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The Fact Semantics for Ramified Type Theory and the Axiom of Reducibility

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Abstract This paper uses an atomistic ontology of universals, individuals, and facts to provide a semantics for ramified type theory. It is shown that with some natural constraints on the sort of universals and facts admitted into a model, the axiom of reducibility is made valid.

1 Introduction

Various authors have criticized Russell's use of the axiom of reducibility in *Principia Mathematica*, among them, Ramsey, Quine, and Russell himself. Recently, however, Linsky [11] has justified the acceptance of the axiom on the basis of his reading of Russell's metaphysics of universals. In the present paper I create a formal semantics from Linsky's interpretation of Russell and prove that the axiom of reducibility is valid in this semantics, thus providing a formal counterpart to Linsky's informal argument.

The primitives of the model theory are universals, individuals, and facts. Facts have individuals and universals as constituents. The language contains constants (that refer to individuals or universals) and variables of all types. Types are divided into two classes: hereditarily predicative types and nonhereditarily predicative types. Hereditarily predicative types are the type of singular terms, predicative relations expressions that hold of individuals, and predicative relations between other entities that are represented by expressions of hereditarily predicative type. There are constants of hereditarily predicative type only. Individual constants denote individuals. Function constants denote universals. Nonhereditarily predicative function expressions do not denote anything. The core of the semantics is a substitutional treatment of quantification for quantifiers that take nonhereditarily predicative variables. Using the substitutional theory of quantification, we give the truth conditions for formulas in terms of formulas containing only function constants and names of individuals.

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In its use of the substitutional theory of quantification, this paper follows Leblanc and Weaver [9], Leblanc [8], and especially Hazen and Davoren [3]. We show that this semantics, combined with some natural constraints on models, makes valid the axiom of reducibility.

This paper is not primarily a piece of Russell scholarship. It is an attempt to provide a reasonable model for Russell's logic (or, more exactly, one formalization of Russell's logic). I do, however, occasionally motivate elements of the model by reference to Russell's work. In addition, the reader will notice that I make heavy use of set-theoretic concepts in this paper. One of the central motivations of *Principia* was to reconstruct all of classical mathematics without the existence of sets. My goal here is only to show that Russell's logic, including the axiom of reducibility, can be given a reasonable semantics. Whether we can reconstruct this semantics without the use of set theoretic concepts is an interesting topic, but one for another paper.

2 Individuals, Universals, and Facts

We begin with ontology. On our theory, there are three broad sorts of entities—universals, individuals, and facts. Facts have universals and individuals as constituents. A version of simple type theory (the theory of "s-types") dictates how entities can be combined into facts. Like Russell, we assume that the class of individuals and the class of universals of any given type each is a set (see Whitehead and Russell [17], *24).[1]

The type of individuals, ι, is a primitive type both of s-types and of r-types (ramified types). Complex s-types are determined by the following inductive definition. The set of s-types is the smallest set such that

1. ι is an s-type;
2. if $t_1, \ldots, t_n$ are s-types (for some n, $0 < n < \omega$), then $(t_1, \ldots, t_n)$ is also an s-type.

A fact is a structure $\langle R, a_1, \ldots, a_n \rangle$, where R of s-type $(t_1, \ldots, t_n)$ and $a_1, \ldots, a_n$ are entities of type $t_1, \ldots, t_n$, respectively.[2]

Note that we include a whole hierarchy of universals in our ontology. This runs contrary to some interpretations of Russell. Hazen and Davoren [3] hold that, with a few exceptions, Russellian facts are all of the form $\langle R, i_1, \ldots, i_n \rangle$ where each i_j is an individual. The exceptions all concern intentional relations, such as belief. It would seem, however, that Russell does think that there are nonintentional relations of higher type:

> Between universals, as between particulars, there are relations of which we may be immediately aware. We have just seen that we can perceive that the resemblance between two shades of green is greater than the resemblance between a shade of green and a shade of red. Here we are dealing with a relation, namely "greater than", between two relations. (Russell [12], pp. 102–3)

Thus it would seem that, at the period when Russell wrote [12], which was at the time he wrote the first edition of *Principia*, he held that there were nonintentional higher-order universals.[3]

Also note that we have excluded facts from themselves being constituents of facts. Facts containing other facts are complicated to incorporate into my proof of the axiom of reducibility because sentences that contain other sentences as arguments are difficult to incorporate into the present framework. I think it is possible to do so, but

I will leave this task to another paper. It is, however, an important issue since Russell thought that certain relations connect individuals with facts. In particular, Russell held that we perceive facts ([14], pp. 82–84). Thus, perceptual relations connect individuals and facts.

We will also use the notion of a *frame*. A frame is a set of facts. The class of frames that we consider have the following property: *For every homogeneous collection of finite sequences of entities, there is a relation in which all and only members of that set stand.* A collection of sequences of entities is homogeneous if there is a sequence of s-types, $(t_1, \ldots, t_n)$, such that the members of each sequence of entities in that set are of these types in that order. Thus, our condition says that for any homogeneous set of sequences X there is some relation R such that if $\langle a_1, \ldots, a_n \rangle \in X$, then $\langle R, a_1, \ldots, a_n \rangle$ is in our frame.

