable, and for every choice of $M, \epsilon > 0$ and every interval I, the quantifier-free expression $C(x_1, \ldots, x_n; J_1, \ldots, J_n)$ of Definition 4.3 can be taken to be a Boolean combination of expressions of the form

$$\|x + x_i\| \in J, \qquad (i = 1, \ldots, n),$$

where

1. the interval J is arbitrarily close to I;
2. the norm of the parameters $x_1, \ldots, x_n$ is arbitrarily close to the norm of τ;
3. the quantifier-free expression $C(x_1, \ldots, x_n; J)$ is positive (see Definition 4.1).

Proposition 8.2 *The following conditions are equivalent for a type τ over a Banach space X.*

1. *τ is strongly quantifier-free definable;*
2. *For every choice of $M, \epsilon > 0$ and every interval I of the form $[0, \alpha]$ there exist a quantifier-free expression $C(x_1, \dots, x_n; [0, \beta])$ in X and $\delta > 0$ such that*
   (a) *$[C(x_1, \dots, x_n; [0, \beta])]$ is (ϵ, δ)-equivalent to $\tau^{-1}[I]$ in $B(M)$;*
   (b) *$|\beta - \alpha| \le \epsilon$ and $|\,\|x_i\| - \|\tau\|\,| < \epsilon$ for $i = 1, \dots, n$.*

Proof Immediate. □

Proposition 8.3 *Let τ be a type over a separable Banach space X, let $*$ be a convolution on the scalar multiples of τ, and let (x_ν) be an approximating sequence for $\operatorname{span}(\tau, *)$ in X. Then, if $\operatorname{span}(\tau, *)$ is stable, for every finite dimensional subspace E of X, every type over $E + \overline{\operatorname{span}}\{x_\nu \mid \nu \in \mathbb{N}\}$ is strongly quantifier-free definable.*

Proof Let $Y = \overline{\operatorname{span}}\{x_\nu \mid \nu \in \mathbb{N}\}$ and let σ be a type over $E+Y$. Fix $M, \epsilon > 0$, and an interval $[0, \alpha]$. We will define a quantifier-free expression $C(c_1, \dots, c_r; [0, \beta])$ and $\delta > 0$ such that for $x \in B_E(M) + B_Y(M)$,

(I) $\sigma(x) \in [0, \alpha]$ implies $x \in [\, C(c_1, \dots, c_r; [0, \beta])\,]$;
(II) $x \in [\, C(c_1, \dots, c_r; [0, \beta + \delta])\,]$ implies $\sigma(x) \in [0, \alpha + \epsilon]$.

Since M is arbitrary, this will suffice to show that σ is quantifier-free definable. Let K be an upper bound for the norms of the projections of $B_{E+Y}(M + \epsilon)$ onto E and Y. We will now construct, by induction on n,

(a) a sequence $(e_{n,k})_k$ in E and a sequence $(y_{n,k})_k$ of blocks of (x_ν) such that $e_{n,k} \in B_E(K)$, $y_{n,k} \in B_Y(K)$, and

$$|\|e_{n,k} + y_{n,k}\| - \|\sigma\|| \le \epsilon$$

for every pair n, k;

(b) for $i = -1, 0, 1, 2, \dots, n$,
   (i) sequences $(\, S(i,k)\,)_k$ and $(\, T(i,k)\,)_k$ of subsets of $\{\,0, \dots, n\,\}$,
   (ii) sequences $(u^s_{i+1,k})_k$ in $B_E(M) + B_Y(M)$ for $s \in S(i,k)$, and sequences $(v^t_{i+1,k})_k$ in $B_E(M) + B_Y(M)$ for $t \in T(i)$.

(The fact that $|\|e_{n,k} + y_{n,k}\| - \|\sigma\|| \le \epsilon$ for every n together with the fact that β was chosen arbitrarily close to α will ensure that the quantifier-free definability of σ is strong, because, toward the end of the proof, we will define $C(c_1, \dots, c_r; [0, \beta])$ as a positive linear combination of expressions of the form $\|c + x\| \in [0, \beta]$, where $c \in \{c_1, \dots, c_n\}$, and each of the parameters $c_1, \dots, c_r$ will be of the form $e_{n,k}+y_{n,k}$, for some n and some k.)

Take β and δ such that

$$\alpha < \beta < \beta + \delta < \alpha + \epsilon.$$

Without loss of generality, we can take δ such that

$$\delta < \min\{\, \beta - \alpha, (\alpha + \epsilon) - (\beta + \delta)\,\}.$$

Take also positive numbers $\eta, \eta_0, \eta_1, \dots$ such that

$$\delta = \eta_{-1} < \eta_0 < \eta_1 < \cdots < \eta < \min\{\, \beta - \alpha, (\alpha + \epsilon) - (\beta + \delta)\,\}.$$

Suppose that we have defined

$$
\begin{gathered}
(e_{0,k})_k, (e_{1,k})_k, \dots, (e_{n,k})_k, \\
(y_{0,k})_k, (y_{1,k})_k, \dots, (y_{n,k})_k, \\
\big(S(-1,k) \big)_k, \dots, \big(S(n-1,k) \big)_k, \\
\big(T(-1,k) \big)_k, \dots, \big(T(n-1,k) \big)_k,
\end{gathered}
$$

and

$$(u^s_{i,k})_k, (v^t_{i,k})_k, \quad \text{for } i = 0, \dots, n \text{ and } s \in S(i,k),\ t \in T(i,k).$$

