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Hyperimmunity in $2^{\mathbb{N}}$

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Abstract We investigate the notion of hyperimmunity with respect to how it can be applied to Π^0_1 classes and their Muchnik degrees. We show that hyperimmunity is a strong enough concept to prove the existence of Π^0_1 classes with intermediate Muchnik degree—in contrast to Post's attempts to construct intermediate c.e. degrees.

1 Introduction

1.1 Motivation This work is an attempt to develop and explore a computability theory on Π^0_1 classes of $2^{\mathbb{N}}$ in direct analogy to the study of c.e. Turing degrees. The two primary concepts of that study are c.e. subsets of $\mathbb{N}$ and Turing reducibility—we assume the reader to be very familiar with both.

The analogous concepts in $2^{\mathbb{N}}$ that we deal with are Π^0_1 subclasses of $2^{\mathbb{N}}$ and *Muchnik* reducibility. We ask ourselves how concepts developed in the study of c.e. Turing degrees can be profitably applied to our developing understanding of Π^0_1 Muchnik degrees. This paper is meant to be read as much as a suggestion of a course of study as a record of results.

It could, of course, be argued that c.e. subsets of $\mathbb{N}$ are more properly analogous to Σ^0_1 rather than Π^0_1 subclasses of $2^{\mathbb{N}}$. One response to this is that it is really a historical artifact that c.e. (i.e., Σ^0_1) subsets of $\mathbb{N}$ rather than co-c.e. (i.e., Π^0_1) were studied. Indeed, most of the properties of c.e. sets that we are concerned with are usually defined explicitly in terms of their complements. But really the analogy that we draw here is not meant to be exact but rather a guide to research, and often it is where the analogy fails that the real research interest lies.

Π^0_1 subclasses of $2^{\mathbb{N}}$ are already an established and ongoing area of research in computability theory (see, for example, [4]). One fruitful way to conceive of a Π^0_1 class is as the set of paths through some computable binary tree. Muchnik reducibility is a less studied but completely natural concept. Just as Turing reducibility is an

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idea that applies to arbitrary subsets of $\mathbb{N}$, Muchnik reducibility can be applied to arbitrary subsets of $\mathbb{N}^{\mathbb{N}}$. $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Muchnik reducible to $B \subseteq \mathbb{N}^{\mathbb{N}}$ (written $A \leqslant_w B$) if, for all $f \in B$, there is a $g \in A$ such that $f \geqslant_T g$. The w here stands for *weak* (as opposed to *strong* or *Medvedev* reducibility) and Muchnik reducibility is often called *weak reducibility*. The idea is that A and B are the respective sets of solutions to two mathematical (*mass*) problems and every solution to the problem represented by B computes a solution to the problem represented by A. In our case, the problems involved will simply be those of finding paths through given computable trees. Two problems are Muchnik equivalent if any solution to either computes a solution to the other. The resulting structure of Π^0_1 classes modulo Muchnik equivalence is called the *Muchnik lattice* and is denoted $\mathcal{P}_w$. A (Π^0_1) Muchnik *degree* is the equivalence class of some (Π^0_1) subset of $2^{\mathbb{N}}$. Just as for c.e. Turing degrees, the Muchnik lattice has a maximum degree and a minimum degree. The standard representative of the maximum degree is the set

$$\mathrm{DNR}_2 = \{f \in 2^{\mathbb{N}} : \forall n\ f(n) \neq \{n\}(n)\};$$

however, one could also use the set PA of completions of Peano arithmetic. The standard representative of the minimum degree is $2^{\mathbb{N}}$. Following the terminology for the c.e. degrees, any Π^0_1 class representing the maximum degree is called *Muchnik complete*.

Our basic program is to study properties of Π^0_1 classes and to see how this influences their Muchnik degrees. However, we will not be concerned with arbitrary properties of Π^0_1 classes but only those properties that have a strong computability theoretic character, namely, those properties that are preserved by computable permutations of $2^{\mathbb{N}}$ (any such property we refer to as being a *computably topological property* because any computable permutation automatically respects the topology on $2^{\mathbb{N}}$). This is the same criterion we use when we define a computability theoretic property of subsets of $\mathbb{N}$. According to the characterization by Rogers of Klein's program in [11], Chapter 4, this specification of the class of objects studied and the characterization of the type of properties studied specifies a mathematical subject.

In this paper we look at the analogy of Post's problem in $\mathcal{P}_w$. Post was the first to ask if there existed a c.e. set of intermediate Turing degree. He tried to create such a set by describing properties that he hoped would guarantee incompleteness while not requiring computability. Properties such as *immunity* and *hyperimmunity* were tried, along with various others. These properties focused on creating a c.e. set with a sufficiently attenuated complement, but none of these attempts succeeded in describing an intermediate Turing degree and Post's problem was solved later by other methods.

Here we revive Post's method in another context—that of the Muchnik lattice, and here his ideas are a lot more fruitful. We use Post's idea of hyperimmunity and use it to define computable topological properties of Π^0_1 classes. This we do in five different ways to get five distinct properties. These properties are mainly concerned with the nature of the set of *branching nodes* of P. That is the set of binary strings σ with the property that $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ have extensions in P. The nature of this set (and other similar sets) has implications for the Muchnik degree of a Π^0_1 class P. The most straightforward result is that if P has no computable element and the set of branching nodes of P is hyperimmune then P is of intermediate Muchnik degree.

At the same time that we consider Muchnik reducibility, we also look at the stronger idea of *Medvedev* reducibility. This is also applicable to arbitrary subsets of $\mathbb{N}^{\mathbb{N}}$, one class A being Medvedev reducible to another class B if there is a computable functional from B into A. This is the uniform version of Muchnik reducibility. Of the five properties defined, three imply Muchnik incompleteness. The other two properties imply Medvedev incompleteness and, as it is known that there are Π^0_1 classes that are Muchnik complete but not Medvedev complete, this is a weaker result. We conjecture, however, that all five imply Muchnik incompleteness, but this is not established. We also apply the stronger property of *dense immunity* to Π^0_1 classes and try to show where measure and the previously studied property of *thinness* fit into the scheme of things.

These ideas create a panoply of open questions—some of which should be reasonably easy to answer and some of which will probably require significantly different methods than those used here. We end with a section on some directions for further research.

1.2 Basics Most of the notation we use is standard. Novel notation for this paper is introduced in this section. The other material in this section can be found in more detail in [4], [3], [2], or [13].

$2^{\mathbb{N}}$ is the class of infinite binary sequences equipped with the natural product topology making it a totally disconnected Polish space. $2^{<\mathbb{N}}$ denotes the set of all finite binary strings. If $\sigma \in 2^{<\mathbb{N}}$, we denote by U_σ the set $\{f \in 2^{\mathbb{N}} : f \supset \sigma\}$. The collection $\{U_\sigma : \sigma \in 2^{<\mathbb{N}}\}$ forms a basis for the topology on $2^{\mathbb{N}}$. Any finite union of basis elements is clopen. Elements of $2^{<\mathbb{N}}$ will usually be denoted by σ, or τ, and infinite binary sequences by f or g, or X or Y. Subsets of $\mathbb{N}$ will be identified with their characteristic function without further mention. $\sigma\tau$ and $\sigma^\frown f$ will denote the concatenation of σ with τ or f.

All unexplained computability theory terminology and notation is standard and can be found in [14] or [11]. We review the concepts that will be particularly important here.

1. If $X = \{x_0 < x_1 < x_2 \dots\} \subseteq \mathbb{N}$ then the map $i \mapsto x_i$ is called the *principal function of* X and is denoted p_X.
2. If f and g are two functions from $\mathbb{N}$ to $\mathbb{N}$ and for all n, $f(n) \geqslant g(n)$ then f is said to *dominate* g. We say f dominates $X \subseteq \mathbb{N}$ if f dominates the principal function of X.
3. If $X \subseteq \mathbb{N}$ is infinite and p_X is not dominated by any computable function then X is called *hyperimmune*.

There is another useful characterization of hyperimmunity. Every finite subset $F = \{x_0 < x_1 < x_2 < \cdots < x_n\}$ of $\mathbb{N}$ can be indexed canonically by $\prod_{i=0}^n p_i^{x_i}$, where p_i is the ith prime number. D_n will denote the finite set canonically indexed by n. A *strong array* is a sequence of finite sets whose canonical indices are given by a computable function. A *disjoint strong array* is a strong array whose elements are pairwise disjoint. $\langle D_{f(n)}\rangle$ will denote a strong array with computable indexing function f.

A well-known theorem (Kuznecov, Medvedev, Uspenski, [14], V.2.3) states that $X \subseteq \mathbb{N}$ is hyperimmune if and only if there is no disjoint strong array $\langle D_{f(n)}\rangle$

such that, for all n, $D_{f(n)} \cap X \neq \varnothing$. This is actually used as the definition of hyperimmunity and the equivalence to our text definition is the theorem.

A *tree* is a subset of $2^{<\mathbb{N}}$ that is closed under taking initial segments. The elements of a tree are called *nodes*. A tree is *computable* precisely when its set of nodes is. A *path* through a tree T is an element f of $2^{\mathbb{N}}$ such that, for all n, $f|_n \in T$. A Π^0_1 *class* is the set of paths through some infinite computable tree. We will thus always assume that Π^0_1 classes are nonempty. If T is a computable tree the associated Π^0_1 class will be denoted $[T]$. If T is a tree and $\sigma \in T$ has the property that there exists $f \in [T]$ such that $f \supset \sigma$, then σ is called *extendible*. The set of extendible nodes of T is denoted $\mathrm{Ext}(T)$. Similarly, if P is a Π^0_1 class and T any tree such that $P = [T]$, then by $\mathrm{Ext}(P)$ we mean $\mathrm{Ext}(T)$ (it is not hard to check that this is well-defined).

In general, computable trees will have nonextendible nodes but in constructing a Π^0_1 class we can view it as a nested computable intersection of trees with no nonextendible nodes. In other words, a Π^0_1 class is the set of paths through some co-c.e. tree that has no nonextendible nodes.

It is also very useful to view Π^0_1 classes syntactically. $P \subseteq 2^{\mathbb{N}}$ is a Π^0_1 class if and only if for some computable predicate $R \subseteq \mathbb{N} \times 2^{\mathbb{N}}$,

$$P = \{f : \forall n \; R(n, f)\}.$$

The equivalence of these different ideas is set out in detail in [4].