This condition is rather strong, so we need a motivation for it. To do so we need some definitions. An entity a has a property P in a frame $\mathcal{F}$ if and only if $\langle P, a \rangle \in \mathcal{F}$. A frame is called *Leibnizian* if for each pair of distinct individuals i and j, there is some property that i has that j does not have. A frame is called *super-Leibnizian* if for each pair of individuals *or universals*, e and e', there is some property that e has and e' lacks. Finally, a frame is said to be *super-duper-Leibnizian* if for any sequence of entities, there is some relation in which the entities in the sequence and only they stand.

I follow Linsky [11] in reading Russell as holding the following two views: (1) relational properties are genuine properties; (2) the class of properties of individuals is closed under Boolean combinations (Linsky [11], p. 106).

Relational properties are entities like *being three miles from Paekakariki* or *takes Zermela for walks*.[4] On Linsky's reading of Russell, an entity might not be distinguished by its monadic properties, like *being green*, *being square*, and so on. But instead, the property that distinguishes between two objects can be a relational property. Thus, in formulating Leibniz's thesis of the identity of indiscernibles, Russell quantifies over both monadic and relational properties.

Linsky argues for the adoption of Boolean combinations of properties on the grounds that the inclusion of these combinations helps to justify the axiom of reducibility. The axiom of reducibility says that any propositional function is coextensive with a predicative function. For Linsky's Russell, first-order function expressions pick out universals or constructions out of universals. Consider the extension of some propositional function that takes individuals as arguments. Linsky suggests that any such extension be characterized in terms of a Boolean combination of universals, such as "being F and G but not $H\ldots$" (Linsky [11], p. 107).

In order to prove that the axiom of reducibility holds generally (not just for functions taking individuals as arguments), we accept the Boolean combination thesis in a strong form. That is, for any set of properties of the same type, Φ, the conjunctive property $\bigwedge \Phi$ and disjunctive property $\bigvee \Phi$ both exist. We also accept relational properties in our model. Suppose that Φ is the set of properties that is had by an entity, a. We call $\bigwedge \Phi$ the *Leibnizian concept* of a. Now consider some set of entities Ψ of the same type. Let Ψ' be the set of Leibnizian concepts of members of Ψ. Then $\bigvee \Psi$ is a property that has all and only members of Ψ in its extension. Thus, our form of the Boolean combination thesis allows us to say that any set of entities of the same type is characterized by some property.

In addition, I treat these Boolean combinations as universals properly speaking. There are facts about them, that is, they enter into facts as arguments and not just as properties of other things. Moreover, the super-Leibnizian thesis holds for Boolean combinations and relational properties as well as for simple universals. That is, for any two distinct properties F and G, F has some property that G lacks. In our formal model theory we do not make any distinction between simple universals and Boolean combinations or relational properties. They are all called "universals" and are treated as elements in our models.

Accepting infinite Boolean combinations helps to motivate our use of super-duper-Leibnizian frames. Suppose that we have a sequence of entities $\langle e_1, \ldots, e_n \rangle$, which have Leibnizian properties $\varphi_1, \ldots, \varphi_n$, respectively. Then the property $\lambda x_1 \ldots x_n(\varphi_1(x_1) \wedge \cdots \wedge \varphi_n(x_n))$ is a property that is had only by that sequence and no other. Thus, the existence of Leibnizian concepts, together with Boolean combinations and a device such as lambda abstraction allows us to construct super-duper-Leibnizian frames from super-Leibnizian frames. Moreover, given Boolean combination and lambda abstraction, from super-duper-Leibnizian frames, we obtain frames in which there is a property corresponding to each homogeneous set of sequences of entities. Let Φ be a homogeneous set of sequences of entities and let us call the set of universals which correspond to the sequences in Φ, Φ'. The property $\lambda x(\vee_{F \in \Phi'} F(x))$ clearly is had by all and only members of Φ.[5]

It is clear that considering only super-duper-Leibnizian frames makes a weak form of the axiom of reducibility valid. This weak form says that every open formula that has only free variables of hereditarily predicative type is equivalent to some predicative predicate expression of those variables. We need the theorems that follow to prove the more general thesis that any open formula is equivalent to a predicative predicate expression.

3 The Language

Our treatment of type theory is based on Church [2]. We also follow Anderson [1], Hazen [4], and Hazen and Davoren [3] in using lambda abstracts, but only in the metalanguage. As we shall see, we also assume that every individual has a name and that there is a name for every universal; thus our language needs as many constants as our model has entities.[6]

Before we set out the other elements or our vocabulary and our formation rules, we will need to state the theory of ramified types (r-types). Our formulation of ramified types is Church's. That is, we begin with the type of individuals, ι, and then for each finite sequence of types, $\tau_1, \ldots, \tau_m$, also include $(\tau_1, \ldots, \tau_m)/n$ as a type (where n is a finite natural number greater than 0). $(\tau_1, \ldots, \tau_m)/n$ is the type of relations between entities of types $\tau_1, \ldots, \tau_m$ and of level n. If the relation is of level one, it is said to be *predicative*.[7] Propositions have a type $(\,)/n$, where $(\,)$ is the empty sequence. Propositions and their types play a very small role in the present theory, since I do not include properties of or relations between propositions in my language.

