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Book Review

Abstract J. P. Mayberry. *Foundations of Mathematics in the Theory of Sets*. Cambridge University Press, Cambridge, 2000. xx+424 pages.

J. P. Mayberry's *The Foundations of Mathematics in the Theory of Sets* is an invigorating call to foundational arms, which should strike readers as either revolutionary, reactionary, or both. I intend the above claim normatively: if it elicits either or both of these opinions from the reader the book will be a success, and it is only if the book is counted as a localized "area of mathematical research" that Mayberry's project will have failed. However strenuous my disagreement in certain regards with Mayberry's more specific philosophical position, in this essay-review I will make the strongest possible case for the goad which Mayberry supplies the mathematical and philosophical communities in this regard. Indeed, even my serious disagreements with Mayberry affirm the significance of the foundations of mathematics which he claims.

1 Introduction

Mayberry's general viewpoint, as presented in the preface to the book, is both sweeping and, in certain regards, quite damning of mathematicians' and formal logicians' neglect of the foundations of mathematics. According to Mayberry there is "widespread misunderstanding among mathematicians concerning the underlying logic of the axiomatic method," and "[i]ndeed, mathematical logicians are as prone to confusion over the foundations of the axiomatic method as their colleagues" (p. xi). This confusion lies, in particular, in the tendency of a large part of the mathematical/logical community to view "foundations of mathematics as a branch of logic" (p. xi), whereas for Mayberry it is a fundamental principle that the axiomatic method cannot itself be justified in a (formally) logical manner, but rather relies on set-theoretic foundations. It is Mayberry's aim to present and analyze these set-theoretic foundations in this book.

As Mayberry puts it, his "approach to set theory rests on one central idea, namely, that the modern notion of *set* is a refined and generalised version of the classical

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Greek notion of *number (arithmos)*, the notion of number found in Aristotle and expounded in Book VII of Euclid's *Elements*" (p. xiii). As quickly becomes clear, Mayberry understands this conception of number as an exclusively cardinal conception, and much of his book is preoccupied, in a variety of ways, with analysing what sense we can make of the notion of ordinal on the basis of a foundational conception of sets as collections determined by their membership and cardinality, that is, independently of ordering. In presenting his conception of cardinal number, Mayberry appeals to Jakob Klein's discussion of the Greek conception of *arithmos* as a finite totality composed of units, with the concomitant commitment to the finiteness of quantity being understood in terms of Euclid's axiom that the whole is greater than the part (Klein [5], pp. 46–60). Given that Mayberry views the conception of set in such close connection with the ancient conception of number, it might not seem entirely out of place to suggest that what Mayberry is providing would more adequately be described as "The Foundations of Mathematics in the Theory of Arithmetic," and indeed Mayberry himself states that "the point of view embodied in this book [is that] all of mathematics is rooted in *arithmetic*, for the central concept in mathematics is the concept of a plurality limited, or bounded, or determinate, or definite—in short, *finite*—in size, the ancient concept of *number* (*arithmos*)" (p. xix).

Since Euclid's axiom is clearly violated by Cantor's transfinite quantities, Mayberry goes on to characterize Cantor's set-theory as *non-Euclidean* (p. xiv). As such, Mayberry sees Cantorian set theory as fundamentally committed to a conception of (unordered) totalities composed of units, but adopting a different, non-Euclidean conception of the "finite." Consequently, what Cantor refers to as the "transfinite" Mayberry will refer to as the "Cantorian finite," and this puts Mayberry in a position to see the justification of set-theoretic foundations in terms of finiteness principles. What will distinguish Cantorian finitism from Euclidean finitism will be precisely which principles of finiteness each accepts.

Mayberry clearly, if tacitly, acknowledges that his construal of set-theory in terms of the provision of finiteness principles is only one possible way of promoting set-theoretic foundations. What are its advantages? For "orthodox" foundations (these will include Euclidean and Cantorian variants of set-theoretic foundations) the primary advantage is that "it enables us to see that the central principles—axioms—of set theory are really *finiteness principles* which, in effect, assert that certain multitudes (pluralities, classes, species) are finite in extent and *for that reason* form sets" (p. xv). That is, the construal of set theory in Mayberry's way allows us to generate a systematic program for the *justification* of set-construction principles. However, understanding the sense in which Mayberry intends to supply a justification will require investigating arguments Mayberry gives later in the book.

A related motivation for Mayberry's construal of set-theory in terms of finiteness principles is that it gives him a vantage point from which to criticize various forms of operationalism which he sees infecting not only competing, non-set-theoretic foundational programs, but also the standard presentation of the Cantorian position as well. All of these positions are criticized, in particular, in terms of the operationalist conception of the natural numbers which they adopt:

> I am convinced that this operationalist conception of natural number is the central fallacy that underlies all of our thinking about the foundations of mathematics. It is not confined to heretics, but is shared by the orthodox Cantorian majority. This *operationalist fallacy* consists in the assumption that the mere

> *description* of the natural number system as "what we obtain from zero by successive additions of one" suffices *on its own* to define the natural number system as a unique mathematical structure—the assumption that the operationalist description of the natural numbers is itself what provides us with a *guarantee* that the system of natural numbers has a unique, fixed structure. (p. xvii)

Although Mayberry will show that the orthodox Cantorian position may be "shored up" by demonstrating the uniqueness of the natural numbers, if, on the other hand, "we abandon Cantorian orthodoxy we thereby abandon the means with which to prove these things" (p. xvii). This is not the only casualty of such a turn: the adoption of Mayberry's strict perspective (whether pursued along Euclidean or Cantorian lines) will lead us to modify our straightforward "orthodox" commitment to the availability of proofs by induction and definition by recursion. These two latter commitments rely on operations which "too must be analysed in terms of more fundamental notions" (p. xvii).

Before proceeding to issues which will allow me to illuminate the difference between Cantorian and Euclidean set-theoretic perspectives, it is worth pointing out that although Mayberry's cardinal-based attitude is consonant with modern expositions of Cantorian set theory such as ZF(C) or NBG(C), historically, there is in fact an evolution of Cantor's thinking in the direction of making the cardinal conception of set primary, but Cantor's early work in set theory was clearly dominated by an ordinal-driven conception of sets (see, for example, Hallett [3], pp. 120–64). Furthermore, when Cantor moves in the direction of a foundationally cardinal conception he is left with dilemmas about the differentiation of fundamental "units" which he was unable to address adequately. It is, then, already a nontrivial orientating decision on Mayberry's part to align himself with a fundamentally nonordered conception of collectivity. This decision is an integral (and consistent) part of Mayberry's own anti-operationalist stance, but as I will suggest below, is at a deeper level a function of even more fundamental commitments in Mayberry's approach to foundations of mathematics.