We introduce some notation and definitions that will be useful. Throughout, P is a Π^0_1 class, σ an element of $2^{<\mathbb{N}}$, X an element of $2^{\mathbb{N}}$, and T a tree.

Notation

1. $P(\sigma) = \{f \in P : f \supset \sigma\}$.
2. If $n \leqslant |\sigma|$, $\sigma[n] = \sigma|_n = \langle \sigma(0), \sigma(1), \dots, \sigma(n-1)\rangle$ (with $\sigma[0] = \varnothing$),
   $$f[n] = f|_n = \langle f(0), f(1), \dots f(n-1)\rangle,$$
   $$P[n] = \{f[n] : f \in P\} = \{\sigma \in \mathrm{Ext}(P) : |\sigma| = n\}.$$
3. $\{e\}^X[n]$ is the partial sequence $\langle x_i\rangle_{i=0}^{n-1}$ where $x_i = \{e\}^X(i)$ whenever the right-hand side is defined, and undefined otherwise. In particular, $\{e\}^X[n] \in T$ implies $\{e\}^X(m)\downarrow$ for all $m \leqslant n-1$. As above, $\{e\}^X[0] = \varnothing$.
4. We will be particularly concerned with a subset of the extendible nodes of P—namely, the *branching nodes of* P. σ is a branching node if $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ are both in $\mathrm{Ext}(P)$. The set of branching nodes of P is denoted $\mathrm{Br}(P)$. If $X \in P$, then by $\mathrm{Br}_X(P)$ we mean the set $\{n \in \mathbb{N} : X|_n \in \mathrm{Br}(P)\}$. The concept of a branching node can also be applied to any subset of $2^{\mathbb{N}}$.
5. The *set of branching levels* of a Π^0_1 class P is the set
   $$\mathrm{Brl}(P) = \{n : \exists \sigma \in \mathrm{Br}(P),\ |\sigma| = n\}.$$
6. An important type of Π^0_1 class is a separating class. If $A, B \subseteq \mathbb{N}$ are disjoint c.e. sets, then the *separating class of A and B*, denoted $\mathcal{S}(A, B)$, is the set
   $$\{f \in 2^{\mathbb{N}} : \forall n[(n \in A \Rightarrow f(n) = 1) \text{ and } (n \in B \implies f(n) = 0)]\}.$$
   It is straightforward to show using the syntactical viewpoint above that $\mathcal{S}(A, B)$ is a Π^0_1 class.

1.3 The Muchnik lattice of Π^0_1 classes If $A, B \subseteq 2^{\mathbb{N}}$, then *$A$ is Muchnik reducible to B* (sometimes called *weakly reducible*), written $A \leqslant_w B$ if

$$\forall y \in B \exists x \in A\big[y \geqslant_T x\big].$$

If $A \leqslant_w B$ and $B \leqslant_w A$, then we write $A \equiv_w B$ and say A and B are *Muchnik equivalent*. The relation $\leqslant_w$ is a pre-order on $2^{\mathbb{N}}$ and it can be made into a partial order in the familiar way.

The *Muchnik degree* of $A \subseteq 2^{\mathbb{N}}$ is the set

$$\deg_w(A) = \{B \subseteq 2^{\mathbb{N}} : B \equiv_w A\}.$$

$\deg_w(A) \leqslant \deg_w(B)$ if $A \leqslant_w B$. This relation is now a partial order on the collection of Muchnik degrees.

If $\mathbb{P}$ is the collection of nonempty Π^0_1 subclasses of $2^{\mathbb{N}}$, then the structure

$$\mathcal{P}_w = \langle\{\deg_w(P) : P \in \mathbb{P}\}, \leqslant\rangle$$

we call the *Muchnik lattice*. To show it is in fact a lattice it is necessary to demonstrate that every two Muchnik degrees have an infimum and supremum. They are as follows. If $P, Q \in \mathbb{P}$ then define

$$P \vee Q = \{f \oplus g : f \in P \text{ and } g \in Q\},$$
$$P \wedge Q = \{\langle 0\rangle^\frown f : f \in P\} \cup \{\langle 1\rangle^\frown g : g \in Q\},$$

and then

$$\deg_w(P) \vee \deg_w(Q) = \deg_w(P \vee Q),$$
$$\deg_w(P) \wedge \deg_w(Q) = \deg_w(P \wedge Q).$$

These operations in $\mathcal{P}_w$ are distributive over each other as can easily be confirmed. Furthermore, $\mathcal{P}_w$ has maximum and minimum elements denoted $\mathbf{1}_w$ and $\mathbf{0}_w$, respectively. Any Π^0_1 class with a computable element is a representative of $\mathbf{0}_w$. One representative of the maximum Muchnik degree is

$$\mathrm{DNR}_2 = \{f \in 2^{\mathbb{N}} : \forall n\ [\{n\}(n) \neq f(n)]\}.$$

This is not immediately obvious but it is proved in [13].

A similar reducibility relation on $\mathbb{P}$ is called *Medvedev reducibility* (sometimes *strong* reducibility). If $P, Q \in \mathbb{P}$ and if there is a computable functional $\Phi : P \longrightarrow Q$, then Q is said to be *Medvedev reducible* to P, written $P \geqslant_M Q$. This gives rise in the same manner as above to the *Medvedev lattice $\mathcal{P}_M$* which is also distributive. If $P, Q \in \mathbb{P}$, then $P \vee Q$ and $P \wedge Q$ are also representatives of the supremum and infimum of their Medvedev degrees. Furthermore, DNR_2 is a representative of $\mathbf{1}_M$. Any such representative is called Medvedev (respectively, Muchnik) complete.

2 Five Computably Topological Properties

We now define the five properties mentioned in the introduction and prove that they are invariant under computable homeomorphisms.

Definition 2.1 A Π^0_1 P class is *small* if $\mathrm{Br}(P)$ is hyperimmune.

Definition 2.2 A Π^0_1 class P is *pathwise hyperimmune (p.h.i.)* if, for some $X \in P$, $\mathrm{Br}_X(P)$ is hyperimmune.

Definition 2.3 A Π^0_1 class is *everywhere pathwise hyperimmune (e.p.h.i.)* if, for all $X \in P$, $\mathrm{Br}_X(P)$ is hyperimmune.

Definition 2.4 A Π^0_1 class is *uniformly pathwise hyperimmune (u.p.h.i.)* if there is no computable function φ such that for all $X \in P$, φ dominates $\mathrm{Br}_X(P)$.

There is a direct counterpart in $2^{\mathbb{N}}$ to the notion of disjoint strong array. If D_n is a finite set of (the Gödel numbers of) finite binary strings, then we define

$$\begin{aligned} D_n^* &= \{g \in 2^{\mathbb{N}} : \exists \sigma \in D_n \; g \supset \sigma\} \\ &= \bigcup \{U_\sigma : \sigma \in D_n\}. \end{aligned}$$

n is then the *canonical index* of the clopen set D_n^*.

We now define a property most directly analogous to the property of hyperimmunity of subsets of $\mathbb{N}$.

Definition 2.5 A Π^0_1 class P is *hyperimmune (h.i.)* if there is no disjoint strong array $\langle D^*_{f(n)} \rangle$ such that, for all n, $P \cap D^*_{f(n)} \neq \varnothing$.

To further emphasize the relatedness of hyperimmunity in $2^{\mathbb{N}}$ and hyperimmunity in $\mathbb{N}$, we make the following observation. If f is a computable function, then we call $\langle D_{f(n)} \rangle$ an *incomparable strong array* if, for all n, $D_{f(n)} \subseteq 2^{<\mathbb{N}}$ and, for all $\sigma, \tau \in \bigcup_n D_{f(n)}$, if $\sigma \neq \tau$, then σ and τ are incomparable.

We make the following definition now which will be useful later on.

Definition 2.6 If $C \subseteq 2^{\mathbb{N}}$ is clopen, then the *root set of* C, $\mathrm{rt}(C)$, is the unique finite subset of $\mathrm{Br}(C)$ of smallest cardinality such that $C = \{f \in 2^{\mathbb{N}} : \exists \sigma \in \mathrm{rt}(C) \; f \supset \sigma\} = \bigcup \{U_\sigma : \sigma \in \mathrm{rt}(C)\}$.

Theorem 2.7 *P is an h.i. Π^0_1 class if and only if there is no incomparable strong array $\langle D_{f(n)} \rangle$ such that, for all n,*

$$\mathrm{Ext}(P) \cap D_{f(n)} \neq \varnothing.$$

Proof This is straightforward using the fact that two clopen subsets of $2^{\mathbb{N}}$ are disjoint if and only if their two root sets are pairwise incomparable. □

The property of smallness has other quite natural characterizations as shown in the next theorem. The two following definitions will be useful.

Definition 2.8 If P is a perfect closed subset of $2^{\mathbb{N}}$, then let Φ_P be the canonical, order-preserving map from $2^{<\mathbb{N}}$ onto $\mathrm{Br}(P)$. That is,

$$\begin{aligned} \Phi_P(\varnothing) &= \text{the unique element of } \mathrm{Br}(P) \text{ of minimum length}, \\ \Phi_P(\sigma^\frown\langle 0\rangle) &= \text{the unique element of } \mathrm{Br}(P) \text{ of minimum length extending } \Phi_P(\sigma)^\frown\langle 0\rangle, \\ \Phi_P(\sigma^\frown\langle 1\rangle) &= \text{the unique element of } \mathrm{Br}(P) \text{ of minimum length extending } \Phi_P(\sigma)^\frown\langle 1\rangle. \end{aligned}$$

Theorem 2.9 *The following are equivalent:*

1. *P is small,*
2. *the function from $\mathbb{N}$ to $\mathbb{N}$ given by*

$$n \mapsto \min\{|\Phi_P(\sigma)| : |\sigma| = n\}$$

   is not dominated by any computable function,
3. $\mathrm{Brl}(P)$ *is hyperimmune*
   (recall that, $\mathrm{Brl}(P) = \{n : \exists \sigma \in \mathrm{Br}(P) \; |\sigma| = n\}$*),*

4. *there is no computable function* f *such that* $\forall n \, \|P[f(n)]\| \geqslant n$,
5. *there is no computable function* f *such that*

$$\forall n \exists \sigma \in \mathrm{Br}(P) \left[f(n) \leqslant |\sigma| < f(n+1) \right].$$

Proof The proofs of most of the above can be found in [1]. The remaining part is straightforward. □

Uniform pathwise hyperimmunity also has the following alternative characterizations which emphasize its relationship to smallness.