An important concept in our theory is that of a hereditarily predicative expression. An expression is hereditarily predicative if and only if it is of a hereditarily predicative type. *The set of hereditarily predicative r-types is defined inductively as the smallest set such that it contains* ι *and if* $\tau_1, \ldots, \tau_n$ *(for finite* $n > 0$*) are hereditarily predicative, then* $(\tau_1, \ldots, \tau_n)/1$ *is hereditarily predicative.*

Another concept that we will use in this paper is that of the *height* of an expression. The idea behind the notion of height is that we need a measure for the distance an expression is from having all of its arguments being hereditarily predicative. The height of a function of type $(\tau_1, \ldots, \tau_n)/n$, where all of $\tau_1, \ldots, \tau_n$ are hereditarily predicative, is zero. Where at least one of $\tau_1, \ldots, \tau_n$ is not hereditarily predicative, the height of a function of type $(\tau_1, \ldots, \tau_n)/n$ is 1 plus the greatest height of any of $\tau_1, \ldots, \tau_n$.

Now we turn to the vocabulary and formation rules of the language. Our language includes names for individuals and universals. It also includes variables of all types except propositions[8] (there are $\aleph_0$ variables of each type), the connectives $\wedge$ and $\sim$, the quantifier $\forall$, as well as parentheses. We define $\vee$, $\supset$, $\equiv$, and $\exists$ as usual.

Our formation rules are fairly standard:

1. If F is a constant of type $(\tau_1, \ldots, \tau_n)/1$ and $e_1, \ldots, e_n$ are names or variables of type $\tau_1, \ldots, \tau_n$, respectively, then $F(e_1, \ldots, e_n)$ is a sentence;
2. If f is a variable of type $(\tau_1, \ldots, \tau_n)/n$ and $e_1, \ldots, e_n$ are names or variables of type $\tau_1, \ldots, \tau_n$, respectively, then $f(e_1, \ldots, e_n)$ is a sentence;
3. If A is a sentence, then $\sim A$ is a sentence;
4. If A and B are sentences, then $A \wedge B$ is a sentence;
5. If A is a sentence, then $\forall x A$ is a sentence.

A formula is a sentence with no free variables. In our models, only formulas are given truth conditions.

We also utilize a piece of Russellian terminology. Where A and B have at most $x_1, \ldots, x_n$ free, then we say that A and B are *formally equivalent* in a model (as are the propositional functions $\lambda x_1 \ldots x_n A$ and $\lambda x_1 \ldots x_n B$) if and only if $\forall x_1 \ldots \forall x_n (A \equiv B)$ is true in that model.

I use a notion of order due to Anderson [1], which slightly (but usefully) modifies that of Church [2].

1. The order of ι is 0;
2. The order of $(\tau_1, \ldots, \tau_m)/n$ is $n + N$, where N is the highest order of the $\tau_1, \ldots, \tau_m$.

Where a is an individual name or free variable, it has the order of its type. For a variable x bound by a quantifier, the order of x is the order of the type of x plus one. We can now define the order of a sentence.

1. $\text{Order}(F(e_1, \ldots, e_m)) = N + n$, where n is the level of F and N is the highest order of $e_1, \ldots, e_m$;
2. $\text{Order}(A \wedge B) = \max\{\text{Order}(A), \text{Order}(B)\}$;
3. $\text{Order}(\sim A) = \text{Order}(A)$;
4. $\text{Order}(\forall x A) = k$, where k is either the order of x plus one or the order of A, whichever is higher.

One reason why we need this notion of order is so that we can define the type of a lambda abstract. Although lambdas are not in our object language, we need them to state the truth conditions for quantified statements and to state the central theorems of this paper. Where A is a sentence in which at most $x_1, \ldots, x_m$ occur free, the type of the abstract $\lambda x_1 \ldots x_m A$ is $(\tau_1, \ldots, \tau_m)/n$, where

$$n = \begin{cases} \text{Order}(A) - \text{Order}(x_i), & \text{if Order}(A) > \text{Order}(x_i) \\ 1 & \text{otherwise} \end{cases}$$

and x_i has the highest order of $x_1, \ldots, x_m$.

The axiom of reducibility is the scheme,

$$\exists f^{(\tau_1,\ldots,\tau_n)/1} \forall x_1 \ldots \forall x_n (A \equiv f(x_1, \ldots, x_n)).$$

This axiom says that for any sentence with at most $x_1, \ldots, x_n$ free, there is a predicative propositional function that is coextensive with it.

The axiom of reducibility is important to Russell's theory of classes. It acts as a comprehension principle of sorts. For Russell, classes are logical fictions. *20.1 of *Principia* defines the expression 'the class of ψs is f' as meaning that there is a predicative predicate φ that is formally equivalent to ψ and φ is f. Thus, in particular, $a \in \hat{z}(\psi z)$ (x is a member of the class of ψs) if and only if there is a predicate expression φ such that φa and $\forall x(\psi x \equiv \varphi x)$. By the axiom of reducibility, therefore, all predicate expressions, predicative or not, determine classes.