A second remark about Mayberry's overall attitude toward set-theoretic foundations is that the program of justifying set-theoretic foundations in terms of finiteness principles carries with it specific ambitions that many other advocates of set-theory as a foundation for mathematics either fail to supply or, indeed, have no desire to supply. In the assessment of Mayberry's position, then, it is important to be as clear as possible about the strength of the claims he is making in defense of set-theoretic foundations and what is necessary in order for these claims to be established. In a section of the first chapter of his book entitled "What the foundations of mathematics consists in," Mayberry draws a distinction between the *exposition* and the *justification* of the foundations of mathematics which is crucial for his understanding of these issues, remarking that "the task of *expounding* the foundations of mathematics must be kept separate from the task of *justifying* them: this is required by the logical role that those foundations are called upon to play" (p. 10). Indeed, Mayberry goes on to point out that there can be "no question of a *rigorous* justification of proposed foundations" (ibid.) on pain of a regress. Justification must proceed, then, "by persuasion rather than by demonstration: it must be dialectical rather than apodeictic" (ibid.).

Given that the concepts to which we appeal in *expounding* a foundation of mathematics are taken to be *basic*—for otherwise they would not be foundational—they must then be justified in the absence of the canonical form of mathematical justification, that is, the provision of proofs themselves based on (justified) definitions. But this means that their justificatory status must in some sense be self-evident.[1] It is imperative, then, to be as clear as possible about what sort of support Mayberry is claiming to provide in terms of appeals to self-evidence. In correspondence with the author of this review, Mayberry has clarified his position in the following terms:

> even though we cannot carry out a proper *process of justification* in these cases—that would be a mathematical definition or a mathematical proof—at least we can clear away misconceptions and prejudices that cloud the judgement and prevent our seeing the basic concepts and principles *as* self-evidently justified. Such misconceptions and prejudices can be *historical* (as when our notion of natural number is conceived to be the original notion of number), *technical* (as when formal mathematical logic is deemed to be an autonomous discipline, independent of, and antecedent to, mathematics), *epistemological, ontological, sociological*, etc. (personal correspondence, December 28, 2002)

In the context of his book, it seems that Mayberry's "justificatory program," in this passive sense of clearing away misconceptions, is primarily organized in terms of the power which accrues from taking the concept of finiteness, and consequently, the distinction between the finite and the infinite, as our guide for organizing our vision of the foundations of mathematics in a coherent way.

Here, however, two points must be made. First, insofar as the concept of the finite serves this organizing function, Mayberry's agnosticism regarding Euclidean versus Cantorian foundations registers a sense in which his own foundational program remains incomplete: "My own view is that *we do not yet know enough about how mathematics can be developed in Euclidean set theory to make an informed choice between that theory and Cantorian orthodoxy*" (p. 387). The way Mayberry puts this point at least makes it sound like pragmatic evidence regarding the success or failure in developing these theories could serve as a guide for resolving our dilemma. But at best it remains unclear how to reconcile this sort of appeal with Mayberry's insistence that set-theoretic foundations be *justified* in terms of the *proper* conception of the distinction between the finite and the infinite: we have, in particular, no reason on Mayberry's account to think that what *works* is what is *proper*, and this seems like a particular deficit in the case of someone arguing that foundations be supplied in the way Mayberry requires.

The second point to introduce here is that although Mayberry does supply persuasive reasons for thinking that the sort of set-theoretical foundations he promotes will provide a philosophical context for resolving problems associated with the operationalist fallacy he criticizes, he is unable, in this reviewer's opinion, to provide equally persuasive reasons for viewing set-theoretic foundations as the exclusive or even the best vantage point from which to understand the distinction between the finite and the infinite which drives his specifically justificatory program. I will discuss this second point and further general criticisms of Mayberry's program below, but this further discussion will involve a greater sense of his position than I have been able to develop at this point in the essay. To this end, I turn now to the issue which is perhaps most illuminating in understanding the consequences of adopting Cantorian versus Euclidean foundations: the status of the natural numbers and the way this is reflected, in particular, in the status of the exponential operation.

2 Natural Numbers and Exponentiation

Because the status of the exponential operation throws into relief both the most significant discrepancies between the Cantorian and Euclidean set-theoretic approaches, and because, as I will argue below, the status of exponentiation is a fundamental issue for foundations for which Mayberry's work constitutes a major contribution, I will focus in this section on the treatment of exponentiation in the two approaches Mayberry considers. I begin with some motivating examples which will illustrate points key for the development of Mayberry's position. Consider the issue of string growth in decimal representation: let's begin by looking at some simple examples. In base 10 decimal representation,

$$\begin{aligned}
2+4 &= 6,\\
8+9 &= 17,\\
103+104 &= 207,\\
846+372 &= 1218,\\
2\times 4 &= 8,\\
8\times 9 &= 72,\\
103\times 104 &= 10712,\\
846\times 372 &= 314712.
\end{aligned}$$

These examples suggest that in the case of addition we may bound the length of the product decimal representation by adding 1 to the longer decimal representation length, and in the case of multiplication we may bound the length of the product decimal representation by the sum of the decimal representation lengths. This can easily be seen sufficient by representing the decimal expansions algebraically. If we have $x = a_0 + a_1 10^1 + a_2 10^2 + \cdots + a_m 10^m$ and $y = b_0 + b_1 10^1 + \cdots + b_n 10^n$, with a_m and b_n nonzero, then $x + y$ is bounded by $2 \max\{a_m, b_n\} \max\{10^m, 10^n\}$, and this is bounded by $2 \times 10^{\max\{m,n\}+1}$, which has $\max\{m, n\} + 1$ places. In the case of multiplication, $x \times y < (a_m + 1)10^m (b_n + 1)10^n \leq 10^{m+n+2}$, and since x has length $m + 1$ and y length $n + 1$, we are done (since 10^{m+n+2} is the *smallest* number having $m + n + 3$ places in its decimal representation and the inequality is strict).