Theorem 2.10 *The following are equivalent:*

1. P *is u.p.h.i.,*
2. *the function from* $\mathbb{N}$ *to* $\mathbb{N}$ *given by*

$$n \mapsto \max(\{|\Phi_P(\sigma)| : |\sigma| = n\})$$

is not dominated by any computable function (compare 2.9(2)*),*

3. *there is no computable function* f *such that*

$$\forall n \forall \sigma \in P[f(n)] \exists \tau \in \mathrm{Br}(P) \left[\tau \supseteq \sigma \text{ and } |\tau| < f(n+1) \right]$$

(compare 2.9(5)*).*

Proof The proofs are similar to the proofs of Theorem 2.9. □

It will also be useful to note the following characterization of e.p.h.i. and p.h.i. analogous to Theorem 2.9(5).

Theorem 2.11 *A* Π^0_1 *class* P *is p.h.i. (e.p.h.i.) if and only if for some (all)* $X \in P$ *there is no strictly increasing computable function* f *such that for all n there is an m such that* $f(n) \leqslant m < f(n+1)$ *and* $X|_m \in \mathrm{Br}(P)$.

Proof See [1], Theorem 2.27($\Leftarrow$). □

We show in Section 4 that e.p.h.i. Π^0_1 classes are necessarily Muchnik incomplete, and it follows immediately that small Π^0_1 classes are also Muchnik incomplete. U.p.h.i., p.h.i., and h.i. classes will be shown to be Medvedev incomplete in the same section. Actually Medvedev incompleteness is relatively straightforward to prove, as Simpson has shown (see Lemma 4.2) that any two Medvedev complete Π^0_1 classes are computably homeomorphic. We show in Section 3 that all five of the previous properties are invariant under computable homeomorphisms, so to show, for example, that all p.h.i. classes are Medvedev incomplete it is enough to show that some representative of the maximum Medvedev degree is not p.h.i. This we do for DNR_2.

2.1 Invariance under computable homeomorphisms Now that these properties are defined, we will prove that they are all computable topological properties.

Theorem 2.12 *Smallness is a computably topological property.*

This was proved in [3]. In fact, the stronger result was proved that if P and Q are Π^0_1 classes and $\{e\} : P \longrightarrow Q$ is surjective, then if P is small so is Q. This stronger property is shared by h.i. Π^0_1 classes but for p.h.i., e.p.h.i., and u.p.h.i. Π^0_1 classes injectivity seems to be needed. The necessity of injectivity in these cases has yet to be established however.

Theorem 2.13 *Hyperimmunity is a computably topological property.*

Proof Suppose $P, Q \subseteq 2^{\mathbb{N}}$ are Π^0_1, $\{e\} : P \longrightarrow Q$ is a computable surjection and Q is not hyperimmune. Let $\langle D^*_{f(n)}\rangle$ witness this last fact. We will find a computable function g so that $\langle D^*_{g(n)}\rangle$ witnesses the fact that P is not hyperimmune.

We first define two functions that will be useful later. Let $n \in \mathbb{N}$ and $n \mapsto l_n = l$ and $n \mapsto t_n = t$ be two functions with the property that $\forall \tau \in P_t[l]\ \{e\}^\tau_t[n] \in Q_t[n]$. Such numbers l and t exist because P is compact. Thus l and t can be found by a computable search, and the functions $n \mapsto l_n$ and $n \mapsto t_n$ can be taken to be computable. We also assume for later purposes that l_n is strictly increasing.

Now let $m = m(n) = \max\{|\sigma| : \sigma \in D_{f(n)}\}$; $l = l_m, t = t_m$, and let $D_{g(n)} = \{\tau \in P_t[l] : \exists \sigma \in D_{f(n)}\ \{e\}^\tau_t[m] \supseteq \sigma\}$. g is computable and $\langle D^*_{g(n)}\rangle$ is pairwise disjoint as $\langle D^*_{f(n)}\rangle$ is. $\square$

This next lemma is key to a lot of what follows.

Lemma 2.14 *Suppose that $P, Q \subseteq 2^{\mathbb{N}}$ are Π^0_1 classes and that $\Phi : P \to Q$ is a computable homeomorphism. Let $X \in P$ and $Y = \Phi(X)$. Suppose $f \in \mathbb{N}^{\mathbb{N}}$ is strictly increasing and such that*

$$\forall n\ \mathrm{Br}_Y[f(n), f(n+1)) \neq \varnothing.$$

Then there is a strictly increasing $g \leqslant_T f$ such that

$$\forall n\ \mathrm{Br}_X[g(n), g(n+1)) \neq \varnothing.$$

Proof Let T_P and T_Q be two computable trees such that $[T_P] = P$ and $[T_Q] = Q$. As Φ is a computable homeomorphism, there is a total computable function $\varphi : T_P \to 2^{<\mathbb{N}}$ such that

1. $\forall \sigma \in \mathrm{Ext}(P)\ \left[\varphi(\sigma) \in \mathrm{Ext}(Q)\right]$,
2. $\forall \sigma, \tau \in T_P\ \left[\sigma \subseteq \tau \implies \varphi(\sigma) \subseteq \varphi(\tau)\right]$,
3. $\forall X \in P\ \left[\bigcup_n \varphi(X[n]) = \Phi(X)\right]$,
4. $\forall X, Y \in P\ \left[X \neq Y \implies \exists n \in \mathbb{N}\ [\varphi(X[n]) \neq \varphi(Y[n])]\right]$,
5. $\forall \tau \in \mathrm{Ext}(Q) \exists \sigma \in \mathrm{Ext}(P)\ \left[\varphi(\sigma) \supseteq \tau\right]$.

φ will not be unique, but if e is an index for Φ, then φ can be taken to be the function $\sigma \mapsto \{e\}^\sigma$, where $\{e\}^\sigma$ is the longest total string of the form

$$\langle \{e\}^\sigma_{|\sigma|}(0), \{e\}^\sigma_{|\sigma|}(1), \{e\}^\sigma_{|\sigma|}(2), \ldots, \{e\}^\sigma_{|\sigma|}(m)\rangle$$

with $m < |\sigma|$. There will be a computable function satisfying the first three properties for any computable functional from P to Q. Property 4 will be true if the functional is one-to-one, and property 5 is true for surjections.

For any $n \in \mathbb{N}$, let $l_n \in \mathbb{N}$ have the property $\forall \sigma \in T_P[l_n]\ |\varphi(\sigma)| \geqslant n$. To see that such an l_n must exist, notice that by compactness there is a $\lambda_n \in \mathbb{N}$ such that $\forall \sigma \in P[\lambda_n]\ |\varphi(\sigma)| \geqslant n$. Now take $l_n \geqslant \lambda_n$ such that $\forall \sigma \in T_P[l_n]\ \sigma[\lambda_n] \in P[\lambda_n]$. That is, all nonextendable nodes of length less than or equal to λ_n have terminated by level l_n. As T_P is computable, we can take the function $n \mapsto l_n$ to be computable.

Now suppose f is given as in the statement of the theorem. We proceed by induction to compute a g from f with the required property. Let $g(0) = l_{f(0)}$. Suppose now that we have computed $g(k)$ for all $k \leqslant n$. For all $k \in \mathbb{N}$, $g(k)$ will be chosen to be of the form $l_{f(k')}$ for some $k' \in \mathbb{N}$. In particular, let $g(n) = l_{f(n')}$. We now use f to choose $g(n+1)$ of the form $l_{f(m)}$ for some $m > n'$. For any $k \in \mathbb{N}$ and $\sigma \in P[l_{f(k)}]$, we define the set

$$S_k(\sigma) = \{\tau \in P[l_{f(k)}] : \varphi(\tau)[f(k)] = \varphi(\sigma)[f(k)] \in Q[f(k)]\}.$$

It is immediate, of course, that $\sigma \in S_k(\sigma)$.

Fix $X \in P$ and $Y = \Phi(X)$ and consider the set $S_{n'}(X[g(n)])$. The argument divides into three different cases, and it will be in general impossible to f-effectively decide which case pertains. However, it will be shown that there is a uniformly f-effective choice for $g(n+1)$ which will suffice for all three cases. That is, we can choose $g(n+1)$ so that X has a branching node in the interval $[g(n), g(n+1))$.

Case 1 $S_{n'}(X[g(n)]) = \{X[g(n)]\}$. Consider $f(n'+1)$. As there is a branching node on Y in the interval $[f(n'), f(n'+1))$, there must be two distinct extensions τ_0 and τ_1 of $Y[f(n')]$ in $Q[f(n'+1)]$. By properties 2 and 5 of φ, there are two incomparable elements in $\mathrm{Ext}(P)$ that get mapped via φ to extensions of τ_0 and τ_1. Hence, by the definition of l_n, there are two distinct elements of $P[l_{f(n'+1)}]$ that get mapped via φ to extensions of τ_0 and τ_1. Call two such elements γ_0 and γ_1. As τ_0 and τ_1 both extend $Y[f(n')]$ we must have that, for $i \in \{0, 1\}$,

$$\varphi(\gamma_i[g(n)]) \supseteq Y[f(n')]$$

(as each $\varphi(\gamma_i[g(n)])$ must extend some element of $Q[f(n')]$, and if it did not extend $Y[f(n')]$, then $\varphi(\gamma_i)$ would not extend $Y[f(n')]$).

But then $\gamma_i[g(n)] \in S_{n'}(X[g(n)])$ for each $i \in \{0, 1\}$ (as $\varphi(X[g(n)]) \supseteq Y[f(n')]$). As $S_{n'}(X[g(n)])$ has only one element, both γ_0 and γ_1 extend $X[g(n)]$. As both are elements of $P[l_{f(n'+1)}]$, X must have a branching node in the interval $[g(n), l_{f(n'+1)})$. Hence $g(n+1) = l_{f(n'+1)}$ suffices for this case.