4 Denotation and Truth Conditions

Every constant in our language refers to an entity. Also, our models are constrained so that every entity has a name in our language.

We have the following correspondence between the r-types of constants and s-types of entities. Where τ is an r-type, τ^* is the corresponding s-type:

1. $\iota^* = \iota$;
2. $[(\tau_1, \ldots, \tau_n)/1]^* = (\tau_1^*, \ldots, \tau_n^*)$.

A model, then, is a pair $\langle \mathcal{F}, \mu \rangle$, where $\mathcal{F}$ is a frame and μ is a function from constants in the language to entities in the frame such that where c is a constant of type τ, $\mu(c)$ is of type τ^*. Now we can give the recursive truth conditions for formulas:

1. where F is a function name and $a_1, \ldots, a_n$ are names of appropriate type, $\models_\mu F(a_1, \ldots, a_n)$ if and only if $\langle \mu(F), \mu(a_1), \ldots, \mu(a_n) \rangle \in \mathcal{F}$;
2. $\models_\mu \sim A$ if and only if $\not\models_\mu A$; $\models_\mu A \wedge B$ if and only if $\models_\mu$ and $\models_\mu A$ and $\models_\mu B$;
3. where x is hereditarily predicative, $\models_\mu (\forall x)A$ if and only if $\models_\mu A[n/x]$, for all n, names of the same r-type as x;
4. where x is nonhereditarily predicative, $\models_\mu (\forall x)A$ if and only if $\models_\mu A'$ for all A' where A' is the normalization of $A[\lambda x_1, \ldots, x_n(B)/x]$ and $\lambda x_1, \ldots, x_n(B)$ is a closed abstract of the same r-type as x.

An easy double induction on the height and complexity of formula proves that if e and e' are names and denote the same entity and A is an arbitrary formula, then $\models_\mu A$ if and only if $\models_\mu A'$, where A' differs from A only in that one or more occurrences of e are replaced by occurrences of e'.

Before we go on to the proof that the axiom of reducibility holds in this model, I would like to discuss briefly the differences between this model and other models for the ramified theory of types. This model differs from, say, those of Kaplan [6] and Anderson [1] in that Kaplan's and Anderson's models contain propositional functions and propositions other than facts. Russell claimed during the period during which he wrote *Principia* to be an antirealist about propositions and propositional functions and so my model is in this respect closer to Russell's view. My model avoids commitment to propositional functions by incorporating a substitutional theory of quantification for nonhereditarily predicative variables. My model, however, is not novel in employing the substitutional theory of quantification. Lebanc and Weaver [9] and Leblanc [8] use the substitutional theory of quantification, as do Hazen and Davoren [3]. The difference between my model and Leblanc and Weaver's model is that the latter utilizes in its substitutional instances predicate constants of all types, not just of hereditarily predicative type. My model is closest to that of Hazen and Davoren, and in fact takes many of its basic features from their model. The difference, however, is that my model places the Leibnizian constraints on the model in order to validate the axiom of reducibility.

5 Decomposition Trees

A decomposition tree for a formula has some superficial similarities to a truth-tree (as in, e.g., Jeffrey [5]). There are, however, some important differences. First, both conjunctions and negated conjunctions (and universal quantified statements and their negations) cause decomposition trees to branch. Second, decomposition trees contain points which have infinitely many children (immediate successors). Every instance of a universal quantified statement is represented in the tree.

The rules for decomposition trees are as follows. The rules of conjunction decomposition and negated conjunction decomposition are

$$\begin{tikzcd} & A \wedge B \arrow[d] & \\ & \wedge \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd} \qquad \begin{tikzcd} & \sim(A \wedge B) \arrow[d] & \\ & \vee \arrow[dl] \arrow[dr] & \\ \sim A & & \sim B \end{tikzcd}$$

Where x is a variable of hereditarily predicative type, we have the following rules for the universal quantifier:

$$\begin{tikzcd} & \forall x A \arrow[d] & \\ & \forall \arrow[dl] \arrow[d] \arrow[dr] & \\ A[i/x] & A[j/x] & A[k/x] \ldots \end{tikzcd} \qquad \begin{tikzcd} & \sim\forall x A \arrow[d] & \\ & \exists \arrow[dl] \arrow[d] \arrow[dr] & \\ \sim A[i/x] & \sim A[j/x] & \sim A[k/x] \ldots \end{tikzcd}$$

Where there are infinitely many constants of the relevant type, there are infinitely many branches under the $\forall$ and $\exists$ in the above rules. The rules for quantification over nonhereditarily predicative functions are

$$
\begin{tikzcd}
 & \forall f A \arrow[d] & & & \sim\forall f A \arrow[d] & \\
 & \forall \arrow[dl] \arrow[d] \arrow[dr] & & & \exists \arrow[dl] \arrow[d] \arrow[dr] & \\
A' & A'' & A''' \ldots & \sim A' & \sim A'' & \sim A''' \ldots
\end{tikzcd}
$$

where A', A'', A''', and so on, result from replacing previously bound f with a closed abstract of the same type as f and normalizing. Finally, we have a rule for double negation decomposition:

$$
\begin{tikzcd}
\sim\sim A \arrow[d] \\
A
\end{tikzcd}
$$

If a node has no children, that is, if the formula labeling the node admits of no further decomposition, then it is called a *terminal node*. The nature of formulas at terminal nodes will be of particular interest to us. Clearly, in a full decomposition tree, terminal nodes will be labeled either by atomic formulas or by negated atomic formulas.