However, when we consider exponentiation, the situation is suddenly different. Indeed, $x^y = (a_0 + a_1 10^1 + \cdots + a_m 10^m)^{(b_0+\cdots+b_n 10^n)} \geq ((a_m)10^m)^{(b_0+\cdots+b_n 10^n)} \geq (10^m)^{(b_0+\cdots+b_n 10^n)} = 10^{(m)(b_0+\cdots+b_n 10^n)} \geq 10^{(m)(b_n)(10^n)} \geq 10^{10^n}$, and so the growth rate is bounded below by 10^n digits in this way. Here the point is simply that as we exponentiate the number of digits is growing in an exponential fashion. (This should not be particularly surprising, since the decimal system itself is based on exponential representation; however, I will say more about the significance of such representation systems below.) It is this basic sort of behavior which distinguishes exponentiation, and this behavior witnesses something which bears on the "definability" of exponentiation, as I will proceed to explain. I will begin this process by first presenting Mayberry's picture of Cantorian finitism, in which the definability of exponentiation causes no problems. This presentation will then allow me to highlight exactly what changes when I turn to Mayberry's exposition of Euclidean finitism, that is, Euclidean set theory (EST).

In the following definitions a local relation is a set of ordered pairs and a local function is a local relation with the additional property that for each element of the domain there is exactly one element in the range to which the first element stands in relation. Mayberry's presentation of Cantorian finitism is then grounded in the notions of Dedekind structures and morphisms between such structures, which Mayberry defines as follows:

> (i) A *Dedekind structure* is an ordered triple (N, s, a), where N is a non-empty set, $s : N \to N$ is a function from N to N, and $a \in N$ is a distinguished element of N.
>
> (ii) Let (N, s, a) and(M, t, b) be Dedekind structures, and let $m : N \to M$ be a local function. Then, by definition, m is a morphism from (N, s, a) to (M, t, b) : iff : $m`a = b$ and $(\forall x \in N)[m`s`x = t`m`x]$; a bijective morphism is called an isomorphism. (pp. 153–54)

A Dedekind structure (*N,s,a*) is then a *simply infinite system* if it satisfies the following three axioms:

I. $(\forall x \in N)[s`x \neq a]$
II. $(\forall xy \in N)[s`x = s`y \text{ implies } x = y]$
III. $(\forall S \subseteq N)[a \in S \text{ and } (\forall x \in S)[s`x \in S] : \text{implies} : S = N]$. (p. 155)

In terms of these definitions Mayberry proves the following theorem, which guarantees what Mayberry calls "definition by recursion along a simply infinite system":

> Let (N, s, a) be a simply infinite system and (M, t, b) an arbitrary Dedekind system. Then there is exactly one function $f : N \to M$ which is a morphism of Dedekind structures. (pp. 156–57)

In terms of this result we are able to guarantee that for a simply infinite system there are unique binary functions satisfying the usual definitions of addition, multiplication and exponentiation, and the existence of these functions in turn makes it possible to define a Peano system as an ordered quintuple $(N, s, a, +, \cdot)$ which satisfies the axioms for a simply infinite system and axioms which characterize addition and multiplication in the usual way. The result establishing definition by primitive recursion along a simply infinite system then guarantees that any two simply infinite systems are isomorphic as Dedekind structures, and that any two Peano systems are isomorphic as structures of their type. Mayberry is thus able to establish categoricity results for both simply infinite systems and Peano systems. This result does not guarantee the existence of Peano systems, because it does not *require* Cantor's axiom of infinity. But in the presence of the axiom of infinity, the existence of Peano systems is assured (pp. 159 ff.).

What about exponentiation? We are essentially asking a question about the representability of exponentiation in the formal system Q which is a fragment of Peano Arithmetic (PA) (Epstein and Carnielli [2], pp. 180–81). In PA we take the functions $s, +$, and $\times$ as primitive, but not exponentiation, and therefore showing the representability of the recursive functions in PA requires using a coding which does not rely on the exponential function. Given that coding in the presence of the exponential function is quite straightforward, the chief technical difficulty in this program is to find a way to code functions without appeal to the exponential function (supplied by Gödel's β-function, see [2], pp. 190–91), and this perhaps suggests that there is something "difficult" about the exponential function. But whether this is the case or not, it does not indicate in what regard the introduction of exponentiation depends on the existence or categoricity of Peano systems. Mayberry's work on Euclidean set theory provides a context in terms of which these issues may be discussed.

The Euclidean perspective is most easily approached by comparing the Cantorian and Euclidean perspectives. In particular, the following problem poses itself: in the Cantorian case we have a successor function which is defined on a simply infinite system (N, s, a); in the Euclidean case, without the Cantorian axiom of infinity, we do not have such a collection N with the status of a set. What are we to do? Mayberry handles this by introducing a distinction between local functions and global functions. As Mayberry says,

> By a *local function* I mean, as a first approximation, a function whose domain of definition is a set (and whose range is therefore, by Replacement, also a set). It is thus to be contrasted with the Fregean conception of *global* function. This notion of local function plays so central a role in modern mathematics that it is scarcely possible to imagine what the subject might be without it. (p. 130)

More exactly, Mayberry defines a (local) relation as a set of ordered pairs, and then defines a local function as a local relation that satisfies the expected functionality criterion. Although this all seems standard enough, the distinction between local and global function is critically related for Mayberry to his espousal of what he calls "Brouwer's principle":

> **Brouwer's Principle** (i) Conventional (i.e. what Brouwer calls "classical") logic is the logic of finite domains. In particular the conventional laws of quantification apply only when the domains of quantification are finite.
>
> (ii) Propositions that require global quantification for their expressions cannot be assigned conventional truth values, true or false. They can only be classified as justified or unjustified. (p. 89)

Brouwer's principle indicates the manner in which Mayberry understands the relation between logical and set-theoretic concerns. In particular, the notion of local function is linked via Brouwer's principle to the commitment to a particular brand of finiteness, either Cantorian or Euclidean in the cases Mayberry considers. Furthermore, by Brouwer's principle, the notion of global function, whatever that turns out to be on any particular view, effectively requires a treatment in terms of intuitionistic logic, although this treatment will be carried out at the foundational rather than at the axiomatic level: it is this which requires us to distinguish between justified and unjustified propositions, on the one hand, and true and false ones, on the other (here see also Mayberry [7] and [8]).