Case 2 $S_{n'}(X[g(n)]) \supsetneq \{X[g(n)]\}$. We claim that there must exist an $n'' > n'$ such that for all elements σ_0, σ_1 of $T_P[l_{f(n'')}]$ if $\sigma_0[g(n)] \neq \sigma_1[g(n)]$, then $\varphi(\sigma_0)[f(n'')] \neq \varphi(\sigma_1)[f(n'')]$. To see this, first notice that, because Φ is one-to-one, for all distinct $\tau_0, \tau_1 \in P[g(n)]$,

$$\forall X_0 \in P(\tau_0) \forall X_1 \in P(\tau_1) \exists k \left[\varphi(X_0[l_{f(k)}])[f(k)] \neq \varphi(X_1[l_{f(k)}])[f(k)]\right].$$

Using the compactness of $P(\tau_0)$ and $P(\tau_1)$ we get, for all distinct $\tau_0, \tau_1 \in P[g(n)]$,

$$\exists k \forall X_0 \in P(\tau_0) \forall X_1 \in P(\tau_1) \left[\varphi(X_0[l_{f(k)}])[f(k)] \neq \varphi(X_1[l_{f(k)}])[f(k)]\right].$$

If we then let k' be the maximum of such ks over all distinct pairs $\tau_0, \tau_1 \in P[g(n)]$, we have (for all such τ_0, τ_1)

$$\forall X_0 \in P(\tau_0) \forall X_1 \in P(\tau_1) \left[\varphi(X_0[l_{f(k')}])[f(k')] \neq \varphi(X_1[l_{f(k')}])[f(k')]\right].$$

Finally, we take $n'' \geqslant k'$ to be such that $\forall \sigma \in T_P[l_{f(n'')}]\ \sigma[l_{f(k')}] \in P[l_{f(k')}]$. Then

$$\forall \sigma_0, \sigma_1 \in T_P[l_{f(n'')}] \left[\sigma_0[g(n)] \neq \sigma_1[g(n)] \rightarrow \varphi(\sigma_0)[f(n'')] \neq \varphi(\sigma_1)[f(n'')]\right]$$

as required. Furthermore, we can then find such an n'' f-computably as T_P is computable.

Case 2 now divides into two separate cases.

Case 2a $S_{n'}(X[l_{f(n'')}]) = \{X[l_{f(n'')}]\}$. This is essentially Case 1 with $l_{f(n'')}$ taking the place of $l_{f(n')} = g(n)$. As argued in Case 1 there must be a branching node on X between $l_{f(n'')}$ and $l_{f(n''+1)}$ and hence between $g(n)$ and $l_{f(n''+1)}$. Thus we choose $g(n+1) = l_{f(n''+1)}$.

Case 2b $S_{n'}(X[l_{f(n'')}]) \supsetneq \{X[l_{f(n'')}]\}$. Let γ_0 and γ_1 be two distinct elements of $S_{n'}(X[l_{f(n'')}])$ with $\gamma_0 = X[l_{f(n'')}]$. Therefore, $\varphi(\gamma_0)[f(n'')] = \varphi(\gamma_1)[f(n'')] = \varphi(X[l_{f(n'')}])[f(n'')]$. We choose n'' so that this would imply that both γ_i extend $X[g(n)]$. Thus X has a branching node above $g(n)$ and below $l_{f(n'')}$. Thus we can choose $g(n+1) = l_{f(n'')}$.

As $n'' > n'$, the choice of $g(n+1) = l_f(n''+1)$ suffices for all three cases. □

We are particularly interested in the situation when f in the previous lemma is computable. This gives immediately the following.

Theorem 2.15 *E.p.h.i., p.h.i., and u.p.h.i. are all computably topological properties.*

Proof Suppose P and Q are computably homeomorphic Π^0_1 classes. If $Y \in Q$ and f a computable function such that

$$\forall n\ \mathrm{Br}_Y(Q) \cap [f(n), f(n+1)) \neq \varnothing.$$

Then Lemma 2.14 constructs a g, also computable, such that

$$\forall n\ \mathrm{Br}_X(P) \cap [g(n), g(n+1)) \neq \varnothing,$$

where X is the pre-image of Y under the homeomorphism. This proves (using Theorems 2.10 and 2.11) that if Q is not e.p.h.i., (p.h.i., u.p.h.i.) then neither is P. □

2.2 Lattice operations The final lemmas in this section will be useful later on for constructing Π^0_1 classes with required properties.

Theorem 2.16 (see [1], Theorem 2.27) *If $P, Q \subseteq 2^\omega$ are Π^0_1, then $P \vee Q$ is small if and only if $P \wedge Q$ is small if and only if both P and Q are small.*

Lemma 2.17 *If $X \subseteq \omega$ and $Y \subseteq \omega$ are co-c.e., then $X \oplus Y = \{2x : x \in X\} \cup \{2x+1 : x \in Y\}$ is hyperimmune if and only if both X and Y are.*

Proof If X or Y were not h.i., it would be straightforward to construct a disjoint strong array witnessing the fact that $X \oplus Y$ were not h.i. So suppose that $X \oplus Y$ was not h.i. Let f be a computable function such that for all n $D_{f(n)} \cap (X \oplus Y) \neq \varnothing$. Let $\left(D_{f(n)}\right)_0 = \{m/2 : m \text{ is even and } m \in D_{f(n)}\}$, and let $\left(D_{f(n)}\right)_1 = \{(m-1)/2 : m \text{ is odd and } m \in D_{f(n)}\}$. For every n either $\left(D_{f(n)}\right)_0 \cap X \neq \varnothing$ or $\left(D_{f(n)}\right)_1 \cap Y \neq \varnothing$. Therefore, if for infinitely many n, $\left(D_{f(n)}\right)_0 \cap X = \varnothing$, then for infinitely many n, $\left(D_{f(n)}\right)_1 \cap Y \neq \varnothing$, and an infinite sequence of such ns could be computed (because X is co-c.e.), contradicting the hyperimmunity of Y. So for some N, and for all $n \geqslant N$, $\left(D_{f(n)}\right)_0 \cap X \neq \varnothing$, contradicting the hyperimmunity of X. □

Theorem 2.18 *If P and Q are Π^0_1 classes and both are e.p.h.i. (p.h.i., u.p.h.i., h.i.), then so is $P \wedge Q$.*

Proof The proofs for e.p.h.i., p.h.i., and u.p.h.i. are very straightforward. The proof for h.i. is analogous to the proof of Lemma 2.17. □

Theorem 2.19 *If $P \wedge Q$ is e.p.h.i. (h.i.), then so are P and Q. This is not the case for u.p.h.i. and p.h.i.*

Proof The first part is immediate. The second is done by noticing that $S \wedge 2^{\mathbb{N}}$ is both u.p.h.i. and p.h.i. if S is small. $\square$

Theorem 2.20 *If S is a small Π^0_1 class and P, a u.p.h.i. Π^0_1 class, then $S \vee P$ is u.p.h.i.*

Proof We show the contrapositive. Let f be a computable even-valued function witnessing the fact that $S \vee P$ is not u.p.h.i. Then for all n and $\sigma \oplus \tau \in (S \vee P)[f(n)]$ and for all $X \oplus Y \in (S \vee P)(\sigma \oplus \tau)$ either $\mathrm{Br}_X(S) \cap \big[f(n)/2, f(n+1)/2\big) \neq \varnothing$ or $\mathrm{Br}_Y(P) \cap \big[f(n)/2, f(n+1)/2\big) \neq \varnothing$. As S is a small Π^0_1 class, there must be an infinite computable set $\{n_i : i \in \mathbb{N}\}$ such that, for all i, $\mathrm{Brl}(S) \cap \big[f(n)/2, f(n+1)/2\big) = \varnothing$. Therefore, for all i and for all $\sigma \in P[f(n_i)]$, there is a $\tau \supseteq \sigma$ such that $\tau \in \mathrm{Br}(P)$ and $|\tau| \leqslant f(n+1)/2$. Hence $f(n_i)/2$ witnesses the fact that P is not u.p.h.i., contradiction. $\square$

Theorem 2.21 *If P and Q are Π^0_1, then P and Q are (e.)p.h.i. if and only if $P \vee Q$ is.*

Proof Straightforward using Lemma 2.17. $\square$

Theorem 2.22 *If $P \vee Q$ is h.i., then both P and Q are.*

Proof Without losing generality, assume that P is not h.i. If f is computable and if $\forall n\ D^*_{f(n)} \cap P \neq \varnothing$, then $\forall n\ (D^*_{f(n)} \vee 2^{\mathbb{N}}) \cap P \vee Q \neq \varnothing$. $\square$

The converse to the previous theorem has not been proved. It is analogous to the theorem that the disjoint union of two co-c.e. and hyperimmune subsets of $\mathbb{N}$ is hyperimmune. We conjecture that it is false in this context. We also conjecture that the join of two u.p.h.i. Π^0_1 classes is not necessarily u.p.h.i.

2.3 Comparisons to measure and each other In [3] it is shown that all small Π^0_1 classes have measure zero. Here we improve this result to show that it holds for all e.p.h.i. classes.

Theorem 2.23 *If P is an e.p.h.i. Π^0_1 class then $\mu(P) = 0$.*

Proof Suppose P is a Π^0_1 class and $\mu(P) > 0$. We will describe an $X \in P$ and a computable function f with f dominating $\mathrm{Br}_X(P)$. Let k be the least positive integer such that $1/2^k < \mu(P)$. f will be the function $n \mapsto n \cdot k$.

There must be at least two extendible nodes on $P[k]$ (or else $1/2^k \geqslant \mu(P)$) and so there must be a branching node of length strictly less than k. There also must exist a $\sigma \in P[k]$ such that $2^k \cdot \mu(P(\sigma)) \geqslant \mu(P)$. Let this σ be σ_1. Iterating the process, there is a $\sigma \in P[(n+1) \cdot k]$ extending σ_n such that $2^k \cdot \mu(P(\sigma)) \geqslant \mu(P(\sigma_n))$. Let this σ be σ_{n+1}. As above, there must be a branching between σ_n and σ_{n+1}.

Then $X = \bigcup_{n=1}^{\infty} \sigma_n$ and $f(n) = n \cdot k$ are as required. $\square$

However, u.p.h.i. Π^0_1 classes need not have measure zero. For example, if S is small and $\mu(Q) > 0$, then $S \wedge Q$ is u.p.h.i. and $\mu(S \wedge Q) > 0$. H.i. Π^0_1 classes are also not necessarily of measure zero as the following shows.