It is clear that if a branch point below a formula is labeled with an existential quantifier or a disjunction, then that formula is true if and only if at least one of its children are true. And if a branch point below a formula is labeled with a universal quantifier or a conjunction, then that formula is true if and only if all of its children are true.

In addition to decomposition trees, we use the notion of a *partial decomposition tree*. Whereas the top node of a decomposition tree is labeled with a formula, the top node of a partial decomposition tree is labeled with a sentence which has one or more free variables. The decomposition rules are applied as usual. As in full decomposition trees, the decomposition rules are applied as far as possible.

6 The Extensionality of Nonhereditarily Predicative Functions

We say that two expressions φ and ψ of the same type τ are *indistinguishable* in a model if and only if for all natural numbers $n > 0$ and for all functions χ of type τ/n, the normalization of $\chi(\varphi)$ is true if and only if the normalization of $\chi(\psi)$ is true. In this section, we show that any two nonhereditarily predicative functions of the same type that are formally equivalent are also indistinguishable.

We need one more definition before we can proceed with the proof. A sentence is atomic if it is of the form $F(a_1, \ldots, a_n)$, where F and $a_1, \ldots, a_n$ are constants. A sentence is *quasi-atomic* if it is of the form $f(a_1, \ldots, a_n)$, where at least one of f and $a_1, \ldots, a_n$ is a variable.

Lemma 6.1 *If f is a variable of nonhereditarily predicative type and A is a sentence with at most f free, then in every terminal node of the partial decomposition of*

A, f either does not occur or it occurs in predicate position either in a quasi-atomic sentence or a negated quasi-atomic sentence.

Proof By reductio. First we note that there are no atomic sentences of the form $F(\ldots f \ldots)$ since only hereditarily predicative functions can be subjects in atomic formulas. So no terminal nodes in which f occurs as an argument are atomic (or negated atomic). Thus, if f occurs as an argument in a terminal node of the tree, in say $\varphi(\ldots f \ldots)$, φ must be complex. But if φ is complex, further decomposition is possible and the node is thus not terminal. □

Lemma 6.2 *If $\lambda x_1 \ldots x_n \varphi$ and $\lambda x_1 \ldots x_n \psi$ are nonhereditarily predicative functions of the same type and $\forall x_1 \ldots \forall x_n(\varphi \equiv \psi)$ holds in a model, then $\lambda x_1 \ldots x_n \varphi$ and $\lambda x_1 \ldots x_n \psi$ are indistinguishable in that model.*

Proof Suppose that $\lambda x_1 \ldots x_n \varphi$ and $\lambda x_1 \ldots x_n \psi$ are functions of the same type and $\forall x_1 \ldots \forall x_n(\varphi \equiv \psi)$ holds in the model. Let A be a sentence with at most f free, where f is of the same type as $\lambda x_1 \ldots x_n \varphi$ and $\lambda x_1 \ldots x_n \psi$.

Now consider the partial decomposition tree of A. We take two copies of this tree. In one copy, we replace free occurrences f throughout with $\lambda x_1 \ldots x_n \varphi$ and normalize and in the other tree we replace free f with $\lambda x_1 \ldots x_n \psi$ and normalize. By Lemma 6.1, in every terminal node in this tree in which f occurs, it occurs as a predicate in a quasi-atomic sentence or negated quasi-atomic sentence. Thus, because φ and ψ are formally equivalent, every terminal node of one copy is equivalent to the corresponding terminal node of the other copy. We can see that this implies that every node of a given copy of the tree will be equivalent in the model to the corresponding node of the other tree. Thus, we can say that the normalization of $A[\lambda x_1 \ldots x_n \varphi / f]$ is equivalent in the model to the the normalization of $A[\lambda x_1 \ldots x_n \psi / f]$.

Generalizing on A, this shows that $\lambda x_1 \ldots x_n \varphi$ and $\lambda x_1 \ldots x_n \psi$ are indistinguishable. □

Now we apply the same reasoning to the general case.

Theorem 6.3 *Let A and B be sentences of the same order in which at most $x_1, \ldots, x_n$ are free and let $\mathcal{M}$ be a model in which A and B are formally equivalent. Suppose that $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ are either closed lambda expressions (only in cases in which the expression is not hereditarily predicative) or constants of the same type as $x_1, \ldots, x_n$ and are such that if e_i is a constant then e_i and e'_i are identical and if e_i is not a constant then e_i and e'_i are formally equivalent. Then, in $\mathcal{M}$, $A[e_1/x_1, \ldots, e_n/x_n] \equiv B[e'_1/x_1, \ldots, e'_n/x_n]$.*

Proof By induction on the order of A and B. The base case, in which A and B are first-order sentences is obvious. For then $x_1, \ldots, x_n$ are individual variables, and the corresponding $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ are all constants and are such that $e'_1, \ldots, e'_n$ are just $e_1, \ldots, e_n$, respectively. Then the theorem follows from the hypothesis that A and B are formally equivalent and the truth condition for the universal quantifier.