So far as EST is concerned, the importance of this distinction lies chiefly in the fact that simply infinite systems are not sets (so that, in particular, the above-stated categoricity result does not apply) and functions defined on simply infinite systems are global, not local, functions. In this setting, we work from the fact that we have a theory of (Euclidean finite) linear ordering, and in terms of this we may define, for any object a and σ any global first level function and for any linear ordering r, "σ generates r from a" (in the obvious way, resting on the ordering r). Then we say that σ generates a simply infinite system from a if for all linear orderings x, x is generated from a by σ and x is nonempty implies that $\sigma(\mathrm{Last}(x)) \notin \mathrm{Field}(x)$ (p. 326). The *elements* of a SIS will be the linear orders lying in it and the corresponding *terms* will be the terms of the constituent linear orderings. Heuristically, we may write this as

$$0 = [], 1 = [\vec{0}], 2 = [\vec{0}, \vec{1}], 3 = [\vec{0}, \vec{1}, \vec{2}],$$

and in general if $n \neq 0$, $n = [\vec{0}, \vec{1}, \ldots, \overrightarrow{n-1}]$ but we must note that because of Mayberry's commitment to anti-operationalism we must not take the dots used in the above notation to mean any more than that n is a linear order generated from 0 by σ; in particular, we should not view this as a shorthand notation meant to represent an explicit, that is, concrete, list of equations of the form $\sigma(0) = 1, \sigma(1) = 2, \ldots, \sigma(n-1) = n$ (see a related discussion at p. 158). Even more importantly for Mayberry's purposes, we must not assume that all finite linear orderings are elements of one canonical ongoing collection of finite linear orderings.

In the context of EST Mayberry is able to prove a "principle of mathematical induction" for SIS's and show that a necessary and sufficient condition for addition, multiplication, or exponentiation to be defined on a SIS N is that "the recursion equations succeed in defining a function on N" (p. 335). This parallels the requirement that a theorem be proved in Cantorian set theory establishing definition by recursion along a SIS, but here the analogous result is much more difficult to obtain since we now are not able to work with SIS's as sets. Consequently we need to include along with the defining functions of the recursion a *bounding* function which guarantees that the recursion functions not grow so fast that their joint growth outstrips any function. This is because, in particular, we do not have any *set* upon which to project, in advance, the values of the recursion, and so must guarantee that we can generate a sufficiently large range to "capture the recursion at each stage."

We may illustrate the distinction between local and global quantification and the concomitant application of Brouwer's principle in the context of EST by considering the statement that "(all) simply infinite systems are not sets." In EST we may demonstrate something which corresponds to this statement, in a sense which I will specify below, by showing that given any simply infinite system and any set we may exhibit an element of the simply infinite system that is not in the set. But the force of this demonstration is, first, logically schematic: what it shows is that *any* infinite system is not *any* set. Second, it is formal: the term 'simply infinite system' is not to be understood to refer to an object per se, but rather to the function which "generates" the simply infinite system, and this function is not itself to be understood as an object, but only in terms of the objects which are its course of values. In this way we are able to understand the result in a way which avoids global quantification over all simply infinite systems, and as Mayberry has ackowledged in correspondence, any attempt to make a point about the *collection* of simply infinite systems would run seriously afoul of his program, since he does "not think we can legitimately speak of *species* of higher level 'entities' " (December 20, 2002).

As mentioned above, even speaking of a simply infinite system as an object is something that can only be understood as shorthand, and accordingly, the informal statement I, "(all) simply infinite systems are not sets" may be more accurately expressed in terms of the following schematic proposition II: The hypotheses (i) and (ii) entail the conclusion (iii):

(i) σ generates a simply infinite system;
(ii) S is a set all of whose members are elements of the simply infinite system generated by σ;
(iii) the linear ordering $\mathrm{L} * \sigma(\mathrm{T})$, where L is the longest linear ordering in S and T is its last term, is an element of the simply infinite system generated by σ but does not belong to S. (December 20, 2002)

Even here, Mayberry concedes,

> [t]he wording of (i), (ii), and (iii) in II is perhaps misleading in that, on the face of it, they contain reference to "things" that are really not "things" at all. But I do not intend for this wording to be taken literally as referential; it is intended merely to help the reader to grasp my *intention* in laying down these definitions. In fact, II can be translated into a proposition in the conceptual notation of the form A, B $\Rightarrow$ C, where A is a Π_1-formula expressing (I), B is a local formula (i.e., containing no global quantifiers) expressing (ii), and C is a local formula expressing (iii). (ibid., modified at Mayberry's request)

The above example illustrates what might be called the *ontological* character of Mayberry's approach to EST. The need for such care in the formulation of this aspect of his presentation of EST is a function of the fact that the appeals to simply infinite systems which support the result Mayberry establishes concerning mathematical induction cannot themselves be treated as appeals to objects within the theory itself. This leads to, but must be conceptually distinguished from, the second major "wrinkle" introduced by EST: because of the unavailability of simply infinite systems within the theory there is no way to establish the categoricity results for simply infinite systems and Peano systems which are available within the Cantorian theory. In the absence of these results we cannot guarantee the existence of a universal numeration system, that is, a scale against which quantity may be "measured" in general, and this leads to the necessity of examining carefully the nature of numeration systems. It is this investigation which demonstrates that in the context of EST, the operationalist assumption of the uniqueness (up to isomorphism) of "the" natural numbers is not simply lacking demonstration, as it typically has been in the context of presentations of Cantorian set theory, but is indeed ill-founded.

Since the trajectory of Mayberry's treatment of this issue becomes quite technical, it is helpful at this point to think back to our original examples. Suppose that I need to define multiplication on decimal strings. We start with a simply infinite system N, which we may think of as a stroke system: |, ||, |||, ||||, ... Next we select a numeration base, ||||||||||, and form decimal strings. This gives us a *new* simply infinite system, which we may call $N[10]$. Now if addition is defined on $N[10]$ we may define multiplication on N by

$$a \times_{10} b = i\{x \in \mathrm{NUM}(k+1) : [\cdots x) \cong [\cdots a) \times_c [\cdots b)\},$$

where $k = \mathrm{length}(a) +_N \mathrm{length}(b)$, and $\mathrm{NUM}(k+1)$ is the collection of all decimal notations of length at most $k+1$ digits. Then $\times_{10}$ satisfies the recursion equations,

$$a \times_{10} \langle 0 \rangle_{10} = \langle 0_{10} \rangle$$

and

$$a \times (b +_{10} \langle 1_{10} \rangle) = (a \times_{10} b) +_{10} a.$$

But, as we have seen, a similar maneuver will not go through for exponentiation, and in fact we obtain 'exp is definable in $N[10]$ if and only if it is definable in N' (p. 343).

Mayberry's Euclidean approach requires us to take into consideration *extensions* of simply infinite systems by what he calls "the method of S-ary expansion," of which the extension of the stroke notation system to decimal notation base 10 given above would be a specific example (see p. 338ff.). This is because, in general, we do not know whether a given function can be represented in a given SIS. There is thus some analogy with the problem of definability in Q, Robinson's fragment of Peano

Arithmetic, and in both cases it is essentially inclusion of definition by recursion that causes problems.

