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Nonconstructive Properties of Well-Ordered T_2 Topological Spaces

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Abstract We show that none of the following statements is provable in Zermelo-Fraenkel set theory (ZF) answering the corresponding open questions from Brunner in "The axiom of choice in topology":

(i) For every T_2 topological space (X, T) if X is well-ordered, then X has a well-ordered base,

(ii) For every T_2 topological space (X, T), if X is well-ordered, then there exists a function $f : X \times W \to T$ such that W is a well-ordered set and $f(\{x\} \times W)$ is a neighborhood base at x for each $x \in X$,

(iii) For every T_2 topological space (X, T), if X has a well-ordered dense subset, then there exists a function $f : X \times W \to T$ such that W is a well-ordered set and $\{x\} = \cap f(\{x\} \times W)$ for each $x \in X$.

1 Introduction Let (X, T) be a T_2 topological space and let $\mathcal{B}$ be a base for X. Clearly,

$$|T| \leq |2^X| \tag{1}$$

and

$$|X| \leq |2^{\mathcal{B}}|. \tag{2}$$

(The map $f : X \to \mathcal{P}(\mathcal{B})(=$ the powerset of $\mathcal{B})$, $f(x) = \{B \in \mathcal{B} : x \in B\}$ is obviously $1:1$). We then have the following proposition.

Proposition 1.1 *In Fraenkel-Mostowski permutation models, a T_2 topological space (X, T) is well-ordered if and only if X has a well-ordered base.*

Proof: From (1) and the fact that in every permutation model Form 91 in Howard and Rubin [4], PW : *The powerset of a well-ordered set can be well-ordered* holds, we have that if X is well-ordered, then T is well-ordered. Similarly from (2) it follows that if X has a well-ordered base, then X is well-ordered. □

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In Cohen models however Proposition 1.1 may fail. Indeed, in the basic Cohen model, model $\mathcal{M}1$ of [4], the real line $\mathbb{R}$ with the standard topology has a countable base, but $\mathbb{R}$ is not well-ordered. There remains the question:

If (X, T) is a well-ordered T_2 *topological space, then does X have a well-ordered base?*

Motivated by this question, Brunner [1] defined the following statements:

(A1) Form 148 in [4]: For every T_2 topological space (X, T), if X is well-ordered, then X has a well-ordered base.

(A2) For every T_2 topological space (X, T), if X is well-ordered, then there exists a function $f : X \times W \to T$ such that W is a well-ordered set and $f(\{x\} \times W)$ is a neighborhood base at x for each $x \in X$.

(A3) For every T_2 topological space (X, T), if X is well-ordered, then each open cover of X has a well-ordered open refinement.

(A4) For every T_2 topological space (X, T), if X is well-ordered,then X satisfies $(*)$: if $O \subseteq T$ covers X, there is a mapping $f : X \to T$ such that $x \in f(x)$ and $f[X]$ refines O.

(A5) For every T_2 topological space (X, T), if X is well-ordered, then $(*)$ is a hereditary property of X.

(A6) For every T_2 topological space (X, T), if X has a well-ordered dense subset, then there exists a function $f : X \times W \to T$ such that W is a well-ordered set and $\{x\} = \cap f(\{x\} \times W)$ for each $x \in X$.

Clearly, each of the above statements is a theorem of ZFC (ZF with the axiom of choice AC, Form 1 in [4]). Brunner [1] asks whether these statements are provable in ZF minus the axiom of regularity (ZF^0) and Howard and Rubin [4] ask whether 148 implies AC. The aim of this paper is to show that none of (Ai), i = 1, 2, 6, is a theorem of ZF and that 148 does not imply AC in ZF^0. In particular, we show that

(A1), (A2), and (A6) are equivalent to AC in ZF.

The set-theoretic status of (Ai), i = 3, 4, 5 still eludes us.

Before setting out with proofs let us make a straightforward remark on the interrelation between the statements (A1) up to (A5).

(i) (A1) $\Longleftrightarrow$ (A2).
(ii) (A1) $\Longrightarrow$ (A3).
(iii) (A3) $\Longleftrightarrow$ (A4) $\Longleftrightarrow$ (A5).

For any undefined topological notion the reader is referred to Willard [9].

2 Results

We begin by observing the following.