We now define the sequences $(\,S(n,k)\,)_k$, $(\,T(n,k)\,)_k$ and $(u^s_{i+1,k})_k$, $(v^t_{i+1,k})_k$. Let

$$
\begin{aligned}
S(n,k) = \Big\{ s \subseteq \{0,\dots,n\} \;\Big|\; & \exists x \in B_E(M) + B_Y(M) \\
& \Big(\sigma(x) \in [0, \alpha + \eta_n] \wedge \bigwedge_{\substack{i \in s \\ 0 \le j \le k}} \|e_{i,j} + y_{i,j} + x\| \in [\beta, \infty) \Big) \Big\}.
\end{aligned}
$$

For each $s \in S(n,k)$, let $u^s_{n+1,k} \in B_E(M) + B_Y(M)$ be such that

(a) $\sigma(u^s_{n+1,k}) \in [0, \alpha + \eta_n]$;
(b) $\bigwedge_{\substack{i \in s \\ 0 \le j \le k}} \|e_{i,j} + y_{i,j} + u^s_{n+1,k}\| \in [\beta, \infty)$;
(c) $u^s_{n+1,k} = \pi_1(u^s_{n+1,k}) + \pi_2(u^s_{n+1,k})$, where
  (i) $\pi_1(u^s_{n+1,k}) \in B_E(M)$,
  (ii) $\pi_2(u^s_{n+1,k}) \in B_Y(M)$,
  (iii) $(\,\pi_2(u^s_{n+1,k})\,)_k$ is a sequence of blocks of (x_ν),
  (iv) $(\,\pi_2(u^s_{n+1,k})\,)_k$ approximates a type in $\operatorname{span}(\tau, *)$.

Similarly, let

$$
\begin{aligned}
T(n,k) = \Big\{ t \subseteq \{0,\dots,n\} \;\Big|\; & \exists x \in B_E(M) + B_Y(M) \\
& \Big(\sigma(x) \in [\alpha + \epsilon - \eta_n, \infty) \wedge \bigwedge_{\substack{i \in t \\ 0 \le j \le k}} \|e_{i,k} + y_{i,k} + x\| \in [0, \beta + \delta] \Big) \Big\},
\end{aligned}
$$

and for each $t \in T(n,k)$ let $v^t_{n+1,k} \in B_E(M) + B_Y(M)$ be such that

(a) $\sigma(v^t_{n+1,k}) \in [\beta + \epsilon - \eta_n, \infty)$;
(b) $\bigwedge_{\substack{i \in t \\ 0 \le j \le k}} \|e_{i,j} + y_{i,j} + v^t_{n+1,k}\| \in [0, \beta + \delta]$;
(c) $v^t_{n+1,k} = \pi_1(v^t_{n+1,k}) + \pi_2(v^t_{n+1,k})$, where
  (i) $\pi_1(v^t_{n+1,k}) \in B_E(M)$,
  (ii) $\pi_2(v^t_{n+1,k}) \in B_Y(M)$,
  (iii) $(\,\pi_2(v^t_{n+1,k})\,)_k$ is a sequence of blocks of (x_ν),
  (iv) $(\,\pi_2(v^t_{n+1,k})\,)_k$ approximates a type in $\operatorname{span}(\tau, *)$.

We now define $e_{n+1,k}$ and $y_{n+1,k}$. Let

$$
\begin{aligned}
F(k) = & \bigcup_{0 \le j \le k} \{\, u^s_{i+1,j} \mid -1 \le i \le n,\ s \in S(i,j) \,\} \\
\cup & \bigcup_{0 \le j \le k} \{\, v^t_{i+1,j} \mid -1 \le i \le n,\ t \in T(i,j) \,\}.
\end{aligned}
$$

Since $F(k)$ is finite, there exists $a \in E + Y$ with $|\|a\| - \|\sigma\|| \leq \epsilon$ such that

$$x \in F(k) \text{ and } \sigma(x) \in [0, \alpha + \eta_n] \quad \text{implies} \quad \|a + x\| \in [0, \alpha + \eta_{n+1}]$$

and

$$x \in F(k) \text{ and } \sigma(x) \in [\alpha + \epsilon - \eta_n, \infty)] \quad \text{implies} \quad \|a + x\| \in [\alpha + \epsilon - \eta_{n+1}, \infty).$$

We take $e_{n+1,k} \in B_E(K)$ and $y_{n+1,k} \in B_Y(K)$ such that $a = e_{n,k} + y_{n,k}$, and so that the sequence $(y_{n+1,k})_k$ is a sequence of blocks of (x_ν) and approximates a type in $\mathrm{span}(\tau, *)$.