3 Foundations and Mayberry's Anti-Operationalism

As has already been mentioned above, Mayberry views a certain "operationalist conception of natural number" as "the central fallacy that underlies *all* of our thinking about the foundations of mathematics" (p. xvii). It will be my thesis in this section that although the core (at least as I will understand it) of Mayberry's critical point is sound, the implications of this point for foundations of mathematics are other than those which Mayberry himself draws. This will have significant implications for Mayberry's own foundational project, since on my construal the implictions of Mayberry's critical point bear equally on set-theoretic and non-set-theoretic foundations, and so cannot be used to argue in favor of set-theoretic foundations, as Mayberry himself wishes to do.

In order to prosecute this agenda we must first gain a more thorough sense of Mayberry's anti-operationalism. As a primary framework, I begin by considering three fundamental theses Mayberry states in the preface to his work, the third of which comprises the passage referred to above.

Thesis 1 "If . . . we see the notion of natural number as a secondary growth on the more fundamental notion of *arithmos* . . . then the principles of proof by induction and definition by recursion are no longer just 'given' as part of the raw data, so to speak, but must be established from more fundamental, set-theoretical principles." (pp. xvi–xvii)

Thesis 2 "Nor are the operations of counting out or calculating to be taken as primary data: they too must be analysed in terms of more fundamental notions." (p. xvii)

Thesis 3 "[The] operationalist conception of natural number is the central fallacy that underlies all our thinking in the foundations of mathematics." (p. xvii)

Thesis 1 affirms Mayberry's commitment to the notion of *arithmos*, "finite plurality in the original Greek sense of 'finite' " (p. xvii), as basic. This thesis plays (at least) two key roles in Mayberry's enterprise which must be distinguished. Insofar as Mayberry provides a parallel development of Euclidean and Cantorian set theories (what I referred to above as Mayberry's "agnosticism"), the Greek notion of *arithmos* serves as a common conceptual core on the basis of which variant notions of finite collection may be grounded. But in line with the *historical* connection between the Greek conception of *arithmos* and Euclid's axiom, the primary nature of the commitment to *arithmos* also reinforces Mayberry's own admitted excitement about EST (see, e.g., p. 387). From the perspective of any decision to pursue EST, the strength of the commitment to *arithmoi* as the fundamental, given mathematical "data" can only increase.

Thesis 2 expresses Mayberry's anti-operationalism most succinctly: operations, whether of counting or calculating, must be analysed in terms of more fundamental data. In conjunction with Mayberry's commitment to Thesis 1, it is clear that operations are to be analysed in terms of our primary ontological commitment to *arithmos*, and whether or not this analysis is to be reductive in any thoroughgoing sense, it is clear that it must be effectively reductive so far, at least, as foundational purposes are concerned. No undischarged appeal to operations may be made in providing foundational justifications.

Thesis 3 has already been considered, but it is perhaps appropriate to reiterate here that after stating this thesis Mayberry goes on to remark that residual operationalism infects the Cantorianism of the "orthodox Cantorian majority" as well. Given that such residual operationalism infects standard (i.e., "non-Mayberrian") presentations of Cantorian set-theory, in what sense are we to distinguish Mayberry's set-theoretic perspective not only from "heretics" (p. xvii), that is, non-set-theoretic foundations, but also from the Cantorian orthodoxy? Here, I believe, Mayberry's answer is clear: we must *purify* set theory of the residual operationalism found in its standard expression while retaining a commitment to the fundamental set-theoretic approach which survives such purification. Only then will we be in a position to establish a set-theoretic foundation (whether it be of a Cantorian or Euclidean, or perhaps even some other, variety) which does not fall prey to the "operationalist fallacy." The debate about foundations, on this picture, will then be one about justifying a particular approach to the set-theoretic characterization of the finite.

Having characterized Mayberry's set-theoretic foundationalism, I am now in a position to offer my criticism of it. The criticism is not, at least directly, one about the potential for success of the foundational program Mayberry suggests, but rather a criticism directed at Mayberry's claims to the uniqueness, or at the very least the primacy, of the set-theoretic perspective. Simply put, the problem is this: there are other, non-set-theoretic perspectives which seem perfectly capable of respecting the point Mayberry makes in his critique of operationalism along the lines of Thesis 3. Even more damningly, it is difficult if not impossible to characterize some of these alternative programs as anything but operationalist in the sense of Thesis 2. If this is so, as I believe it is, then Mayberry's so-called critique of operationalism is not exclusively about operationalism as a philosophical position at all, but about something else. But before attempting to ascertain what this "something else" might be, let me describe these alternative approaches.

To begin with, it should be noted that Mayberry himself recognizes that there are proponents of non-set-theoretic foundations who respect his "critique of operationalism." Mayberry mentions, in particular, Petr Vopenka and Edward Nelson's respective rejections of the idea of a unique natural number system and goes on to remark:

> Neither Vopenka nor Nelson approaches the natural number problem from the foundational standpoint I have adopted here. Indeed, both are formalists of a sort (in Nelson's case, a self-described formalist), but their formalism is not, like Bourbaki's, a mere device for avoiding a serious confrontation with foundational issues. On the contrary, it functions more like a working hypothesis that allows them to get to grips, each in his own fashion, with the very difficult business of developing mathematics without presupposing a unique natural number sequence. (pp. 387–88)

I am not sure whether either Vopenka or Nelson would be happy with Mayberry's characterization of their respective (philosophical) formalisms as "working hypotheses," but in any case, it seems clear that from Mayberry's own set-theoretic perspective such formalism should look, at best, like a provisional investigation which brackets ultimate concerns of justification which must refer to (set-theoretic) ontology. If, indeed, this is all that is going on in the cases of Vopenka and Nelson, then we might still be willing to concede to Mayberry the unique relevance of set-theoretic foundations as the only perspective which gets to the ontological bottom of things.[2] That this is *not* all that is going on follows from my description of further, essentially operationalist, positions which nonetheless respect the point Mayberry makes in what I have called Thesis 3; it is to these that I now turn.

Perhaps Mayberry's most explicit characterization of the "operationalist fallacy," and certainly the one most relevant for present purposes, is embodied in his specific claim that

> The assumption that simply by giving the conventional operationalist *description* of the natural numbers, we have *thereby* characterised them uniquely will simply not stand up under serious scrutiny. (p. 387)

With this statement of the "operationalist fallacy," in particular, I could not be in more wholehearted agreement. But this point has been made, most strongly, and prior to the work of either Vopenka or Nelson, in the so-called ultra-intuitionism of Yessenin-Volpin, whose philosophical position is simultaneously as anti-set-theoretical and as pro-operationalist as any position is likely to come. Although Yessenin-Volpin's position is admittedly challenging to ascertain with confidence, his treatment of the noncategoricity of natural number systems in particular has been studied and presented with great care in a series of papers by Isles, and in Isles's work one finds the commitment to the noncategoricity of natural number notation systems specifically motivated by an operationalist (or, as I would ultimately prefer, "praxiological") philosophical stance.