Theorem 2.1 *(A1) does not imply AC in ZF^0.*

Proof: Let $\mathcal{N}$ be the basic Fraenkel model (model $\mathcal{N}1$ in [4]). By Proposition 1.1 we have that (A1) holds in $\mathcal{N}$. On the other hand, AC fails in $\mathcal{N}$ (see [4]) meaning that (A1) does not imply AC in ZF^0 as required. □

However in ZF, (A1) is equivalent to AC as Theorem 2.3 clarifies. In particular, we show that both (A1) and (A6) are equivalent to the set-theoretic principle PW (see the introduction) which in ZF is known to be equivalent to AC (see Felgner and Jech [3]). We recall first the notion of an independent family of sets.

Definition 2.2 Let $\theta \geq \omega$ be an ordinal number. A family $\mathcal{A} \subseteq \mathcal{P}(\theta)$ is said to be *independent* if and only if for any finite collection $A_1, \ldots, A_m, B_1, \ldots, B_n$ of distinct elements of $\mathcal{A}$, $|A_1 \cap \cdots \cap A_m \cap (E \backslash B_1) \cap \cdots \cap (E \backslash B_n)| = |\theta|$.

Theorem 2.3 *In ZF the following statements are equivalent:*

(i) PW,

(ii) (A1),

(iii) (A6).

Proof: (i) $\rightarrow$ (ii) This is straightforward.

(ii) $\rightarrow$ (i) Fix an ordinal number $\kappa \geq \omega$ and let $\mathcal{A} = \{a_i : i \in 2^\kappa\} \subseteq \mathcal{P}(\kappa)$ be an independent family (see Kunen [6], Exercise (A6), p. 288). The existence of such a family can be proved in ZF^0. We show that 2^κ is well-ordered.

For each $i \in 2^\kappa$, let $G_i = \{x \in \mathcal{P}(\kappa) : |x \triangle a_i| < \omega\}$ where $\triangle$ denotes the operation of symmetric difference. Since for all $i, j \in 2^\kappa$, $i \neq j$, $a_i \triangle a_j$ is infinite, we have that $G_i \cap G_j = \varnothing$. Put $G = \cup\{G_i : i \in 2^\kappa\}$. For each $x \in [\kappa]^{<\omega} (= \{x \in \mathcal{P}(\kappa) : |x| < \omega\})$, $i \in 2^\kappa$ and $g \in G_i$, put

$$B(x, i, g) = \{y \in [\kappa]^{<\omega} : x \subseteq y \text{ and } y \cap g = \varnothing\}. \tag{3}$$

Claim 2.4 *The family $\{B(x, i, g) : x \in [\kappa]^{<\omega}, i \in 2^\kappa, g \in G_i\}$ is a cover of $[\kappa]^{<\omega}$.*

Proof of Claim 2.4: Fix $x \in [\kappa]^{<\omega}$ and let $i \in 2^\kappa$. Then $a_i \backslash x \in G_i$ and $x \in B(x, i, a_i \backslash x)$ finishing the proof of the Claim 2.4. □

Let $\mathcal{B} = \{B(x, g) : x \in [\kappa]^{<\omega}, g = \cup Q, Q \in [G]^{<\omega}\}$ where $B(x, g) = \{y \in [\kappa]^{<\omega} : x \subseteq y \text{ and } y \cap g = \varnothing\}$.

Claim 2.5 *$\mathcal{B}$ is a base for a T_2 topology $T_{\mathcal{B}}$ on $[\kappa]^{<\omega}$.*

Proof of Claim 2.5: By Claim 2.4 we have that $\mathcal{B}$ is a cover of $[\kappa]^{<\omega}$. On the other hand, if $x \in B(x_1, g_1) \cap B(x_2, g_2)$, then since $x \cap g_1 = x \cap g_2 = \varnothing$, we have that $B(x, g_1 \cup g_2) \in \mathcal{B}$ and clearly, $x \in B(x, g_1 \cup g_2) \subseteq B(x_1, g_1) \cap B(x_2, g_2)$. Therefore, $\mathcal{B}$ is a base. We show now that $\mathcal{B}$ generates a T_2 topology on $[\kappa]^{<\omega}$. Fix $x, y \in [\kappa]^{<\omega}$ with $x \neq y$ and let $g \in G$ be such that $(x \cup y) \cap g = \varnothing$ (take, for example, any $i \in 2^\kappa$ and put $g = a_i \backslash (x \cup y)$). Then $V_x = B(x, g \cup (y \backslash x))$ and $V_y = B(y, g \cup (x \backslash y))$ are disjoint neighborhoods of x and y, respectively. Assume otherwise and let $z \in V_x \cap V_y$. Then $x \subseteq z$, $z \cap (g \cup (y \backslash x)) = \varnothing$, and $y \subseteq z$, $z \cap (g \cup (x \backslash y)) = \varnothing$. Thus,