For the inductive case we take the partial decomposition tree of A and, as in Lemma 6.2, we take two copies of it. This time, in the only cases in which nonhereditarily predicative free variables appear in terminal nodes, they are either in predicate position or they are arguments of predicates that are free variables in quasi-atomic or negated quasi-atomic sentences. The reasoning for this is the same as in Lemma 6.1.

As in Lemma 6.2, we take two copies of this partial decomposition tree. We consider a terminal node in which at least one nonhereditarily predicative variable is free and in argument position. This node will be either of the form $x_j(b_1, \ldots, b_m)$ or $\sim x_j(b_1, \ldots, b_m)$, where each b_i is either a constant or one of $x_1, \ldots, x_n$ (distinct from x_j).

By the inductive hypothesis, since e_j and e'_j are of lower order than A and B, in our model $e_j(c_1, \ldots, c_m) \equiv e'_j(c'_1, \ldots, c'_m)$, where c_k is e_i if b_k is x_i and c_k is b_k if b_k is a constant. Thus, repeating the argument of Lemma 6.2, we can show that $A[e_1/x_1, \ldots, e_n/x_n] \equiv A[e'_1/x_1, \ldots, e'_n/x_n]$ is true in $\mathcal{M}$.

Moreover, we repeat the argument using the decomposition tree for B to show that $B[e_1/x_1, \ldots, e_n/x_n] \equiv B[e'_1/x_1, \ldots, e'_n/x_n]$ is true in $\mathcal{M}$. Since, by hypothesis, A and B are formally equivalent in $\mathcal{M}$ and by the transitivity of material equivalence, $A[e_1/x_1, \ldots, e_n/x_n] \equiv B[e'_1/x_1, \ldots, e'_n/x_n]$ is true in $\mathcal{M}$, ending the proof of the theorem. □

7 The Axiom of Reducibility

In this section, we show that all instances of the axiom of reducibility are valid in our class of models. We first show how to construct *reduction formulas* for propositional functions. A reduction formula for a function φ that takes only hereditarily predicative expressions as arguments is just

$$\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv F(x_1, \ldots, x_n)),$$

where F is a name of a universal that has the same extension as φ. We call 'F' the *final constant* of this formula and '$F(x_1, \ldots, x_n)$' the *reduct* of '$\varphi(x_1, \ldots, x_n)$'.

Now consider a function φ some arguments of which are not hereditarily predicative. We begin with a sequence of expressions $\langle e_1, \ldots, e_n\rangle$ such that the normalization of $\varphi(e_1, \ldots, e_n)$ is true in the model. Let $e_\alpha, \ldots, e_\nu$ be the nonhereditarily predicative expressions from this sequence and $A_\alpha, \ldots, A_\nu$ be their reducts. We then replace the final constant in each of the $A_\alpha, \ldots, A_\nu$ with variables of the same type, say, $f_\alpha, \ldots, f_\nu$. We call the resulting sentences, $A'_\alpha, \ldots, A'_\nu$. We now can construct a reduction sentence for φ:

$$\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv \\ \exists f_\alpha \ldots \exists f_\nu(\forall \mathbf{z}_\alpha(x_\alpha(\mathbf{z}_\alpha) \equiv A'_\alpha) \wedge \cdots \wedge \forall \mathbf{z}_\nu(x_\nu(\mathbf{z}_\nu) \equiv A'_\nu) \wedge G(x'_1, \ldots, x'_n)),$$

where x'_i is x_i if x_i is hereditarily predicative and f_i otherwise. Here, for brevity, we use boldface $\mathbf{z}$ to indicate a sequence of variables. In this formula, 'G' is the final constant and it stands for a relation that holds of all and only sequences of entities $\langle a_1, \ldots, a_n\rangle$ such that φ is true of the sequence of expressions (which may include closed lambda terms) $\langle e_1, \ldots, e_n\rangle$ and a_i is the referent of e_i, where e_i is a constant, and where e_i is not a constant, a_i is the referent of the final constant in a reduction formula of e_i.

Before we go on, let us look at an example of a reduction formula. Suppose that φ is a unary propositional function that takes as arguments unary functions that are not hereditarily predicative, but the arguments of which are hereditarily predicative. Thus, we begin with ψ, which is a propositional function such that the normalization of $\varphi(\psi)$ is true in a given model. We know that there is a property name F such that $\forall x(\psi(x) \equiv F(x))$ is true. We then generalize. For each ψ_i such that the normalization of $\varphi(\psi_i)$ is true, there is at least one corresponding F_i. We then choose a property name G such that $G(F_i)$ is true if and only if F_i is the final constant of a reduction formula of one of these ψ_i. Then our reduction formula is $\forall f(\varphi(f) \equiv \exists g(\forall x(f(x) \equiv g(x)) \wedge G(g)))$.

The construction of reduction formulas, in effect, shows that the axiom of reducibility is valid in our models. But to see this, we show that, where $\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv A)$ is a reduction formula, $\lambda x_1 \ldots x_n A$ is predicative and that each reduction formula is valid.