From Isles's competing perspective we may see, indeed, that much of the force militating against recognizing any foundational *problem* with establishing the categoricity of the natural numbers derives largely from arguments stemming directly from realist, and in particular, set-theoretic, commitments. Isles presents three arguments for the uniqueness of the natural numbers which he takes to be common. The first makes an appeal to induction in order to produce an isomorphism between any two natural number series; this Isles rejects as circular, and indeed it is just Mayberry's point that in the absence of some antecedent foundational defense of such an application of a principle of induction it must indeed fail on just this ground. The second and third arguments Isles considers rely on "definitions or 'constructions' of the natural numbers" ([4], p. 113), and both commit to a notion of mathematical realism which Isles, in particular, finds philosophically prejudicial. Isles describes first the set-theoretic definition or characterization of the natural numbers "as the intersection of all sets which contain a zero and which are closed under a successor function" ([4], p. 113). As Isles points out, and as Mayberry's juxtaposition of Cantorian and Euclidean set theory dramatically demonstrates, the availability of such a characterization will depend on the set-theoretic axioms adopted, and so will rely on a principled argument for selecting one particular version of set theory. Furthermore, philosophically this approach arguably requires a realist commitment to set-theoretic collections, since the definition itself is impredicative.

The more simple-minded characterization, which Isles presents last, is the one that he takes to underlie most mathematicians' belief in the existence of a unique collection of natural numbers. Isles calls this the "counting description" and characterizes it in terms of the following three rules:

> R1: Write down a stroke 1.
> R2: Given a set of strokes (call it X) write down X1.
> R3: Now apply R1 once and then apply R2 again and again. ([4], p. 113)

Here we encounter what Mayberry would presumably be willing to call the "naive operationalism" of the working mathematician. Isles does not contest R1 and R2 himself, but, as he remarks, R3 is "in a different category":

> It does not determine a unique method of proceeding because *that* determination is contained in the words "apply R2 again and again." But these words make use of the very conception of natural number and indefinite repetition whose explanation is being attempted: in other words, this description is circular. ([4], p. 113)

Isles aligns the two positions of the working mathematician and the "orthodox" set-theorist in a way which is reminiscent of Mayberry's remark about the pervasiveness of the operationalist fallacy among both working mathematicians and mathematical logicians studying axiomatic set theory as a mathematical domain. But whereas Mayberry will attempt to resolve this dilemma by purifying set theoretic realism (or, if one prefers, foundations of mathematics in the ontology of sets), Isles will respond to the situation by abandoning just these sorts of ontological commitments. It is in the process of abandoning these ontological commitments that Isles comes to question the categoricity of the natural numbers.

To this end, Isles considers one final potential argument for the uniqueness of the natural numbers: "Nonsense. I understand R3 perfectly well because I understand how to use it" ([4], p. 114). But as Isles points out, this depends on our having a *univocal* conception of our use of the natural numbers, and Isles takes this to be palpably false:

> For the use may be manifested in an enormous variety of forms, using various notation systems, computer, etc., and it may be a difficult matter to see that two such apparently different procedures are, in some sense, isomorphic. ([4], p. 114)

What Isles shows, then, is that the most common arguments both of the ontological and praxiological sort fail to support the uniqueness of the natural numbers. But, in fact, it is most evident in his response to the argument from use how the nonuniqueness of the natural numbers may be motivated, by appeal to the diversity of ways in which we make use of "them." And the appeal to such motivation plays a crucial *and legitimate* philosophical role in the work of Isles and the tradition from which his work derives (Mannoury, van Dantzig, Yessenin-Volpin). This is not to say that all the philosophical or foundational advantages lie on the side of this tradition I would dub "praxiological"; rather, it is to point out simply that there are two philosophically antithetical and prima facie legitimate ways of responding to the state of affairs of which Isles and Mayberry provide us with a largely common diagnosis.

In fact, I believe Nelson's work constitutes a distinct, third alternative. In making this claim I am suggesting a different attitude to Nelson's work than the one which seems to be implied by Mayberry's remarks about Nelson discussed previously. That, in any case, Nelson's work shares praxiological affiliations with the tradition to which Isles subscribes is, I believe, indicated by remarks from Nelson such as the following:

> The intuition that the set of all subsets of a finite set is finite . . . is a questionable intuition. Let A be the set of some 5000 spaces for symbols on a blank sheet of typewriter paper, and let B be the set of some 80 symbols of a typewriter; then perhaps B^A is infinite. Perhaps it is even incorrect to think of B^A as being a set. To do so is to postulate an entity, the set of all possible typewritten pages, and then to ascribe some kind of reality to this entity—for

> example, by asserting that one can in principle survey each possible typewritten page. But perhaps it simply is not so. Perhaps there is no such number as 80^{5000}; perhaps it is always possible to write a new and different page. ([9], p. 50)

What specifically interests me in Nelson's position here is that he is allowing himself to appeal to issues like surveyability in fashioning his argument. In particular, the persuasive force of the passage relies on such issues in much the same way as passages from Isles do. From an ontologizing, set-theoretic perspective, these appeals could only seem psychological and (hence) inessential, but from a praxiological perspective they are susceptible to being recognized as of philosophically fundamental import: in this case equally as fundamental as the distinction between the finite and the infinite (Bassler [1]).

This leaves us, then, with the question already mentioned above: if Mayberry's critique of operationalism is not about operationalism per se, then what is it about? An obvious candidate answer would be: it is about the status of the categoricity of the natural numbers. Up to a first approximation this seems like the right answer, and Mayberry has conceded in correspondence that if it is not "one neologism too far" we might speak here of the "iterationist" fallacy, (August 3, 2003), but the problem with this proposed response in any case is that it leaves us largely in the dark if we wish to compare how foundational philosophical attitudes might bear on this issue. On the other hand, just because I am claiming that Mayberry's set-theoretic foundations don't offer a uniquely relevant perspective on this issue does not mean that philosophical foundations cannot contribute to our investigation. In fact, as I have attempted to demonstrate throughout this essay-review, it is just the fact that Mayberry does take foundations seriously which leads him to recognize the centrality of this issue. But we must now reassess the situation after recognizing that we may appeal to *other* foundational perspectives as well in order to motivate the centrality of this issue.