Theorem 2.24 (Simpson) *Every Π^0_1 class of positive measure contains an h.i. Π^0_1 class of positive measure.*

Proof Suppose $P \subseteq 2^{\mathbb{N}}$ is Π^0_1 and $\mu(P) \geqslant m > 0$ for some computable real m. We will diagonalize against the class of disjoint strong arrays to create an h.i. subclass. Let d be a partial computable function such that, for a given $e \in \mathbb{N}$, $\mu(D^*_{\{e\}(d(e))}) < m/2^{e+1}$. $d(e)$ is defined if (but not only if) the range of $\{e\}$ is infinite. Let $P' = P \setminus \bigcup_{e\in\mathbb{N}} D^*_{\{e\}(d(e))}$. P' is Π^0_1 as $\bigcup_{e\in\mathbb{N}} D^*_{\{e\}(d(e))}$ is Σ^0_1 and it has positive measure because

$$\mu(\bigcup_{e\in\mathbb{N}} D^*_{\{e\}(d(e))}) \leqslant \sum_{e\in\mathbb{N}} m/2^{e+1} \leqslant m/2 < m.$$

It is h.i. because for all e, $D^*_{\{e\}(d(e))} \cap P' = \varnothing$. □

Theorem 2.25 *Small ⇒ e.p.h.i. ⇒ p.h.i. ⇒ u.p.h.i.*

Proof From the definitions it is clear that e.p.h.i. ⇒ p.h.i. ⇒ u.p.h.i. For the first implication suppose P were Π^0_1 and not e.p.h.i.—witnessed by $X \in P$ and computable function f dominating $\mathrm{Br}_X(P)$. $\mathrm{Br}_X(P) \subseteq \mathrm{Brl}(P)$ and so f also dominates $\mathrm{Brl}(P)$. Therefore, P is not small. □

Theorem 2.26 *U.p.h.i. $\not\Rightarrow$ p.h.i.*

Proof We denote by 1^n and 0^n the strings of n ones and zeroes, respectively, with the understanding that $1^0 = 0^0 = \varnothing$. Let f be the principal function of some hyperimmune Π^0_1 subset of $\mathbb{N}$. Let T be the tree generated by the set $\{0^i{}^\frown 1^{f(i)+1}{}^\frown\gamma : i \in \mathbb{N}, \gamma \in 2^{<\mathbb{N}}\}$ and let $P = [T]$. P is Π^0_1 by inspection. For every $X \in P$, $\mathrm{Br}_X(P)$ is cofinite, so P is clearly not p.h.i. However, if Φ_P is the function from Definition 2.8, then for every $n > 0$, $\max\{|\Phi_P(\sigma)| : |\sigma| = n\} \geqslant f(n-1)$ which is not dominated by any computable function. So P is u.p.h.i. □

The P constructed in the previous theorem has a computable path (namely, 0^∞) and so has trivial Muchnik degree. As we will be interested in the Muchnik degrees of the Π^0_1 classes we create, this could be a problem; however, as the next theorem shows, we needn't worry.

Theorem 2.27 *There exists a Π^0_1 class with no computable path (and hence perfect) that is u.p.h.i. but not p.h.i.*

Proof Let P be as constructed in the previous theorem, and let S be a small Π^0_1 class with no computable path. Then Lemma 2.20 says that $P \vee S$ will be u.p.h.i. and Theorem 2.21 says that it will not be p.h.i. □

Theorem 2.28 *P.h.i. $\not\Rightarrow$ e.p.h.i. and h.i. $\not\Rightarrow$ e.p.h.i.*

Proof Any p.h.i. or h.i. class of positive measure illustrates this. □

The following is based on an unpublished construction by Lerman.

Theorem 2.29 *E.p.h.i. $\not\Rightarrow$ h.i.*

Proof We construct an e.p.h.i. class P which is not h.i. by describing a computable sequence T_s of nested computable trees such that $T = \bigcap_s T_s$ and $P = [T]$. P will be countable with exactly one nonisolated path X. We will find a perfect Π^0_1 class with the required properties in a corollary. We adopt the 0^n notation from Theorem 2.26.

To build T_s, we construct a sequence of natural numbers $0 = l_0 \leqslant l_1 \leqslant l_2 \cdots$ with $\lim_{s\to\infty} l_s = \infty$. At each stage s we have

(i) $T_s[l_s] = T[l_s]$,
(ii) $\forall\sigma \in T_s[l_s],\ [\tau \supseteq \sigma \implies \tau \in T_s]$.

To ensure that T has the required properties, we construct, concurrently with T_s, two double sequences of nonnegative integers,

$$e_{0,s} < e_{1,s} < \cdots < e_{n_s,s}$$

and

$$u_{0,s}, u_{1,s}, \ldots u_{n_s,s},$$

with the following properties:

A1 $\lim_s n_s = \infty$.
A2 $\forall i$ $\lim_s u_{i,s}$ and $\lim_s e_{i,s}$ exist and are denoted u_i and e_i.
A3 The unique nonisolated path of P is

$$X = 1^{e_0}{}^\frown 0^{u_0}{}^\frown 1^{e_1}{}^\frown 0^{u_1}{}^\frown \ldots.$$

A4 If we let $\tau_{i,s}$ denote the string

$$1^{e_{0,s}}{}^\frown 0^{u_{0,s}}{}^\frown 1^{e_{1,s}}{}^\frown 0^{u_{1,s}}{}^\frown \ldots {}^\frown 0^{u_{i,s}},$$

then $\forall s$ $\tau_{n_s,s} \in T_s[l_s]$. $\tau_{n_s,s}$ is to be considered an approximation to the path X.

In what follows, l_s will be defined independently and the values of the $u_{j,s}$ will be induced by the requirement in A4 that $l_s = \sum_{j=0}^{n_s} e_{j,s} + u_{j,s}$.

If s is a stage at which $T_{s+1} \neq T_s$, then $l_{s+1} > l_s$ and

$$T_{s+1}[l_{s+1}] = \{\sigma^\frown 0^{l_{s+1}-l_s} : \sigma \in T_s[l_s]\} \cup \{\tau_{n_s,s}{}^\frown 1^p{}^\frown 0^{l_{s+1}-l_s-p} : 0 < p \leqslant l_{s+1} - l_s\}. \quad (1)$$

T_{s+1} is then any string extending or extended by an element of $T_{s+1}[l_{s+1}]$.

All that remains in the construction is to describe the sequences $\langle e_{i,s}\rangle$, $\langle u_{i,s}\rangle$, and $\langle l_s\rangle$ and to determine the stages at which $T_{s+1} \neq T_s$. At stage $s = 0$, we set $n_s = 0$ and $e_{n_s,s} = u_{n_s,s} = 0$. This gives $l_0 = 0$ and $\tau_{0,0} = \varnothing$ by definition. Now let s be arbitrary and suppose n_s and l_s are defined. Also suppose that $e_{i,s}$ and $u_{i,s}$ are defined for all $i \leqslant n_s$. For convenience we begin indexing the partial computable functions at 1. Let e be the least positive integer such that

B1 $e \neq e_{i,s}$ for any $i \leqslant n_s$,
B2 for some $0 < k \leqslant s$, if j is the largest integer such that $e_{j,s} < e$, then

$$|\tau_{j,s}| + e + k \leqslant \{e\}_s(|\tau_{j,s}| + e + k)\downarrow < \{e\}_s(|\tau_{j,s}| + e + k + 1)\downarrow. \quad (2)$$

Then we set

C1 $n_{s+1} = j + 1$,
C2 $e_{n_{s+1},s+1} = e$,
C3 $e_{i,s+1} = e_{i,s}$ and $u_{i,s+1} = u_{i,s}$ for all $i \leqslant j$,
C4 $l_{s+1} = \max\{l_s + 1, \{e\}(|\tau_{j,s}| + e + k + 1)\}$,
C5 $u_{n_{s+1},s+1} = l_{s+1} - |\tau_{j,s}| - e$
(this to ensure that $|\tau_{n_{s+1},s+1}| = l_{s+1}$).

If no such e exists then all values are unchanged. The point is that if $\{e\}$ appears at stage s to be a total increasing function, then we ensure that

$$\mathrm{Br}_X(P) \cap \big[\{e\}(|\tau_{j,s}| + e + k), \{e\}(|\tau_{j,s}| + e + k + 1)\big) = \varnothing.$$

As X is the only element of P that has infinitely many branching nodes on it, this ensures that P is e.p.h.i. via Theorem 2.11.

It remains to show that P is e.p.h.i. and not h.i. This is done in the next few lemmas.

Lemma 2.30 *For all $\tau \in T$, $\tau^\frown 0^\infty \in P$.*

Proof If $\{e\}$ is a total increasing function then there will be a stage s for which 2 is satisfied and $l_{s+1} > l_s$. Thus $\lim_s l_s = \infty$. Let $\tau \in T$ be arbitrary and s such that $l_s \leqslant |\tau| < l_{s+1}$. Then τ is of the form $\sigma^\frown 1^{n\frown} 0^m$ for some $0 \leqslant n, m < l_{s+1} - l_s$, and $\sigma \in P[l_s]$. An inspection of Equation (1) above taking $p = n$ (if necessary) then gives the result. □

Lemma 2.31 *For all s, $\tau_{n_s,s} \in T$.*

Proof By induction. First, $\tau_{n_0,0} = \varnothing \in T$. Now let s be arbitrary and suppose $\tau_{i,s} \in T$ for all $i \leqslant n_s$. We can assume $T_s \neq T_{s+1}$. There are two cases.

Case 1 $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$. In this case, j from (2) is just n_s and

$$\tau_{n_{s+1},s+1} = \tau_{n_s,s}{}^\frown 1^{e_{n_{s+1},s+1}\frown} 0^{u_{n_{s+1},s+1}}.$$

So using Equation (2) and the definition of l_{s+1} we have

$$0 < e_{n_{s+1},s+1} \leqslant l_{s+1} - l_s.$$

Take $p = e_{n_{s+1},s+1}$ in (1). We can do this because $l_{s+1} - l_s = l_{s+1} - |\tau_{n_s,s}| > e_{n_{s+1},s+1}$ by A4, C4, and (2) above.