Claim 8.4 *Suppose that* $0 \leq i \leq n$, $0 \leq j \leq k$, *and* $s \in S(i-1, j)$, $t \in T(i-1, j)$. *Then*

$$\|e_{n,k} + y_{n,k} + u^s_{i,j}\| \in [0, \alpha + \eta_n]$$

and

$$\|e_{n,k} + y_{n,k} + v^t_{i,j}\| \in [\alpha + \epsilon - \eta_n, \infty).$$

Proof Claim 8.4 follows immediately from the definitions. □

Claim 8.5 *Suppose that*

$$0 \leq n(0) < n(1) < \cdots < n(N),$$
$$0 \leq k(0) < k(1) < \cdots < k(L),$$

and

$$\exists x \in B_E(M) + B_Y(M) \left(\sigma(x) \in [0, \alpha] \wedge \bigwedge_{\substack{0 \leq i \leq N \\ 0 \leq j \leq L}} \|e_{n(i),k(j)} + y_{n(i),k(j)} + x\| \in [\beta, \infty) \right).$$

Then there exist sequences

$$(f_{0,j})_{j \leq L}, \ldots, (f_{N,j})_{j \leq L}$$

in E and sequences

$$(z_{0,j})_{j \leq L}, \ldots, (z_{N,j})_{j \leq L}$$

in $\mathrm{span}\{x_\nu \mid \nu \in \mathbb{N}\}$ *such that*

(a) $f_{i,j} \in B_E(M)$ *and* $z_{i,j} \in B_Y(M)$, *for* $0 \leq i \leq N$ *and* $0 \leq j \leq L$;
(b) $(z_{i,j})_{j \leq L}$ *is a finite sequence of blocks of* (x_ν);
(c) $\|e_{n(i),k(j)} + f_{p,q} + y_{n(i),k(j)} + z_{p,q}\| \in [\beta, \infty)$, *for* $0 \leq i \leq p \leq N$ *and* $0 \leq j \leq q \leq L$;
(d) $\|e_{n(i),k(j)} + f_{p,q} + y_{n(i),k(j)} + z_{p,q}\| \in [0, \alpha + \eta]$, *if* $0 \leq p \leq i \leq N$ *and* $0 \leq q \leq j \leq L$.

Furthermore, the correspondence is well behaved under extensions in the following sense. If $0 \leq N < N'$, $0 \leq L < L'$,

$$0 \leq n(0) < n(1) < \cdots < n(N) < n(N+1) < \cdots < n(N'),$$
$$0 \leq k(0) < k(1) < \cdots < k(L) < k(L+1) < \cdots < k(L'),$$

and

$$\exists x \in B_E(M) + B_Y(M)$$
$$\Big(\sigma(x) \in [0, \alpha] \wedge \bigwedge_{\substack{0 \le i \le N' \\ 0 \le j \le L'}} \|e_{n(i),k(j)} + y_{n(i),k(j)} + x\| \in [\beta, \infty), \Big),$$

then the list of sequences

$$(f_{0,j})_{j \le L'}, \dots, (f_{N',j})_{j \le L'},$$
$$(z_{0,j})_{j \le L'}, \dots, (z_{N',j})_{j \le L'}$$

given by the proof for the pair N', L' extends naturally the list of sequences given for the pair N, L, that is, the entries $f_{i,j}$ and $z_{i,j}$ coincide when $0 \le i \le N$ and $0 \le j \le L$.

Proof of Claim 8.5 We construct the sequences $(f_{p,j})_{j \le L}$ and $(z_{p,j})_{j \le L}$ by induction on p. First we note that for every $j \le L$ we have

$$S(n(0) - 1, k(j)) \ne \varnothing;$$

in fact, $\varnothing \in S(n(0) - 1, k(j))$ since, by the hypothesis of our claim,

$$\exists x \in B_E(M) + B_Y(M) \Big(\sigma(x) \in [0, \alpha + \eta_{n(0)-1}]\Big).$$

Take $s \in S(n(0) - 1, k(j))$. Recall the definition of $\pi_1(u^s_{n(0),k(j)})$ and $\pi_2(u^s_{n(0),k(j)})$, and for $j \le L$ let

$$f_{0,j} = \pi_1(u^s_{n(0),k(j)}),$$
$$z_{0,j} = \pi_2(u^s_{n(0),k(j)}).$$

By Claim 8.4,

$\|e_{n(i),k(j)} + f_{0,q} + y_{n(i),k(j)} + z_{0,q}\| \in [0, \alpha + \eta_{n(i)}]$, if $0 \le i \le N$ and $0 \le q \le j \le L$.

Assume that we have $(z_{0,j})_{j \le L}, \dots, (z_{p,j})_{j \le L}$ and $(f_{0,j})_{j \le L}, \dots, (f_{p,j})_{j \le L}$ as desired. Let

$$s = \{n(0), \dots, n(p)\}.$$

By the hypothesis of the claim, for every $j \le L$ we have $s \in S(n(p), k(j))$. Let

$$f_{p+1,j} = \pi_1(u^s_{p+1,k(j)}),$$
$$z_{p+1,j} = \pi_2(u^s_{p+1,k(j)}).$$

Then

$$\|e_{n(i),k(j)} + f_{p+1,q} + y_{n(i),k(j)} + z_{p+1,q}\| \in [\beta, \infty),$$

for $0 \le i \le p$ and $0 \le j \le q \le L$, and by Claim 8.4,

$$\|e_{n(i+1),k(j)} + f_{p+1,q} + y_{n(i+1),k(j)} + z_{p+1,q}\| \in [0, \alpha + \eta_{n(i+1)}],$$

for $0 \le p \le i \le N - 1$ and $0 \le j \le q \le L$. We have proved Claim 8.5. $\square$