Theorem 7.1 *If $\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv A)$ is a reduction formula, then $\lambda x_1 \ldots x_n A$ is predicative.*

Proof By induction on the height of the function to be reduced.

Case 1 Suppose that φ takes only hereditarily predicative expressions as arguments. Then its reduction formulas are all of the form $\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv F(x_1, \ldots, x_n))$, where F refers to a relation. Then clearly F is hereditarily predicative and $\lambda x_1 \ldots x_n F(x_1, \ldots, x_n)$ is predicative.

Case 2 Suppose that φ takes some nonhereditarily predicative expressions as arguments. Then its reduction formulas are all of the form

$$\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv \\ \exists f_\alpha \ldots \exists f_\nu(\forall \mathbf{z}_\alpha(x_\alpha(\mathbf{z}_\alpha) \equiv A'_\alpha) \wedge \cdots \wedge \forall \mathbf{z}_\nu(x_\nu(\mathbf{z}_\nu) \equiv A'_\nu) \wedge G(x'_1, \ldots, x'_n)).$$

By inductive hypothesis, for each A'_i, each corresponding $\lambda \mathbf{z}_i A_i$ is predicative. Thus, the order of each final constant in each A_i (and hence the variable that replaces it in A'_i) is at most one greater than the order of the maximal order of any member of $\mathbf{z}_i$. Thus, for each x_i the variable that replaces the final constant in A'_i has an order less than or equal to that of x_i. Since G is hereditarily predicative, its order is one greater than the order of the greatest order of all the final constants (and the variables that replace them). Hence the order of G is at most one greater than the maximal order of $x_1, \ldots, x_n$. Thus, $\lambda x_1 \ldots x_n(\forall \mathbf{z}_\alpha(x_\alpha(\mathbf{z}_\alpha) \equiv A'_\alpha) \wedge \cdots \wedge \forall \mathbf{z}_\nu(x_\nu(\mathbf{z}_\nu) \equiv A'_\nu) \wedge G(x'_1, \ldots, x'_n))$ is predicative. □

Theorem 7.2 *If $\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv A)$ is a reduction formula, then it is valid.*

Proof Again by induction on the height of φ.

Case 1 φ takes only hereditarily predicative expressions as arguments. Then the reduction formula for φ is just $\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv F(x_1, \ldots, x_n))$, where F is coextensive with φ. Thus, $\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv F(x_1, \ldots, x_n))$ is true.

Case 2 Suppose that φ takes some nonhereditarily predicative expressions as arguments. Then, as in the preceding theorem, its reduction formulas are all of the form

$$\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv \\ \exists f_\alpha \ldots \exists f_\nu(\forall \mathbf{z}_\alpha(x_\alpha(\mathbf{z}_\alpha) \equiv A'_\alpha) \wedge \cdots \wedge \forall \mathbf{z}_\nu(x_\nu(\mathbf{z}_\nu) \equiv A'_\nu) \wedge G(x'_1, \ldots, x'_n))).$$

First, consider a sequence of expressions $\langle e_1, \ldots, e_n \rangle$ such that the normalization of $\varphi(e_1, \ldots, e_n)$ is true in a model, where each hereditarily predicative variable x_i is replaced with a constant of the same type and each nonhereditarily predicative variable is replaced with a closed lambda term of the same type. By the inductive hypothesis, $\forall \mathbf{z}_i(e_i(\mathbf{z}_i) \equiv A_i)$ is true for all i, $\alpha \leq i \leq \nu$. By the construction of reduction sentences, $G(e'_1, \ldots, e'_n)$ is also true, where e'_i is just e_i for all constants e_i and e'_i is the final constant from $\forall \mathbf{z}_i(e_i(\mathbf{z}_i) \equiv A_i)$ otherwise. Then, by the construction of the reduction sentence, we know that $G(e'_1, \ldots, e'_n)$ is also true.

Now we take a sequence of expressions $\langle e_1, \ldots, e_n \rangle$ such that

$$(*) \quad \exists f_\alpha \ldots \exists f_\nu (\forall \mathbf{z}_\alpha(e_\alpha(\mathbf{z}_\alpha) \equiv A'_\alpha) \wedge \cdots \wedge \forall \mathbf{z}_\nu(e_\nu(\mathbf{z}_\nu) \equiv A'_\nu) \wedge G(e'_1, \ldots, e'_n))$$

is true (where e'_i is f_i for all e_i that are not constants). Thus,

$$(**) \qquad \forall \mathbf{z}_\alpha(e_\alpha(\mathbf{z}_\alpha) \equiv A''_\alpha) \wedge \cdots \wedge \forall \mathbf{z}_\nu(e_\nu(\mathbf{z}_\nu) \equiv A''_\nu) \wedge G(h_1, \ldots, h_n)$$

is true where $h_1, \ldots, h_n$ are constants of appropriate types such that h_i is e_i for e_i that are constants and A''_i results from A'_i by replacing f_i with h_i (i.e., (**) is just an instantiation of (*)).