$$y \subseteq z \text{ and } y \cap (y \backslash x) = \varnothing \tag{4}$$

and

$$x \subseteq z \text{ and } x \cap (x \backslash y) = \varnothing \tag{5}$$

By (4) we have that $y \subseteq x$ and by (5), $x \subseteq y$. Therefore, $x = y$, a contradiction. This completes the proof of Claim 2.5. □

Since $([\kappa]^{<\omega}, T_{\mathcal{B}})$ is a well-ordered T$_2$ space, let by (A1) $\mathcal{W} = \{W_j : j \in \aleph\}$ be a well-ordered base. Consider now the open cover $\mathcal{U} = \{B(\varnothing, i, g) : i \in 2^\kappa, g \in G_i\}$ where $B(x, i, g)$ is given by (3). Then $\mathcal{V} = \{V \in \mathcal{W} : V \subseteq U$ for some $U \in \mathcal{U}\}$ is clearly a well-ordered open refinement of $\mathcal{U}$. For every $V \in \mathcal{V}$, let

$$H_V = \{i \in 2^\kappa : \exists g \in G_i, V \subseteq B(\varnothing, i, g)\}. \tag{6}$$

Claim 2.6 *For each $V \in \mathcal{V}$, H_V is finite.*

Proof of Claim 2.6: Assume the contrary and let $V_0 \in \mathcal{V}$ be such that H_{V_0} is infinite. As each G_i can be well-ordered uniformly ($\{a_i \triangle x : x \in [\kappa]^{<\omega}\}$ is a uniform well-ordering of G_i) we may define an infinite set $\{g_i \in G_i : i \in H_{V_0}\}$ such that $V_0 \subseteq B(\varnothing, i, g_i)$ for all $i \in H_{V_0}$. Fix $B(x_0, g)$ a basic open set contained in V_0. Then $g = g_{i_1} \cup g_{i_2} \cup \cdots \cup g_{i_n}$ for some $n \in \omega$ and $g_{i_j} \in G_{i_j}$, $j \leq n$. Since $B(x_0, g) \subseteq \cap\{B(\varnothing, i, g_i) : i \in H_{V_0}\}$, we have that $(\cup\{g_i : i \in H_{V_0}\}) \backslash g = \varnothing$ (otherwise fix $i \in H_{V_0}$ and $y \in g_i \backslash g$, then $x_0 \cup \{y\} \in B(x_0, g) \backslash B(\varnothing, i, g_i)$, a contradiction). Since $|g_i \triangle a_i| < \omega$, it follows immediately that for all $i \in H_{V_0}$, $F_i = a_i \backslash (a_{i_1} \cup a_{i_2} \cup \cdots \cup a_{i_n})$ is finite. This contradicts the fact that $\mathcal{A}$ is an independent family and completes the proof of Claim 2.6. □

Since $\mathcal{A}$ is an independent family, $\mathcal{U}$ has no finite subcover. Furthermore, as $\mathcal{W}$ is a base it is clear that $2^\kappa = \cup\{H_V : V \in \mathcal{V}\}$ and since κ is well-ordered, 2^κ is linearly ordered (e.g., lexicographically). Thus, each H_V is well-ordered and consequently 2^κ is well-ordered finishing the proof of (ii) → (i).

(i) → (iii) Since in ZF, AC $\Longleftrightarrow$ PW, this is straightforward.