4 Assessment and Conclusion

The ultimate strength of Mayberry's attitude to set-theoretic foundations is the force with which he insists that the justificatory power of (philosophical) foundations of mathematics must be a function of our capacity to provide a philosophical accounting for the distinction between the finite and the infinite. On both historical and conceptual grounds Mayberry makes the point, in this reviewer's opinion convincingly, that this distinction lies at the core of our conception of *arithmos*, and hence at the core of our conception of set-theoretic collection. It is in terms of this distinction, then, that arguments must be given for accepting those axiomatic formulations of the basic principles which underlie the existence and formation of such collections.

In the process of pursuing this project with respect to both Cantorian and Euclidean set theory, however, an interesting twist begins to emerge with respect to the distinction between the finite and the infinite. In Cantorian set theory, we are able to say categorically what counts as a finite number (in Cantor's sense, not to be confused with Mayberry's locution "Cantorian finite"), since Mayberry is able to provide a proof of the categoricity of Peano systems. However, the existence of such a collection relies on an appeal to Cantor's so-called axiom of infinity, and Mayberry is able to justify this according to his own standards only if such transfinite (in Cantor's sense) collections are ("Cantorian") finite in Mayberry's sense. The problem, here, is the following: how are we then to take the property which characterizes the *Cantorian* finiteness of the individual natural numbers? In particular, what are we to say about it so far as Mayberry's distinction between the finite and the infinite (in the Cantorian context) is concerned? At the very least it seems we must say that Cantor's own notion of finiteness is not conceptually foundational. Indeed, since Cantor makes *his* distinction between the finite and the transfinite in terms of whether collections can or cannot be put in one-to-one correspondence with a proper subset of themselves, this sense of the finite simply reflects on the distinction between Euclidean and non-Euclidean collections from within the Cantorian perspective! But given that Euclid's axiom is *denied* as a proper characterization of the (Mayberrian) finite in this Cantorian perspective, the distinction cannot have any conceptually fundamental status, and so the distinction between the Cantorian finite and the Cantorian transfinite is lacking any particular foundational significance: what collections are finite *in this sense* turns out not to have much *philosophical* weight.

On the other hand, within EST we encounter a different, although perhaps in some sense complementary, situation. Here we adopt Euclid's axiom as delimiting what counts as Mayberrian, and hence Euclidean, finite; in this case it seems that our traditional distinction between the finite and the infinite aligns with what is to be counted as definite, and hence a set, versus what is indefinite, and hence does not so count. But what we then find is that within the realm of the definite there is no canonical collection of finite quantities! And so, once again, what counts as definite in the sense of being a collection may or may not count as finite in the sense of being "measurable" against any particular numeration system.

The fact, as Mayberry admits, that we are not currently in a good position to decide, on the basis of our understanding of the distinction between the finite and the infinite, between the Cantorian and Euclidean approaches, already suggests a deficiency in our ability to appropriate fully the distinction between the finite and the infinite for foundational decisions. In conjunction with the peculiar twist I have described above, there is perhaps sufficient motivation to ask whether the distinction between the finite and the infinite is itself definite or indefinite. Mayberry's own work suggests that the distinction between the finite and the infinite is indefinite, and Mayberry has indicated in correspondence that this is indeed his view, but it must at the same time be sufficiently *definite* that we are able to appeal to it in order to discriminate between various foundational proposals. In particular, a *complete* exposition and justification of the foundations of mathematics in the sense which Mayberry requires would require that the distinction between the finite and the infinite be *sufficiently* definite that we could at least decide which axioms are self-evident and which are not. This may suggest that we should perhaps give up on completeness "for now," or else establish some sense of an ongoing program for how to add axioms to set theory (as Lavine has attempted to do in [6], pp. 309–28]).

In correspondence, Mayberry suggests he can accommodate vagueness in the distinction between the finite and the infinite by recognizing that global quantification, by Brouwer's principle, doesn't obey the law of excluded middle (LEM):

> for if we take $\exists x \forall y [y \in x \leftrightarrow \Phi(y)]$ to assert the finiteness of the species of all y such that $\Phi(y)$, then the assertion is global (in fact Σ_2) and therefore excluded middle does not apply to it. Thus assertions of finiteness are "vague" in this sense on my view. (December 28, 2002)

But here two points must be made. First, such an accommodation is just that: an accommodation of vagueness ex post facto, which does little to *explain* why the distinction between the finite and the infinite is indefinite or what sort of consequences we should draw from this indefiniteness. Second, as a consequence of this lack of explanatory power, it is not clear that Mayberry's proposal accommodates the *correct* vagueness or how, indeed, we could even decide whether it does. In particular, we are prima facie dealing with two different types of vagueness: the sort of vagueness which emerges by denying LEM for global quantification and the sort of vagueness that arises by virtue of the noncategoricity of SIS's in EST. What we would need then, at the very least, is some story which would *link* together these two different senses of vagueness. But as I've suggested elsewhere in the review, there is insufficient reason, at least at present, to think that these two "platforms" within Mayberry's program mesh together well enough that we should expect such a "pre-established harmony"! If, on the other hand, they could convincingly be shown so to mesh, this would be striking and compelling evidence for the organicity of Mayberry's program. What is needed here are more compelling arguments on Mayberry's part, if possible, to the effect that his set-theoretic ontologism (Thesis 2) and his specific criticism of operationalist justifications for an ontology of number (Thesis 3) must go hand in hand.

The lack of such compelling argument points to potentially the greatest weakness of Mayberry's program: in the process of focusing attention on the key role played by the distinction between the finite and the infinite and the concomitant issues surrounding the status of simply infinite systems, he may have severed our philosophical access to some of the tools and perspectives which will be required for a philosophical consideration of the fundamental significance of this distinction. Mayberry's claims to the effect that set-theoretic foundations are the strongest vantage point for criticizing operationalism in the foundations of mathematics is vitiated as a consequence of Mayberry fusing two senses of 'anti-operationalism' which are in fact conceptually distinct. The first of these, associated with what I have named above 'Thesis 2', commits one to the philosophical position that appeals to mathematical operations such as counting or calculating must ultimately be discharged by appeals to more fundamental notions. The second of these, associated with what I have named above 'Thesis 3', commits one to the position that an "operationalist" conception of natural number "is the central fallacy that underlies all our thinking in the foundations of mathematics" (p. xvii). But, as I have argued above, the fallacy at issue here is one which is avoided by a number of operationalists in the sense of Thesis 2, and besets many set-theorists of the "standard" Cantorian persuasion.