But, by the construction of reduction sentences, if $G(h_1, \ldots, h_n)$ then there are expressions $b_n, \ldots, b_n$ such that

$$\begin{aligned} &\varphi(b_1, \ldots, b_n) \equiv \\ &\quad (\forall \mathbf{z}_\alpha(b_\alpha(\mathbf{z}_\alpha) \equiv B_\alpha) \wedge \cdots \wedge \forall \mathbf{z}_\nu(b_\nu(\mathbf{z}_\nu) \equiv B_\nu) \wedge G(h_1, \ldots, h_n))), \end{aligned}$$

where each $\forall \mathbf{z}_i(b_i(\mathbf{z}_i) \equiv B_i)$ is a reduction sentence. By an easy induction on the complexity of reduction sentences, it can be seen that the only constant in each B_i is h_i. Moreover, every reduct of a propositional function of a given type has the same logical form. Putting these two facts together, we can see that B_i is just A''_i.

Clearly, each $\forall \mathbf{z}_i(e_i(\mathbf{z}_i) \equiv A''_i)$ and $\forall \mathbf{z}_i(b_i(\mathbf{z}_i) \equiv A''_i)$ are themselves reduction formulas. By the inductive hypothesis, every reduction formula of a height lower than the reduction formula for φ is true. Therefore, by standard logical moves, we know that $\forall \mathbf{z}_i(e_i(\mathbf{z}_i) \equiv b_i(\mathbf{z}_i))$ is true.

Moreover, by Theorem 6.3, if $b_1, \ldots, b_n$ and $e_1, \ldots, e_n$ are of the same type, respectively, and are formally equivalent, we know that $\varphi(e_1, \ldots, e_n) \equiv \varphi(b_1, \ldots, b_n)$ is true in the model.

Putting all of this together we get

$$\begin{aligned} &\varphi(e_1, \ldots, e_n) \equiv \\ \exists f_\alpha \ldots \exists f_\nu (\forall \mathbf{z}_\alpha(e_\alpha(\mathbf{z}_\alpha) \equiv A'_\alpha) &\wedge \cdots \wedge \forall \mathbf{z}_\nu(e_\nu(\mathbf{z}_\nu) \equiv A'_\nu) \wedge G(e'_1, \ldots, e'_n)), \end{aligned}$$

for all sequences of expressions of the correct type (where every hereditarily predicative expression is a constant). Therefore, generalizing, every reduction formula is valid. □

Now we know that the axiom of reducibility is valid in our class of models.

Corollary 7.3 *$\exists f^{(\tau_1,\ldots,\tau_n)/1} \forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \equiv f(x_1, \ldots, x_n))$ is valid.*

Proof By Theorem 7.2 and existential generalization. □

8 Concluding Remarks

In this paper I have shown that the axiom of reducibility is valid in the class of frames in which for each homogeneous collection of sequences of entities there is a relation that determines it. There is, however, more work to be done. One important task which we have already mentioned is that we need to extend the current theory to treat relations between individuals and facts and to treat propositional functions—such as truth predicates—that apparently take propositions as arguments. I think both of these tasks can be treated by integrating another Russellian sort of entity—logical forms—into the semantic framework. But this is a large topic and I will leave it to another paper.[9]

Notes

1. *24 in fact says that there is a universal class for every type of expression. This, in effect, means that the elements of a given type constitute a set.

2. Thus we only include singular facts, as opposed to general facts, in our ontology. General facts do appear in *The Philosophy of Logical Atomism* (Russell [15], p. 101). But in *Principia* Russell rejects them: "If φx is an elementary proposition, it is true when it *points to* a corresponding complex. But $(x).\varphi x$ does not point to a single complex: the corresponding complexes are as numerous as all possible values of x" (Whitehead and Russell [17], Volume 1, p. 46).

3. It may be that he changed his mind about this by the time he wrote the second edition of *Principia*, which is the period with which Hazen and Davoren are concerned. Hazen has told me that he does think that Russell had changed his mind on this point between the two editions of *Principia*.

4. Of course, this is a bit misleading. For Russell "Paekakariki" and "Zermela" are not proper names, but we will ignore that point here.

5. Note that the foregoing is meant only to be a motivation for the condition on frames, not a proof that super-duper-Leibnizian frames satisfy this condition.

6. See Lewis [10], p. 145. As Lewis points out, we can always take a thing (or universal) to be its own name. Following Swift, he calls such a language a *Lagadonian language*.

7. Whitehead and Russell give two definitions of "predicativity" in [17]. The first coincides with Church's, namely, "We will define a function as *predicative* when it is the next order above that of its argument…" ([17], Volume 1, p. 53). The other definition (found at [17], *12, p. 164) states that "a function is said to be *predicative* when it is a matrix." A matrix, in Russell's terminology, is an open formula. I cannot make sense of the latter definition, but Landini makes sense of it in his radically different interpretation of *Principia*. See Landini [7], Chapter 10.

8. In a much later work, Russell says: "I suggest that variable propositions are only legitimate when they are an abbreviation for name-variables and relation variables" ([13], p. 199).

9. Church [2], Kaplan [6], and Urquhart [16] think that *Principia* requires propositions. I agree with them insofar as I think that it will take a lot of work and some serious alteration of the theory of ramified theory of types to get rid of propositions.

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