Case 2 Let $j < n_s$ be the largest integer such that $\tau_{n_{s+1},s+1} \supsetneq \tau_{j,s}$ and

$$\tau_{n_{s+1},s+1} = \tau_{j,s}{}^\frown 1^{e_{n_{s+1},s+1}\frown} 0^{u_{n_{s+1},s+1}}.$$

By definition $e_{n_{s+1},s+1} < e_{j+1,s}$ so

$$\tau_{j,s}{}^\frown 1^{e_{n_{s+1},s+1}} \subsetneq \tau_{j,s}{}^\frown 1^{e_{j+1,s}} \subseteq \tau_{n_s,s} \in T.$$

Therefore, $\tau_{j,s}{}^\frown 1^{e_{n_{s+1},s+1}} \in T$, and so, by Lemma 2.30, $\tau_{n_{s+1},s+1} \in T$. □

Lemma 2.32 *For all s such that $T_s \neq T_{s+1}$, $|\tau_{n_{s+1},s+1}| > |\tau_{n_s,s}|$. Either $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$ or $\tau_{n_{s+1},s+1}$ is less than $\tau_{n_s,s}$ lexicographically.*

Proof $|\tau_{n_s,s}| = l_s$ for all s and l_s is increasing in s whenever $T_s \neq T_{s+1}$. Assume that it is not the case that $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$. If j is as (2), then

$$\tau_{n_{s+1},s+1} \supseteq \tau_{j,s}^\frown 1^{e_{n_{s+1},s+1}\frown} 0^1 \text{ and } \tau_{n_s,s} \supseteq \tau_{j,s}^\frown 1^{e_{j+1,s}}.$$

As $e_{n_{s+1},s+1} < e_{j+1,s}$, the result follows. □

Lemma 2.33 *P is not h.i.*

Proof For convenience we (computably) re-index the sequence $\langle T_s \rangle$ so that $T_{s+1} \neq T_s$ for all s. Now consider the disjoint strong array given by $D_{f(s)} = \{\tau_{n_s,s}{}^\frown 1^{l_{s+1}-l_s}\}$ for each s. First notice that $\tau_{n_s,s}{}^\frown 1^{l_{s+1}-l_s} \in T_{s+1}$ for all s (take $p = l_{s+1} - l_s$ in (1)). We claim that the sequence is increasing in length and strictly decreasing in lexicographical order. Hence it is pairwise incomparable. Lemma 2.30 then guarantees that $D_{f(s)} \cap \mathrm{Ext}(P) \neq \varnothing$ for all s and, therefore, P is not h.i. by Theorem 2.7.

To prove the claim consider two cases.

Case 1 $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$. Then $|\tau_{n_{s+1},s+1}| = l_{s+1} = |\tau_{n_s,s}{}^\frown 1^{l_{s+1}-l_s}|$ and $\tau_{n_{s+1},s+1} = \tau_{n_s,s}{}^\frown 1^{e_{n_{s+1},s+1}}{}^\frown 0^{u_{n_{s+1},s+1}}$. But

$$
\begin{aligned}
u_{n_{s+1},s+1} &= l_{s+1} - |\tau_{j,s}| - e_{n_{s+1},s+1} \\
&> |\tau_{j,s}| + e_{n_{s+1},s+1} + k - |\tau_{j,s}| - e_{n_{s+1},s+1} \\
&\quad \text{from (2) and the definition of } l_{s+1} \\
&> 0.
\end{aligned}
$$

Therefore, $\tau_{n_{s+1},s+1}$ is lexicographically less than $\tau_{n_s,s}{}^\frown 1^{l_{s+1}-l_s}$.

Case 2 If it is not the case that $\tau_{n_{s+1},s+1} \supsetneq \tau_{n_s,s}$, then, by Lemma 2.32, $\tau_{n_{s+1},s+1}$ is lexicographically less than $\tau_{n_s,s}$. As $|\tau_{n_{s+1},s+1}| > |\tau_{n_s,s}|$, the two strings must be incomparable. Therefore, any extension of $\tau_{n_{s+1},s+1}$ must be lexicographically less than any extension of $\tau_{n_s,s}$. The result follows a fortiori. □

Lemma 2.34 *X is the only nonisolated path in P.*

Proof It is immediate from the construction that $e_{i,s+1} \leqslant e_{i,s}$ for all i and s. So e_i exists for all i. $u_{i,s} \neq u_{i,s+1}$ only when $e_{i,s} \neq e_{i,s+1}$ so u_i exists as well. And for all i, $\tau_i = \lim_s \tau_{i,s} = 1^{e_0}{}^\frown 0^{u_0}{}^\frown 1^{e_1}{}^\frown 0^{u_1}{}^\frown \ldots {}^\frown 0^{u_i}$ exists. But for each s $\tau_{i,s} \in T$ and so $\tau_i \in T$. $X = \bigcup_i \tau_i$ and so $X \in P$. Furthermore, τ_i is a branching node for all i (as $\tau_i{}^\frown 0^1 \in T$ by Lemma 2.30 and $\tau_i{}^\frown 1^1 \in T$ as it is extended by τ_{i+1}). So there are infinitely many branching nodes along X and X is not isolated.

Let $i > 0$ be arbitrary and let s be such that $\tau_{i,s} = \tau_i$ and $n_t > i$ for all $t \geqslant s$. Then if $Y \in P$ such that $Y \not\supset \tau_i$ then $Y \not\supset \tau_{n_t,t}$ for all $t \geqslant s$. An inspection of (1) shows that for all $\sigma \supsetneq Y[l_s]$, $\sigma = Y[l_s]{}^\frown 0^{|\sigma|-l_s}$ and hence Y is isolated. □

Lemma 2.35 *If $\sigma \in T$ and σ is of the form*

$$1^{e_0}{}^\frown 0^{u_0}{}^\frown \ldots 1^{e_1}{}^\frown 0^q$$

where $0 < q < u_i$, then $\sigma \notin \mathrm{Br}(P)$.

Proof By (1) above, if $\sigma \in T$ then $\sigma^\frown\langle 1\rangle \in T$ only if σ is of the form $\tau_{n_s,s}{}^\frown 1^q$ for some s and $0 \leqslant q < l_{s+1} - l_s$. This is inconsistent with being of the above form. □

Lemma 2.36 *P is e.p.h.i.*

Proof Let $\{e\}$ be any strictly increasing total computable function—a candidate for witnessing the fact that P is not e.p.h.i. Let s be a stage such that

(i) for all $e_i < e$, $e_{i,s} = e_i$.

For all stages $t \geqslant s$ and for all $e_i < e$, $\tau_{i,t} = \tau_i$. In particular, if j is as in (2) then $e_j < e$ and $|\tau_{j,t}|$ is constant for all $t \geqslant s$. We can also assume that s is so large that it also satisfies

(ii) there exists a $0 < k \leqslant s$ such that

$$|\tau_{j,s}| + e + k \leqslant \{e\}_s(|\tau_{j,s}| + e + k)\downarrow < \{e\}_s(|\tau_{j,s}| + e + k + 1)\downarrow .$$

The construction then ensures that $X \supset \tau_j$ and the choice of $u_{n_{s+1},s+1}$ guarantees that

$$\mathrm{Br}_X(P) \cap \big[\{e\}(|\tau_{i,s}| + e + k), \{e\}(|\tau_{i,s}| + e + k + 1)\big) = \varnothing,$$

and so $\{e\}$ does not witness the fact that P is not e.p.h.i. As e was arbitrary, P is e.p.h.i. □

Theorem 2.37 *Small $\Rightarrow$ h.i.*

Proof If we assume that a Π^0_1 class P is not h.i. witnessed by $\langle D^*_{f(n)}\rangle_n$, then the computable function

$$n \mapsto \max\{|\sigma| : \sigma \in D_{f(n)}\}$$

witnesses the fact that P is not small via the characterization 2.9(4), as there must be at least n extendible nodes on P at the level given by $\max\{|\sigma| : \sigma \in D_{f(n)}\}$. □

Corollary 2.38 *There is a Π^0_1 class with no computable elements that is e.p.h.i. but not h.i.*

Proof Let S be any small Π^0_1 class with no computable path and P as in Theorem 2.29. By Lemma 2.21 $P \vee S$ is e.p.h.i. But $P \vee S$ is not h.i. by Lemma 2.22, and it does not have computable elements. □

Corollary 2.39 *There is a Π^0_1 class with no computable elements that is e.p.h.i. but not small.*

Proof $P \vee S$ from Corollary 2.38 is e.p.h.i. but not h.i. By Theorem 2.37 it cannot be small. □

Theorem 2.40 *h.i. $\not\Rightarrow$ u.p.h.i. In fact, any Π^0_1 class of positive measure contains an h.i. Π^0_1 class of positive measure that is not u.p.h.i.*

Proof Let P be any Π^0_1 class of measure $m > 0$. We will create the required $Q \subseteq P$ by adapting the construction of Theorem 2.24. Let $k \in \mathbb{N}$ be such that $m > 2^{-k}$. We will ensure that for all $\sigma \in \mathrm{Ext}(Q)$, $\mu(Q(\sigma)) \geqslant 2^{-2|\sigma|-k-2}$. The function defined recursively by

$$f(0) = 0$$
$$f(n+1) = 2f(n) + k + 3$$

will then witness the fact that Q is not u.p.h.i. via the characterization 2.10(3). This is straightforward to see because for any n and any $\sigma \in Q[f(n)]$, the measure of $Q(\sigma)$ is no less than $2^{-2f(n)-k-2}$. So there must be a branching node above σ of length less than $2f(n) + k + 3$—if there were not, then the measure of $Q(\sigma)$ could be no greater than $2^{-2f(n)-k-3}$.