Claim 8.6 *Suppose that*

$$0 \le n(0) < n(1) < \dots < n(N),$$
$$0 \le k(0) < k(1) < \dots < k(L),$$

and

$$\exists x \in B_E(M) + B_Y(M)$$
$$\Big(\sigma(x) \in [\alpha + \epsilon, \infty) \wedge \bigwedge_{\substack{0\le i\le N\\0\le j\le L}} \|e_{n(i),k(j)} + y_{n(i),k(j)} + x\| \in [0, \beta + \delta]\Big).$$

Then there exist sequences

$$(f_{0,j})_{j\le L}, \ldots, (f_{N,j})_{j\le L}$$

in E and sequences

$$(z_{0,j})_{j\le L}, \ldots, (z_{N,j})_{j\le L}$$

in $\operatorname{span}\{x_\nu \mid \nu \in \mathbb{N}\}$ *such that*

(a) $f_{i,j} \in B_E(M)$ *and* $z_{i,j} \in B_Y(M)$, *for* $0 \le i \le N$ *and* $0 \le j \le L$;
(b) $(z_{i,j})_{j\le L}$ *is a finite sequence of blocks of* (x_ν);
(c) $\|e_{n(i),k(j)} + f_{p,q} + y_{n(i),k(j)} + z_{p,q}\| \in [0, \beta + \delta]$, *for* $0 \le i \le p \le N$ *and* $0 \le j \le q \le L$;
(d) $\|e_{n(i),k(j)} + f_{p,q} + y_{n(i),k(j)} + z_{p,q}\| \in [\alpha + \epsilon - \eta, \infty)$, *if* $0 \le p \le i \le N$ *and* $0 \le q \le j \le L$.

Furthermore, the correspondence is well behaved under extensions, as indicated in Claim 8.5.

Proof of Claim 8.6 The proof is analogous to that of Claim 8.5. □

Claim 8.7 *There exist $m \in \mathbb{N}$ satisfying the following property. If $n(0), \ldots, n(m)$ and $k(0), \ldots, k(m)$ are indices satisfying*

$$0 \le n(0) < \cdots < n(m) \le (m+1)^2, \qquad 0 \le k(0) < \cdots < k(m) \le (m+1)^2,$$

then neither of the following two conditions holds:

$$(\dagger) \quad \exists x \in B_E(M) + B_Y(M)\Big(\sigma(x) \in [0, \alpha] \wedge \bigwedge_{\substack{0\le i\le m\\0\le j\le m}} \|e_{n(i),k(j)} + y_{n(i),k(j)} + x\| \in [\beta, \infty)\Big),$$

$$(\ddagger) \quad \exists x \in B_E(M) + B_Y(M)\Big(\sigma(x) \in [\alpha + \epsilon, \infty) \wedge \bigwedge_{\substack{0\le i\le m\\0\le j\le m}} \|e_{n(i),k(j)} + y_{n(i),k(j)} + x\| \in [0, \beta + \delta]\Big).$$

Proof of Claim 8.7 Suppose that the claim is false. We now define a tree Γ. For each $m \in \mathbb{N}$ the nodes on the mth level of Γ are indexed by the tuples of the form

$$(*) \qquad (\, n(0), k(0), n(1), k(1), \ldots, k(m), k(m)\,)$$

where

$$0 \le n(0) < \cdots < n(m) \le (m+1)^2, \qquad 0 \le k(0) < \cdots < k(m) \le (m+1)^2$$

(so each level of Γ is finite). The node at level m indexed by the tuple $(*)$ is the set of all lists of sequences of the form

$$(\,(f_{0,j})_{j\le m}, (z_{0,j})_{j\le m}, \ldots, (f_{m,j})_{j\le m}, (z_{m,j})_{j\le m}\,)$$

such that either the conclusion of Claim 8.5 or the conclusion of Claim 8.6 holds with m in place of L and N. The partial order $\lhd$ of Γ is defined as follows. If ζ, ζ' are nodes of Γ with

$$\begin{array}{lll} \zeta & \text{indexed by} & (\,n(0), k(0), \dots, n(m), k(m)\,), \\ \zeta' & \text{indexed by} & (\,n'(0), k'(0), \dots, n'(m), k'(m)\,), \end{array}$$

then $\zeta \lhd \zeta'$ if and only if $m < m'$ and ζ is an initial segment of ζ'.

By Claims 8.5 and 8.6 (and our assumption that Claim 8.7 is false), Γ has height ω. Hence, by König's Lemma, there exist double sequences

$$(f_{i,j})_{i,j\in\mathbb{N}}, \qquad (z_{i,j})_{i,j\in\mathbb{N}}$$

such that either the conclusion of Claim 8.5 or the conclusion of Claim 8.6 holds for all N and L. Since, by construction, the sequences $(y_{n,k})_k$ and $(z_{i,j})_j$ are sequences of blocks of (x_ν) and approximate respective types in $\operatorname{span}(\tau, *)$, either case contradicts the stability of $\operatorname{span}(\tau, *)$, by Proposition 7.3. This proves Claim 8.7. $\square$

Fix m as in Claim 8.7. Let

$$\{\,c_1, \dots, c_r\,\} = \{\,e_{n,k} + y_{n,k} \mid 0 \le n, k \le (m+1)^2\,\},$$

and let $C(c_1, \dots, c_r; [0, \beta])$ be the following quantifier-free expression:

$$\bigvee_{0\le i\le m} \ \bigwedge_{(m+1)\cdot i\le n<(m+1)\cdot(i+1)} \ \bigvee_{0\le j\le m} \ \bigwedge_{(m+1)\cdot j\le k<(m+1)\cdot(j+1)} \|e_{n,k}+y_{n,k}+x\| \in [0, \beta].$$

Notice that $C(c_1, \dots, c_r; [0, \beta])$ is positive and its parameters are in $E+Y$. To finish the proof, we only have to show that Conditions (I) and (II), stated at the beginning of the proof, hold.