Rather than denominating the fallacy Mayberry describes in Thesis 3 as 'operationalist', it would be better to recognize it as one involving the conflation of certain operationalist and certain ontological commitments, an "iterationist" fallacy, as Mayberry has alternatively suggested. In particular, the assumption that "the" natural numbers can be described operationally is one which besets only those who would simultaneously make use of an operational description of counting *and* identify this operation with an unproblematic ontological commitment to that object to which the term 'the natural numbers' would refer. As is clear from the examples of Mayberry, on the one hand, and of Yessenin-Volpin and Isles on the other, this conflation can be cleared up by recognizing that an operational description of counting does not amount to a definite description of "the" natural numbers, but in *either* of two ways.

In Mayberry's case this point is made by showing that the natural numbers can only be derived from a conception of abstract finite collections by invoking a controversial axiomatic commitment (namely, Cantor's axiom of infinity), and that in the absence of such a commitment no such categorical object can be derived. In the cases of Yessenin-Volpin and Isles, the recognition that the operation of counting does not endow one with a unified sense of number leads, instead, in the direction of developing alternative natural number notation systems without any further analysis of the operation of counting in terms of more fundamental notions. As such, the position is operationalist in precisely the sense which Mayberry criticizes in Thesis 2.

The comparative advantages and disadvantages of these two approaches must be left for another time; my point here is simply to argue that there are competing irreconcilable perspectives which may contribute to the investigation of the foundations of mathematics insofar as it recognizes the centrality of Mayberry's Thesis 3. Given that there are such competing perspectives, the debate about the foundations of mathematics should not be restricted to one about the proper set-theoretic characterization of the distinction between the finite and the infinite, but should instead be the broader foundational debate about this distinction, and closely related ones, in which set-theoretic foundations participate on a common playing field with other foundational perspectives.

As we have seen, on the issue of the noncategoricity of the natural numbers (or at least the foundational requirement that their categoricity be demonstrated) and the importance of this recognition for the foundations of mathematics, Mayberry is largely allied with such figures as Isles, Vopenka, and Nelson. If we refer to the common target of their criticisms as "the iterationist fallacy," adopting the locution suggested by Mayberry and which this reviewer would strongly recommend, then we may see, as Mayberry has stressed in correspondence, that he sees "Isles and Nelson more as allies than as opponents" (August 3, 2003). Disentangling Mayberry's critique of the iterationist fallacy from his critique of operationalism as a philosophical position would go some way toward establishing the grounds on which this reviewer would recommend that the foundations of mathematics currently be investigated: on the one hand, in terms of the competing perspectives of set-theoretic foundations and operationalism, on the other hand in terms of a foundational denial or promotion (i.e., demonstration) of the categoricity of the natural numbers. Three of the four possible combinations seem not only prima facie conceptually coherent but indeed have all been developed: Mayberry has outlined two distinct alternative set-theoretic foundations which avert the iterationist fallacy, one of which foundationally supports the categoricity of the natural numbers (CST) and one of which denies it (EST), and Yessenin-Volpin, Isles and others have taken an operationalist approach which recognizes the iterationist fallacy and which explicitly argues against the categoricity of the natural numbers. Mayberry's recognition of the deeper sense in which Isles and Nelson serve as allies would be an affirmation that for Mayberry the critique of the iterationist fallacy is currently more pressing, if not more important, than the question of ontological, set-theoretic foundations versus what I would refer to as a "praxiological," operational orientation.[3] However, it must be conceded by the operationalist that Mayberry's program has behind it the weight of a well-developed, indeed, canonical tradition of set-theoretic foundations, and so provides the most powerful source currently available for the critique of the iterationist fallacy. This reviewer, for one, finds Mayberry's argument persuasive that the removal of this fallacy is the most pressing task facing mathematical foundations today.

Notes

1. This point has been stressed by Mayberry in personal correspondence with the author of this review.

2. Some foundationally-oriented category-theoretists might conceivably be unhappy with this claim on independent grounds, but I think they would be wrong to object since, whatever "category-theoretic foundations" might turn out to be, I would favor an approach to them which de-emphasized any ontologically foundational role for categories. Making categories do such work ultimately amounts, I believe, to dressing sets up in categorical clothing, and hence does no justice to what is truly powerful in the category-theoretic approach. But these overtly programmatic remarks must, unfortunately, await another occasion for their defense.

3. Mayberry has insisted, rightly I think, that his theory is constructive "in a very strong sense" (August 3, 2003), and so I have accepted his label 'operationalism' to refer to positions that have historically been characterized in the context of the foundations of mathematics as constructivist. Ultimately, however, I think there are regards in which the positions outlined by Yessenin-Volpin, Isles, and Nelson, among others, are more accurately characterized as praxiological rather than operational. In particular, I do not think they fall prey to some of the traditional objections to operationalism. But I must defer a fuller discussion of this point for now.

References

[1] Bassler, O. B., "The surveyability of mathematical proof: A historical perspective," to appear in *Synthese*. 120

[2] Epstein, R. L., and W. A. Carnielli, *Computability*, 2d edition, Wadsworth/Thomson Learning, Belmont, 2000. Zbl 0951.03001. MR 2001c:03001. 112

[3] Hallett, M., *Cantorian Set Theory and Limitation of Size*, reprint edition, vol. 10 of *Oxford Logic Guides*, The Clarendon Press, New York, 1986. Zbl 0656.03030. MR 86e:03003. 109

[4] Isles, D., "Remarks on the notion of standard nonisomorphic natural number series," pp. 111–34 in *Constructive Mathematics*, vol. 873 of *Lecture Notes in Mathematics*, Springer, Berlin, 1981. Zbl 0461.03017. MR 83b:03067. 118, 119

[5] Klein, J., *Greek Mathematical Thought and the Origin of Algebra*, Dover Publications Inc., New York, 1992. Translated by E. Brann, reprint of the 1968 English translation. Zbl 0159.00302. MR 94b:01010. 108

[6] Lavine, S., *Understanding the Infinite*, Harvard University Press, Cambridge, 1994. Zbl 0961.03533. MR 95k:00009. 121

[7] Mayberry, J., "On the consistency problem for set theory: An essay on the Cantorian foundations of classical mathematics. I," *British Journal for the Philosophy of Science*, vol. 28 (1977), pp. 1–34. Zbl 0395.03032. MR 56:123a. 113

[8] Mayberry, J., "On the consistency problem for set theory: An essay on the Cantorian foundations of mathematics. II," *British Journal for the Philosophy of Science*, vol. 28 (1977), pp. 137–70. Zbl 0395.03033. MR 56:123b. 113

[9] Nelson, E., *Predicative Arithmetic*, vol. 32 of *Mathematical Notes*, Princeton University Press, Princeton, 1986. Zbl 0617.03002. MR 88c:03061. 120

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