Now we construct Q. As usual $Q = \bigcap_s Q_s$—a computable intersection of clopen sets. Let P' be the h.i. Π^0_1 class of positive measure as constructed in Theorem 2.24 and suppose $P' = \bigcap_{s=0}^{\infty} P'_s$—a computable intersection of clopen sets. Let $Q_0 = 2^{\mathbb{N}}$. Suppose Q_s is defined; let

$$Q_{s+1} = (P'_s \cap Q_s) \setminus \bigcup \{Q_s(\sigma) : \mu(Q_s(\sigma)) < 2^{-2|\sigma|-k-2}\}.$$

Then Q is equal to

$$P' \setminus \bigcup_{s=0}^{\infty} \bigcup \{Q_s(\sigma) : \mu(Q_s(\sigma)) < 2^{-2|\sigma|-k-2}\}$$

and it is h.i. as it is a subset of P'. To prove that $\mu(Q) > 0$ first notice that

$$\begin{aligned}
\mu(\bigcup_{s=0}^{\infty} \bigcup \{Q_s(\sigma) : \mu(Q_s(\sigma)) < 2^{-2|\sigma|-k-2}\}) &\leqslant \sum_{n=0}^{\infty} \sum_{|\sigma|=n} 2^{-2|\sigma|-k-2} \\
&= \sum_{n=0}^{\infty} 2^n 2^{-2n-k-2} \\
&< m \sum_{n=0}^{\infty} 2^{-n-2} \\
&= m/2.
\end{aligned}$$

But $\mu(P') \geqslant m/2$ from Theorem 2.24, so $\mu(Q) > m/2 - m/2 = 0$.

Finally, assume for a contradiction that $\tau \in \mathrm{Ext}(Q)$ and $\mu(Q(\tau)) < 2^{-2|\tau|-k-2}$. Then there must exist a t such that $\mu(Q_t(\tau)) < 2^{-2|\tau|-k-2}$. But then $Q_t(\tau) \cap Q_{t+1} = \varnothing$ and $\tau \notin \mathrm{Ext}(Q_{t+1}) \supseteq \mathrm{Ext}(Q)$. Contradiction. □

It will be useful later to note the following.

Theorem 2.41 *If $S = \mathcal{S}(A, B)$ is a separating Π^0_1 class then S is u.p.h.i. if and only if it is small. That is, small, e.p.h.i., p.h.i., and u.p.h.i. are equivalent in the case of separating classes.*

Proof All separating Π^0_1 classes S have the property

$$\forall n \in \mathrm{Brl}(S) \forall \sigma \in S[n] \left[\sigma \in \mathrm{Br}(S)\right].$$

By an easy induction argument it can be seen that for all $n \in \mathbb{N}$ (recalling Definition 2.8)

$$\min\{|\Phi_S(\sigma)| : |\sigma| = n\} = \max\{|\Phi_S(\sigma)| : |\sigma| = n\}.$$

Then the characterizations 2.9(2) and 2.10(2) show that any separating u.p.h.i. class is small. □

Figure 1 sums up the results in this section. The lack of an arrow between two properties indicates that a Π^0_1 counterexample with no computable paths is known.

$$\begin{tikzcd}
 & & \textit{zero measure} & & \\
\textit{small} \arrow[r] \arrow[dr] & \textit{e.p.h.i.} \arrow[ur] \arrow[rr] & & \textit{p.h.i.} \arrow[r] & \textit{u.p.h.i.} \\
 & \textit{h.i.} & & &
\end{tikzcd}$$

Figure 1

3 Thinness and Other Strengthenings

A similar analysis can be carried out using stronger notions than that of hyperimmunity. In this section we briefly consider the notions of *dense immunity* and (*co-*)*maximality*. For convenience I give their definitions here, but also see [14].

Definition 3.1 $X \subseteq \mathbb{N}$ is *dense immune* if p_X dominates every computable function.

Theorem 3.2 *$X \subseteq \mathbb{N}$ is dense immune if and only if for all strong arrays $\langle D_{f(n)} \rangle$ there are at most finitely many n such that*

$$\Big\| \bigcup_{i=0}^{n} D_{f(i)} \cap X \Big\| \geqslant n.$$

Notice that there is no requirement of disjointness in Theorem 3.2.

Definition 3.3 $X \subseteq \mathbb{N}$ is *maximal* if it is coinfinite and for every c.e. set $Y \supseteq X$ either Y is cofinite or $Y \setminus X$ is finite.

We use these well-established ideas to define analogous properties in $2^{\mathbb{N}}$ in the fashion of Section 2. Dense immunity turns out to be the most similar. We define a Π^0_1 class to be *very small (v.small)* in the same way as we defined smallness—but with "dense immunity" replacing "hyperimmunity." This was done in [1] in detail. We shall also define *everywhere pathwise dense immunity* (*e.p.d.i.*), *pathwise dense immunity* (*p.d.i.*), *uniform pathwise dense immunity* (*u.p.d.i.*) in the obvious way. We will then show that these properties are distinct from the ones defined earlier.

In order to define *dense immunity* (*d.i.*) for subsets of $2^{\mathbb{N}}$ we will use the alternative characterization in Theorem 3.2 and follow Theorem 2.7. We now recall Definition 2.6.

Definition 3.4 A Π^0_1 class P is *dense immune* (d.i.) if it is infinite and there is no strong array $\langle D^*_{f(n)} \rangle$ such that for infinitely many n

$$\|\mathrm{rt}\Big(\bigcup_{i=0}^{n} D^*_{f(i)}\Big) \cap \mathrm{Ext}(P)\| \geqslant n. \tag{3}$$

It is necessary to establish that these are all invariant under computable homeomorphisms. This is straightforward. The proof for d.i. Π^0_1 classes is similar to 2.13, the proof for v.small is in [1], and the rest of the proofs follow from Lemma 2.14.

To see how these new properties compare to the ones defined using hyperimmunity we first notice that every dense immune subset of $\mathbb{N}$ is hyperimmune so the following table is evident.

$$\begin{array}{ccccccc} \textit{small} & \longrightarrow & \textit{e.p.h.i.} & \longrightarrow & \textit{p.h.i.} & \longrightarrow & \textit{u.p.h.i.} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \textit{v. small} & \longrightarrow & \textit{e.p.d.i.} & \longrightarrow & \textit{p.d.i.} & \longrightarrow & \textit{u.p.d.i.} \end{array}$$

Figure 2

In fact we now show that all of these classes are distinct and that no further implications in the diagram are required. To do this it will be shown now that no diagonal arrows exist on the diagram (apart from the immediately necessary ones) and hence that the arrows on the bottom row are nonreversible, and the dense immune versions are distinct from their hyperimmune analogues.

To see there are no unnecessary diagonal implications it is sufficient to establish the following four lemmas.

Theorem 3.5 *There is a small Π^0_1 class with no computable path that is not u.p.d.i.*

Proof Let S be a small separating Π^0_1 class that is not v.small. Such an S exists by Theorem 3.16 of [1]. If S were u.p.d.i. then it would be v.small (using an analogous result to Theorem 2.41 and the fact that S is separating). □

Lemma 3.6 *E.p.d.i. $\not\Rightarrow$ small.*

Sketch of the proof The proof is similar to the proof of Theorem 2.29. A Π^0_1 class is constructed with exactly one nonisolated path X which has a dense-immune set of branching nodes on it. As before, every level of P is a branching level. □

Lemma 3.7 *P.d.i. $\not\Rightarrow$ e.p.h.i.*

Proof Take a v.small Π^0_1 class P. $P \wedge 2^{\mathbb{N}}$ will be p.d.i. but not e.p.h.i. □

Lemma 3.8 *U.p.d.i. $\not\Rightarrow$ p.h.i.*

Proof This is the same as Theorem 2.26 with f taken as the characteristic function of a dense immune Π^0_1 subset of $\mathbb{N}$. □

Of course, other concepts of diminutiveness such as hyperhyperimmunity, r-maximality, and so on (see [14], §X, for example) could be studied in a similar way. We do not do this here and questions remain about whether the analogous properties would be computably topological in $2^{\mathbb{N}}$.

The collections of canonically indexed sets D_n and D^*_n form bases for the respective topologies on $\mathbb{N}$ and $2^{\mathbb{N}}$. But whereas $2^{\mathbb{N}}$ is compact in this topology, $\mathbb{N}$ is not. So some array definitions are possible in $2^{\mathbb{N}}$ that have no analogy (or rather no *interesting* analogy) in $\mathbb{N}$. For example, we can require of a Π^0_1 class that for any disjoint strong array $\langle D^*_{f(n)} \rangle$ there are at most finitely many n such that $\langle D^*_{f(n)} \rangle \cap P \neq \varnothing$. The analogous property for subsets of $\mathbb{N}$ is equivalent to a set's being finite. However, for subsets of $2^{\mathbb{N}}$ this property is equivalent to being *thin* in the sense of [5], [7], [8], and elsewhere. This is interesting to note because the usual definition of thinness (to follow) suggests that the correct analogy in $\mathbb{N}$ is maximality.

Definition 3.9 A Π^0_1 class P is *thin* if its only Π^0_1 subclasses are its clopen subclasses (in the relative topology).

It is not straightforward to see relationships between the previously defined properties and thinness but some have been established.

Theorem 3.10 *Thin* $\Longrightarrow$ *d.i.*

Proof Suppose P is not d.i. witnessed by the strong array $D^*_{f(n)}$. Let $D^*_{g(0)} = D^*_{f(0)}$ and

$$D^*_{g(n+1)} = \bigcup_{i=0}^{n+1} D^*_{f(i)} \setminus \bigcup_{i=0}^{n} D^*_{f(i)}.$$

It is easy to see that $\langle D^*_{g(n)} \rangle$ is a disjoint strong array that also witnesses the fact that P is not d.i. As $\|\mathrm{rt}\big(\bigcup_{i=0}^{n} D^*_{g(i)}\big) \cap \mathrm{Ext}(P)\| \geqslant n$ for infinitely many n, it must be the case that $D^*_{g(n)} \cap P \neq \varnothing$ for infinitely many n. As $\langle D^*_{g(n)} \rangle$ is disjoint, P cannot be thin. $\square$

Lemma 3.11 *v.small* $\Longrightarrow$ *d.i.*

Proof If $\langle D_{f(n)} \rangle$ witnesses the fact that P is not d.i. then the function

$$m(n) = \max \left\{ |\sigma| : \sigma \in \mathrm{rt}\Big(\bigcup_{i=0}^{n} D_{f(i)}\Big) \right\}$$

witnesses the fact that $\mathrm{Br}(P)$ is not d.i. and hence that P is not v.small. $\square$

Theorem 3.12 *D.i.* $\not\Longrightarrow$ *thin.*

Proof If P is v.small then so is $P \vee P$ (see [1]), and by Lemma 3.11 $P \vee P$ is d.i. But $P \vee P$ is not thin as $\{f \oplus f : f \in P\}$ is a Π^0_1, nonclopen, proper subset of $P \vee P$. $\square$

Theorem 3.13 *Thin* $\not\Longrightarrow$ *v.small.*

Proof This is Theorem 4.3 in [1] and is a consequence of results in [7]. $\square$

Theorems 3.12 and 3.13 together with Lemma 3.11 show that thinness and v.smallness are independent properties. That is, there are Π^0_1 classes that are v.small but not thin and Π^0_1 classes that are thin but not v.small. In comparison, there are Π^0_1 classes that are small but not thin (if P is small then so is $P \vee P$) but then it is unknown whether every thin class is small. We conjecture the negative.