Condition (II) follows from the fact that (‡) of Claim 8.7 does not hold. To prove (I), suppose that $x \in B(M)$ and $x \notin [C(c_1, \dots, c_r; [0, \beta])]$. Then there exist indices $n(0), \dots, n(m)$ and $k(0), \dots, k(m)$ such that

$$\begin{array}{ll} (m+1)\cdot i \le n(i) < m\cdot(i+1) & \text{for } i = 1, \dots, m, \\ (m+1)\cdot j \le k(j) < m\cdot(j+1) & \text{for } j = 1, \dots, m, \end{array}$$

and

$$\bigwedge_{0\le i\le m} \ \bigwedge_{0\le j\le m} \|e_{n(i),k(j)} + y_{n(i),k(j)} + x\| \notin [0, \beta].$$

Since (†) of Claim 8.7 does not hold, we must have $\sigma(x) \notin [0, \alpha]$. $\square$

Proposition 8.8 *Let τ be a type over a separable Banach space X, let $*$ be a convolution on the scalar multiples of τ, and suppose that (x_ν) is an approximating sequence for* $\operatorname{span}(\tau, *)$ *in X. Then the following conditions are equivalent.*

1. $\operatorname{span}(\tau, *)$ *is stable.*
2. *For every finite dimensional subspace E of X, every type over*

$$E + \overline{\operatorname{span}}\{x_\nu \mid \nu \in \mathbb{N}\}$$

   is quantifier-free definable.
3. *For every finite dimensional subspace E of X, every type over*

$$E + \overline{\operatorname{span}}\{x_\nu \mid \nu \in \mathbb{N}\}$$

   is strongly quantifier-free definable.

Proof (1) $\Rightarrow$ (3) is given by Proposition 8.3. We prove (2) $\Rightarrow$ (1). Suppose that $\operatorname{span}(\tau, *)$ is not stable. By Proposition 7.3, there exist double sequences $(y_{m,k})$, $(z_{n,l})$ in $\operatorname{span}\{x_\nu \mid \nu \in \mathbb{N}\}$, an element $x \in X$, and real numbers α, β such that

$$\lim_{m<n<k<l} \|y_{m,k} + z_{n,l} + x\| \leq \alpha < \beta \leq \inf_{n<m<l<k} \|y_{m,k} + z_{n,l} + x\|.$$

Let

$$Y = \overline{\operatorname{span}}\big\{\{x\} \cup \{x_\nu \mid \nu \in \mathbb{N}\}\big\}.$$

After replacing $(y_{m,k})_k$ with a subsequence if necessary, we can assume that there is an element a_m such that

$$\operatorname{tp}(a_m/Y) = \lim_k \operatorname{tp}(y_{m,k}/Y).$$

Then, by $(*)$,

$$(**) \qquad \lim_{m<n<k<l} \|y_{m,k} + z_{n,l} + x\| \leq \alpha < \beta \leq \lim_{n<m<l} \|a_m + z_{n,l} + x\|.$$

Similarly, by refining $(z_{n,l})_l$, we find an element b_n such that

$$\operatorname{tp}\big(b_n/\overline{\operatorname{span}}\big\{\{a_m \mid m \in \mathbb{N}\} \cup Y\big\}\big) = \lim_l \operatorname{tp}\big(z_{n,l}/\overline{\operatorname{span}}\big\{\{a_m \mid m \in \mathbb{N}\} \cup Y\big\}\big).$$

By $(**)$,

$$\lim_{m<n<k} \|y_{m,k} + b_n + x\| \leq \alpha < \beta \leq \lim_{n<m} \|a_m + b_n + x\|.$$

Notice that $\operatorname{tp}\big(a_m/\overline{\operatorname{span}}\big\{\{b_n\} \cup Y\big\}\big)$ is an heir of $\operatorname{tp}(a_m/Y)$, for every n. Hence, if a and b are such that (after adequate refinements)

$$\operatorname{tp}\big(a/\overline{\operatorname{span}}\big\{\{b_n \mid n \in \mathbb{N}\} \cup Y\big\}\big) = \lim_m \operatorname{tp}\big(a_m/\overline{\operatorname{span}}\big\{b_n \mid n \in \mathbb{N}\} \cup Y\big\}\big)$$

$$\operatorname{tp}\big(b/\overline{\operatorname{span}}\big\{\{a\} \cup Y\big\}\big) = \lim_n \operatorname{tp}\big(b_n/\overline{\operatorname{span}}\big\{\{a\} \cup Y\big\}\big),$$

then $\operatorname{tp}\big(a/\overline{\operatorname{span}}\big\{\{b\} \cup Y\big\}\big)$ is an heir of $\operatorname{tp}(a/Y)$.