(iii) → (i) Fix an ordinal number κ. Since $|\kappa| < |2^\kappa|$ we may assume without loss of generality that $\kappa \subseteq 2^\kappa$. Let $\mathcal{W} = \{W_f : f \in 2^\kappa \backslash \kappa\}$ be an independent family of subsets of κ. Define a topology T on $X = 2^\kappa$ by requiring: All points in κ to be isolated whereas neighborhoods of $f \in 2^\kappa \backslash \kappa$ are all sets of the form

$$V_f = \{f\} \cup (W_f \backslash (\cup Q \cup A)), \; Q \in [\mathcal{W} \backslash \{W_f\}]^{<\omega}, \; A \in [W_f]^{<\omega}.$$

(X, T) is a T$_2$ space. Indeed, let $x, y \in X$, $x \neq y$. We consider the following cases.

Case 1: $x, y \in \kappa$. Then $\{x\}, \{y\}$ are the required disjoint neighborhoods of x and y, respectively.

Case 2: $x \in \kappa$, $y \in 2^\kappa \backslash \kappa$. Then $\{x\}$, $\{y\} \cup (W_y \backslash \{x\})$ are the required disjoint neighborhoods of x and y, respectively.

Case 3: $x, y \in 2^\kappa \backslash \kappa$. Then $\{x\} \cup (W_x \backslash W_y)$, $\{y\} \cup (W_y \backslash W_x)$ are the required disjoint neighborhoods of x and y, respectively.

Thus, (X, T) is a T_2 space having the well-ordered set κ as a dense subset. Adjoin an extra point ∞ to X and extend the topology T by declaring neighborhoods of ∞ to be all supersets of $\{\infty\}$ missing finitely many sets $\{f\} \cup W_f$, $f \in 2^\kappa \backslash \kappa$. Thus, each neighborhood of ∞ misses only finitely many elements of $2^\kappa \backslash \kappa$. Clearly $Y = X \cup \{\infty\}$ with the extended topology T^∞ is a T_2 space having κ as a dense subset.

Let, by (A6), $\{Z_i : i \in \aleph\}$ be a well-ordered family of neighborhoods of $\{\infty\}$ such that $\{\infty\} = \cap\{Z_i : i \in \aleph\}$. Then $2^\kappa \backslash \kappa = \cup\{(2^\kappa \backslash \kappa) \backslash Z_i : i \in \aleph\}$ and by the above each set $(2^\kappa \backslash \kappa) \backslash Z_i$ is finite. As 2^κ is linearly ordered, $(2^\kappa \backslash \kappa) \backslash Z_i$ is well-ordered. Thus, $2^\kappa \backslash \kappa$ is well-ordered finishing the proof of (iii) $\to$ (i) and of the theorem. $\square$

Remark 2.7 The statement "If (X, T) is a T_2 space with a well-ordered dense subset, then each open cover of X has a well-ordered open refinement" has also been considered in [1] where it is shown not to be a theorem of ZF; in the basic Cohen model, the Moore plane (see Steen and Seebach [8], Example 82) is a separable T_2 space having an open cover with no well-ordered open refinement. Via the latter proof, Brunner implicitly suggests that the above statement implies a well-known weak choice principle, namely, the axiom of choice for families of nonempty subsets of $\mathbb{R}$, AC($\mathbb{R}$), and Form [79 A] in [4]. However, following the proof of Theorem 2.3 we deduce that the above statement is equivalent to AC in ZF. Indeed, let (X, T) be the T_2 space of Theorem 2.3 and let $O = \{\{f\} \cup W_f : f \in 2^\kappa \backslash \kappa\} \cup \{\{x\} : x \in \kappa\}$. Clearly, O is an open cover of X. Let $V = \{V_i : i \in \aleph\}$ be a well-ordered open refinement of O. For each $f \in 2^\kappa \backslash \kappa$, let i_f be the least $i \in \aleph$ such that $f \in V_i$. Then $V_{i_f} \subseteq \{g\} \cup W_g$ for some $g \in 2^\kappa \backslash \kappa$. Necessarily, $g = f$ and consequently the function $f \mapsto V_{i_f}$ is $1:1$ meaning that 2^κ is well-ordered.

3 *Summary*

The following diagram summarizes the results of the paper.

$$\boxed{\begin{array}{c} \boxed{\boxed{\mathrm{AC} \equiv (\mathrm{A1}) \equiv (\mathrm{A2}) \equiv (\mathrm{A6})}} \\ \Downarrow \\ \boxed{\boxed{(\mathrm{A3}) \equiv (\mathrm{A4}) \equiv (\mathrm{A5})}} \end{array}}$$

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