Theorem 3.14 *Thin* $\Longrightarrow$ *u.p.h.i.*

Proof The proof is very similar to Simpson's proof that all thin Π^0_1 classes have zero measure. We prove the contrapositive. Suppose a Π^0_1 class P were not u.p.h.i. and this witnessed by the computable function f. That is,

$$\forall n \forall \tau \in P[f(n)] \exists \sigma \supseteq \tau \left[f(n) \leqslant |\sigma| < f(n+1) \text{ and } \sigma \in \mathrm{Br}(P) \right].$$

Define a sequence of elements of $\mathrm{Ext}(P)$ as follows ("left" and "right" here refer to the lexicographical ordering on $2^{\mathbb{N}}$):

$$\sigma_1 = \text{ the rightmost string on } P[f(1)];$$
$$\sigma_{n+1} = \text{ the rightmost string on } P[f(n+1)] \text{ to the left of } \sigma_n.$$

To prove that σ_n exists for all n, we use induction to prove that for all $n > 0$ there is a $\tau \in P[f(n)]$ such that τ is strictly to the left of σ_n. If τ is the rightmost such string in $P[f(n)]$, then σ_{n+1} will be the rightmost element of $P[f(n+1)]$ extending τ.

Base case There is a branching node on P before level $f(1)$ so there must be a $\tau \in P[f(1)]$ strictly to the left of σ_1.

Induction Suppose that τ is the rightmost element of $P[f(n)]$ strictly to the left of σ_n. There must be a branching node above τ before level $f(n+1)$ as P is not u.p.h.i. Therefore, there must be a τ' strictly to the left of σ_{n+1} defined as above.

The set $S = \bigcup_n U_{\sigma_n} \cap P$ is open in the relative topology of P, but it is not closed, as the set $\{U_{\sigma_n} : n \in \mathbb{N}\}$ is pairwise disjoint and P is compact. Furthermore, $\bigcup_n U_{\sigma_n}$ is Σ^0_1 so $P \setminus S$ is a nonclopen Π^0_1 subclass of P, and P is not thin. □

4 Muchnik and Medvedev Degrees

As we do for Turing degrees, if $\mathcal{C}$ is any property of Π^0_1 classes and $\mathbf{d}$ is a Muchnik degree, then we say $\mathbf{d}$ has property $\mathcal{C}$ if $\mathbf{d}$ has a representative with property $\mathcal{C}$. The questions arise now whether the properties defined in this paper describe different classes of Muchnik degrees. Also it can be asked where these classes of Muchnik degrees fit into the known structure of the Muchnik lattice. An analogous type of theorem in the Turing degrees is one from Dekker that states that every c.e. Turing degree has a hyperimmune representative [6].

Here not as much is known as would be liked, but we present some basic results. Some conjectures and open questions are discussed in the following section.

The next two lemmas are very useful in this area. Recall that $\leqslant_w$ refers to Muchnik or *weak* reducibility and $\leqslant_M$ to Medvedev or *strong* reducibility—the uniform version of Muchnik reducibility.

Lemma 4.1 (Simpson; see [12] or [1]) *For all Π^0_1 classes P and Q, if $P \geqslant_w Q$, then there exists a Π^0_1 subclass $P' \subseteq P$ such that $P \geqslant_M Q$.*

Lemma 4.2 (Simpson; see [13]) *If P and Q are Medvedev complete Π^0_1 classes, then P is recursively homeomorphic to Q.*

Recall that $\mathrm{DNR}_2 = \{f \in 2^{\mathbb{N}} : \forall n\ f(n) \neq \{n\}(n)\}$ is Medvedev (and hence Muchnik) complete.

Lemma 4.3 DNR_2 *is neither h.i. nor u.p.h.i.*

Proof Let $e_0 < e_1 < e_2 \cdots$ be a computable sequence of indices for the empty function. For every i, define

$$E_i = \{f \in 2^{\mathbb{N}} : \forall j < i\ f(e_j) = 0 \text{ and } f(e_i) = 1\}.$$

Each E_i intersects DNR_2 as $0 \neq \{e_i\}(e_i) \neq 1$ for all i. They are also pairwise disjoint and so form a disjoint strong array. So DNR_2 is not h.i.

To see it is not u.p.h.i. first notice that $\mathrm{DNR}_2 = \mathcal{S}(A, B)$ where $A = \{e : \{e\}(e) \downarrow= 0\}$ and $B = \{e : \{e\}(e) \downarrow= 1\}$. It is therefore a separating class and if it were u.p.h.i. it would be small by Theorem 2.41. It is not small, however, because $e_0, e_1, e_2, \ldots$ is a computable sequence of branching levels of DNR_2 (Theorem 2.9(3)). □

Theorem 4.4 *If P is an h.i. or e.p.h.i. Π^0_1 class, then it is Muchnik incomplete.*

Proof Suppose P were an h.i. Π^0_1 class and that $P \geqslant_w \mathrm{DNR}_2$. Then by Lemma 4.1 there would be a Π^0_1 $P' \subseteq P$ such that $P' \geqslant_M \mathrm{DNR}_2$. So P' is Medvedev complete. Hyperimmunity is closed under taking subsets so P' is also h.i. But DNR_2 is not h.i. by Lemma 4.3 and so no other Medvedev complete Π^0_1 class can be by Lemma 4.2 and Theorem 2.13. The proof is identical for the e.p.h.i. case. $\square$

It is now immediate that every small Π^0_1 class is Muchnik incomplete.

Theorem 4.5 *If P is u.p.h.i., then it is Medvedev incomplete.*

Proof If it were Medvedev complete then DNR_2 would be u.p.h.i. by Lemma 4.2 and Theorem 2.13. $\square$

It is currently an open question whether every u.p.h.i. or p.h.i. Π^0_1 class is Muchnik incomplete. We conjecture that it is so.

Theorem 4.6 *There is an h.i. Muchnik degree that is not less that any small Muchnik degree.*

Proof There is a Π^0_1 class R consisting entirely of 1-random reals with the property that any Π^0_1 subclass of R has positive measure [10]. R also has the property that if M is any Π^0_1 class of positive measure then $R \geqslant_w M$ (see [12] for an exposition).

Lemma 2.24 implies that R must have an h.i. Π^0_1 subclass R' and the above implies that $R' \equiv_w R$. So R has h.i. Muchnik degree. But if S were any small Π^0_1 class such that $S \geqslant_w R$ then by Lemma 4.1 there would be a Π^0_1 $S' \subseteq S$ such that $S' \geqslant_M R$. That is, there would be a computable functional $\Phi : S' \longrightarrow R$. S' is small as it is a subclass of S and its image under Φ is also small by Theorem 2.12. But every subset of R is of positive measure so the image of Φ must be small and of positive measure—contradicting Theorem 2.23. $\square$

Corollary 4.7 *The class of small Muchnik degrees is strictly contained in the class of h.i. Muchnik degrees.*

The problem of the density of the Muchnik lattice is still the outstanding problem in the area. Partial results have been obtained, for example, Corollary 3.17 in [1]. The next lemma by Simpson in [12] (Corollary 7.5) gives upward density for a large class of Muchnik degrees.

Lemma 4.8 (Simpson) *Let P, Q, and S be Π^0_1 classes such that P is of positive measure and S is a separating class. Then*

$$P \vee Q \geqslant_w S \implies Q \geqslant_w S.$$

Proof See [12]. Note that $P \not\geqslant_w S$ by Theorem 5.3 in [9]. The proof is a relativization and generalization of Theorem 5.3. $\square$

This gives the following theorem as a corollary.

Theorem 4.9 *If P is a small Π^0_1 class, then there is a Π^0_1 class Q such that*

$$P <_w Q <_w \mathrm{DNR}_2.$$

Proof By Lemma 2.37, P must be h.i. and therefore $P <_w \mathrm{DNR}_2$ by Theorem 4.4. In the proof of Theorem 4.6, $R \not\leqslant_w P$. Therefore, $R \vee P >_w P$. But Lemma 4.8 also implies that $R \vee P <_w \mathrm{DNR}_2$ as DNR_2 is a separating class. $\square$

5 Open Questions and Further Directions

Question 5.1 *Does there exist a u.p.h.i. (e.p.h.i., p.h.i.) Muchnik degree that is not small (h.i.)?*

These problems can be solved by constructing a u.p.h.i. (e.p.h.i., p.h.i.) Π^0_1 class that has no small (h.i.) subclass. And then use an argument like the proof of Theorem 4.6. It is not clear how to proceed in other questions of this type—for example, does there exist a u.p.h.i. Muchnik degree that is not p.h.i.?

Question 5.2 *Is every u.p.h.i. (p.h.i.) Muchnik degree Muchnik incomplete?*

An essential property in showing that every small Π^0_1 class (for example) is Muchnik incomplete is the property that every Π^0_1 subclass of a small Π^0_1 class is small. This property is not shared by u.p.h.i. or p.h.i. classes. Another method of showing incompleteness needs to be found.

There are many easily describable intermediate Muchnik degrees. For example, in [12] Simpson defines a transfinite sequence of such degrees related to the diagonally nonrecursive functions. It is unknown how the properties described in this paper relate to such degrees. For example, the following is the obvious question.

Question 5.3 *Is every small (thin) Muchnik degree thin (small)?*

It is known that not every small Π^0_1 class is thin (see [1]) but it is not known if every thin class is small. Whether or not their Muchnik degrees coincide could be answered negatively if one were to construct a small (thin) Π^0_1 class with no thin (small) subclass.

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