Now, further refinement of $(y_{m,k})_k$ yields an element a'_m such that

$$\operatorname{tp}\big(a'_m/\overline{\operatorname{span}}\big\{\{b_n \mid n \in \mathbb{N}\} \cup Y\big\}\big) = \lim_k \operatorname{tp}\big(y_{m,k}/\overline{\operatorname{span}}\big\{\{b_n \mid n \in \mathbb{N}\} \cup Y\big\}\big),$$

and

$$\lim_{m<n} \|a'_m + b_n + x\| \leq \alpha < \beta \leq \lim_{n<m} \|a_m + b_n + x\| = \|a + b + x\|.$$

Let a' and b' be such that (again, after taking refinements)

$$\operatorname{tp}\big(b'/\overline{\operatorname{span}}\big\{\{a'_m \mid m \in \mathbb{N}\} \cup Y\big\}\big) = \lim_n \operatorname{tp}\big(b_n/\overline{\operatorname{span}}\big\{\{a'_m \mid m \in \mathbb{N}\} \cup Y\big\}\big)$$

$$\operatorname{tp}\big(a'/\overline{\operatorname{span}}\big\{\{b'\} \cup Y\big\}\big) = \lim_m \operatorname{tp}\big(a'_m/\overline{\operatorname{span}}\big\{\{b'\} \cup Y\big\}\big).$$

Then

(i) $\operatorname{tp}(a'/Y) = \operatorname{tp}(a/Y)$;
(ii) $\operatorname{tp}(b'/Y) = \operatorname{tp}(b/Y)$;
(iii) $\|a' + b' + x\| \leq \alpha$.

Consider the structure

$$\mathbf{Y} = \big(\, Y,\ \operatorname{tp}(a'/Y)\,\big).$$

By Proposition 3.3, if $(\hat{Y}, f)$ is an ultrapower of $\mathbf{Y}$, then f is a type over $\hat{Y}$ and, furthermore, an heir of $tp(a'/Y)$. Thus, using (i)–(iii), we find an ultrapower of $\mathbf{Y}$ and elements a'', b'' such that

1. $\operatorname{tp}(a''/Y) = \operatorname{tp}(a'/Y) = \operatorname{tp}(a/Y)$;
2. $\operatorname{tp}(b''/Y) = \operatorname{tp}(b'/Y) = \operatorname{tp}(b/Y)$;
3. $\|a'' + b'' + x\| \leq \alpha$;
4. $\operatorname{tp}\left(a''/\overline{\operatorname{span}}\{\{b''\} \cup Y\}\right)$ is an heir of $\operatorname{tp}(a''/Y)$.

By taking an isometry that fixes Y pointwise and maps b'' to b (and relabeling a'' as its image under this isometry) we can assume $b'' = b$. Thus,

$$\|a'' + b + x\| \leq \alpha < \|a + b + x\|.$$

This means that the types

$$\operatorname{tp}\left(a/\overline{\operatorname{span}}\{\{b, x\} \cup \{x_\nu \mid \nu \in \mathbb{N}\}\}\right),$$
$$\operatorname{tp}\left(a''/\overline{\operatorname{span}}\{\{b, x\} \cup \{x_\nu \mid \nu \in \mathbb{N}\}\}\right)$$

are two distinct heirs of $\operatorname{tp}\left(a/\overline{\operatorname{span}}\{\{x\} \cup \{x_\nu \mid \nu \in \mathbb{N}\}\}\right)$; hence this type cannot be quantifier-free definable, by Proposition 5.2. □

9 The Main Theorem

Theorem 9.1 *Let (x_n) be a bounded sequence in a separable Banach space X such that no normalized sequence of blocks of (x_n) converges. Then the following conditions are equivalent.*

1. *There exists a sequence (y_n) of blocks of (x_n) such that for every finite dimensional subspace E of X, every type over $E + \overline{\operatorname{span}}\{y_n \mid n \in \mathbb{N}\}$ is quantifier-free definable.*
2. *There exists a sequence (y_n) of blocks of (x_n) such that (y_n) determines a spreading model whose fundamental sequence is isometric over X to the standard unit basis of either ℓ_p $(1 \leq p < \infty)$ or c_0.*

Let us first observe that (2) $\Rightarrow$ (1) is easy. Fix (y_n) as in (2), and suppose that (y_n) approximates $\operatorname{span}(\tau, *)$ (see Definition 7.1). Then by Propositions 2.4 and 2.5, either

$$\lambda\tau * \mu\tau = \left(|\lambda|^p + |\mu|^p\right)^{1/p} \tau$$

or

$$\lambda\tau * \mu\tau = \max(|\lambda|, |\mu|)\tau.$$

In either case, for every $M > 0$ the operation $*$ is commutative and uniformly continuous on the set of elements of $\operatorname{span}(\tau, *)$ of norm at most M. Hence, $\operatorname{span}(\tau, *)$ is stable by Proposition 7.3, and Condition (1) of the theorem follows from Proposition 8.8.

The rest of this section is devoted to the proof of (1) $\Rightarrow$ (2). We first need to introduce some terminology.

Definition 9.2 Let X be a Banach space, let E be a subspace of X, and let ϵ be a positive number. If $1 \leq p < \infty$, a sequence (x_n) is said to be $(1+\epsilon)$*-equivalent over E to the standard unit basis of ℓ_p* if whenever $x \in E$ and $\lambda_0, \ldots, \lambda_n$ are scalars,

$$(1+\epsilon)^{-1}\left\|x + \sum_{i=0}^{n} \lambda_i x_i\right\| \leq \left\|x + \Big(\sum_{i=0}^{n} |\lambda_i|^p\Big)^{1/p} x_0\right\| \leq (1+\epsilon)\left\|x + \sum_{i=0}^{n} \lambda_i x_i\right\|.$$

The sequence (x_n) is $(1+\epsilon)$*-equivalent over E to the standard unit basis of c_0* if whenever $x \in E$ and $\lambda_0, \dots, \lambda_n$ are scalars,

$$(1+\epsilon)^{-1}\left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\| \leq \left\| x + \big(\max_{0\leq i\leq n} |\lambda_i|\big) x_0 \right\| \leq (1+\epsilon)\left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\|.$$

We will show that (1) of Theorem 9.1 implies one of the following conditions:

(a) there exists $1 \leq p < \infty$ such that for every $\epsilon > 0$ and every finite dimensional subspace E of X there exists a sequence of blocks of (x_n) which is $(1+\epsilon)$-equivalent over E to the standard unit basis of ℓ_p;

(b) for every $\epsilon > 0$ and every finite dimensional subspace E of X there exists a sequence of blocks of (x_n) which is $(1+\epsilon)$-equivalent over E to the standard unit basis of c_0.

Then (2) of the theorem follows by simple diagonalization.

The proof is based on an argument of Bu [4]. However, our argument is completely elementary. (In [4], Bu reproves the main theorem of [18] by invoking a principle from descriptive set theory that Dellacherie in [6] labeled the Kunen-Martin Theorem.)

Let $(\Sigma, \leq)$ be a partially ordered set. For an ordinal α we define a set Σ^α as follows.

1. $\Sigma^0 = \Sigma$.
2. If $\alpha = \beta + 1$,

$$\Sigma^{\alpha+1} = \{\, \xi \in \Sigma^\alpha \ \mid \text{There exists } \eta \in \Sigma^\alpha \text{ with } \eta > \xi \,\}.$$

3. If α is a limit ordinal,

$$\Sigma^\alpha = \bigcap_{\beta<\alpha} \Sigma^\beta.$$

The *rank* of Σ, denoted $\operatorname{rank}(\Sigma)$, is the smallest ordinal α such that $\Sigma^{\alpha+1} = \varnothing$. If such an ordinal does not exist, we will say that Σ has *unbounded rank* and write $\operatorname{rank}(\Sigma) = \infty$.

Proposition 9.3 *Suppose that* $\operatorname{rank}(\Sigma) = \infty$. *Then there exists a sequence (ξ_n) in Σ such that $\xi_0 < \xi_1 < \cdots$.*

Proof Fix an ordinal α such that $\Sigma^\alpha = \Sigma^\beta$ for every $\beta > \alpha$. Take $\xi_0 \in \Sigma^\alpha$. Then $\xi_0 \in \Sigma^{\alpha+1}$, so there exists $\xi_1 \in \Sigma^\alpha$ with $\xi_1 > \xi_0$. Now, $\xi_1 \in \Sigma^{\alpha+1}$, so there exists $\xi_2 \in \Sigma^\alpha$ with $\xi_2 > \xi_1$. Continuing in this fashion, we find (ξ_n) as desired. □

Proof of (1) ⇒ (2) of Theorem 9.1 Without loss of generality, we can assume that (x_n) approximates $\operatorname{span}(\tau, *)$, for some type τ over X (see Definition 7.1). Furthermore, by replacing (x_n) with the sequence (y_n) given by (1) and invoking Proposition 8.8, we can assume that $(\tau, *)$ is stable. Proposition 7.11 now allows us to fix a type $\tau_0 \in \overline{\operatorname{span}}(\sigma, *)$ such that $\mathfrak{p}[\operatorname{span}(\rho, *)] = \mathfrak{p}[\operatorname{span}(\tau_0, *)]$ for every type $\rho \in \overline{\operatorname{span}}(\tau_0, *)$ of norm 1. By replacing τ_0 with $\tau_0 * (-\tau_0)$ if necessary, we may assume that τ_0 is symmetric. Fix $p \in \mathfrak{p}[\operatorname{span}(\tau_0, *)]$.

Let Σ be the set of finite sequences of blocks of (x_n) partially ordered by extension, that is, if $(y_0, \dots, y_k)$ and $(z_0, \dots, z_l)$ are sequences of blocks of (x_n), we have $(y_0, \dots, y_k) < (z_0, \dots, z_l)$ if and only if $k < l$ and $y_i = z_i$ for $i = 0, \dots, k$.

By Proposition 9.3, it suffices to prove that for every $\epsilon > 0$ and every finite dimensional subspace E of X, the set

$$\Sigma[\epsilon, E] = \left\{ (y_0, \dots, y_n) \in \Sigma \;\middle|\; (y_0, \dots, y_n) \text{ is } (1+\epsilon)\text{-equivalent over } E \text{ to the standard unit basis of } \ell_p(n+1) \right\}$$

has unbounded rank.

We construct for every ordinal α a type τ_α over X such that

1. $\|\tau_\alpha\| = 1$;
2. τ_α is symmetric;
3. $\tau_\alpha \in \overline{\text{span}}(\tau_\beta, *)$ for every $\beta < \alpha$;
4. for every $\epsilon > 0$, every finite dimensional subspace E of X, and every $\rho \in (\tau_\alpha, *)$, the set

$$\Sigma[\epsilon, E, \rho] = \left\{ (y_0, \dots, y_n) \in \Sigma \;\middle|\; \text{tp}\Big(\sum_{i=0}^{n} \mu_i y_i / E\Big) \overset{1+\epsilon}{\sim} \Big(\sum_{i=0}^{n} |\mu_i|^p\Big)^{1/p} \rho \restriction E \text{ for all scalars } \mu_0, \dots, \mu_n \right\}$$

   has rank $\geq \alpha$.

Note that (4) implies that $p \in \mathfrak{p}[\text{span}(\tau_\alpha, *)]$ for every ordinal α.

The type τ_0 defined above satisfies (1)–(3). Condition (4) is immediate from the symmetry of τ_0 and the fact every type over a finite dimensional space E can be approximated by types realized in E.

Suppose that τ_α has been defined, let (σ_l) be a sequence of types of norm 1 in $\text{span}(\tau_\alpha, *)$ which witnesses the fact that $p \in \mathfrak{p}[\text{span}(\tau_\alpha, *)]$, and define $\tau_{\alpha+1} = \lim \sigma_l$. Conditions (1)–(3) are clearly satisfied. We prove (4).

Fix $\epsilon > 0$, a type $\rho \in \text{span}(\tau_{\alpha+1}, *)$, a finite dimensional subspace E of X, and a finite sequence $(y_0, \dots, y_n)$ of blocks of (x_n). Take real numbers δ_1, δ_2 such that $0 < \delta_1 < \delta_2 < \epsilon$ and $(1+\delta_2)^2 < 1+\epsilon$.

For scalars $\lambda_0, \dots, \lambda_n$, we have

$$(\dagger) \qquad \Big(\sum_{i=0}^{n} |\lambda_i|^p\Big)^{1/p} \tau_\alpha = \lim_l \, (\lambda_0 \sigma_l * \cdots * \lambda_n \sigma_l).$$

Each σ_l is in $\text{span}(\tau_\alpha, *)$, so using (†) and the fact that the Krivine-Maurey convolution is commutative and separately continuous on $\text{span}(\tau_\alpha, *)$, for every $x \in X$ we find types $\upsilon_0, \dots, \upsilon_n \in \text{span}(\tau_\alpha, *)$ such that

$$\Big(\sum_{i=0}^{n} |\lambda_i|^p\Big)^{1/p} \rho(x) \overset{1+\delta_1}{\sim} \lambda_0 \upsilon_0 * \cdots * \lambda_n \upsilon_n(x)$$

for all scalars $\lambda_0, \dots, \lambda_n$. Let b be a realization of υ_0. Since E is finite dimensional, there exists a sequence $(z_1, \dots, z_n)$ of blocks of (x_n) such that

$$(\ddagger) \qquad \Big(\sum_{i=0}^{n} |\lambda_i|^p\Big)^{1/p} \rho \restriction E \overset{1+\delta_2}{\sim} \text{tp}\Big(\lambda_0 b + \sum_{i=1}^{n} \lambda_i z_i \,/\, E\Big)$$

for all scalars $\lambda_0, \dots, \lambda_n$. Furthermore, $(z_1, \dots, z_n)$ can be taken from any tail of (x_n), and we take it from one whose support is disjoint from the support of

$(y_0, \dots, y_n)$. Let
$$F = \overline{\text{span}}\big\{\, E \cup \{z_1, \dots, z_n, \}\,\big\}.$$
We now prove that
$$(y_0, \dots, y_n) \in \Sigma[\delta_2, F, \upsilon_0] \quad \text{implies} \quad (y_0, \dots, y_n, z_1, \dots, z_n) \in \Sigma[\epsilon, E, \rho].$$
Since $\upsilon_0 \in \text{span}(t_\alpha, *)$, this will conclude the proof of (4).

Fix scalars $\mu_0, \dots, \mu_n, \lambda_1, \dots, \lambda_n$, and suppose $(y_0, \dots, y_n) \in \Sigma[\delta_2, F, \upsilon_0]$. Since $\text{tp}(b/X) = \upsilon_0$, we have
$$\text{tp}\,\Big(\sum_{i=0}^{n} \mu_i y_i + \sum_{i=1}^{n} \lambda_i z_i \;/\; E\Big) \overset{1+\delta_2}{\sim} \text{tp}\,\Big(\Big(\sum_{i=0}^{n} |\mu_i|^p\Big)^{1/p} b + \sum_{i=1}^{n} \lambda_i z_i \;/\; E\Big).$$
Hence, by (‡),
$$\text{tp}\,\Big(\sum_{i=0}^{n} \mu_i y_i + \sum_{i=1}^{n} \lambda_i z_i \;/\; E\Big) \overset{(1+\delta_2)^2}{\sim} \Big(\sum_{i=0}^{n} |\mu_i|^p + \sum_{i=1}^{n} |\lambda_i|^p\Big)^{1/p} \rho \restriction E.$$
Since $(1+\delta_2)^2 < 1+\epsilon$, it follows that $(y_0, \dots, y_n, z_1, \dots, z_n) \in \Sigma[\epsilon, E, \rho]$. If α is a limit ordinal, we take an ultrafilter $\mathcal{U}$ on α and define $t_\alpha = \lim_{\beta<\alpha, \mathcal{U}} t_\beta$. ☐

Note

1. The term *fundamental sequence* is extracted from [1].

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