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A Counterexample in Tense Logic

FRANK WOLTER

Abstract We construct a normal extension of **K4** with the finite model property whose minimal tense extension is not complete with respect to Kripke semantics.

Call a normal bimodal logic in the propositional language with $\Box^+$ and $\Box^-$ a tense logic if it contains the tense axioms

$$\textbf{tense} = \{p \to \Box^+\Diamond^- p,\ \ p \to \Box^-\Diamond^+ p\}.$$

With each normal modal logic Λ containing **K4** we associate its minimal tense extension $\Lambda^+.t$, which is the smallest tense logic containing Λ formulated in $\Box^+$. Recall that a modal logic is called complete (has the finite model property) iff the following is equivalent for all formulas φ: $\varphi \in \Lambda \Leftrightarrow \langle g, R\rangle \models \varphi$, for all (finite) frames $\langle g, R\rangle$ validating Λ. This paper provides a counterexample to the natural assumption that completeness is transferable when moving to the minimal tense extension. The problem whether completeness transfers from Λ to $\Lambda^+.t$ can also be described as an axiomatization problem. Indeed, the existence of a complete logic Λ such that $\Lambda^+.t$ is incomplete is equivalent to the existence of a modally definable class of transitive Kripke-frames $\mathbf{M}$ such that the theory of

$$\mathbf{M}^t = \{\langle g, R, R^{-1}\rangle \mid \langle g, R\rangle \in \mathbf{M}\}$$

is not axiomatizable by a set of formulas formulated in $\Box^+$ and **tense**. (Here the theory of a class of frames F is the set of all formulas which are valid in all frames in F.)

It is easy to construct modal logics containing **K4** with the finite model property (fmp, for short) whose minimal tense extensions do not enjoy fmp. Take for instance provability logic $\mathbf{G} = \mathbf{K4} \oplus \Box(\Box p \to\ p) \to \Box p$. $\mathbf{G}$ is known as the theory of the class of inverse well-founded frames and has fmp (cf. Fine [2]). But $\mathbf{G}^+.t$ does not have fmp since the tense logic determined by the finite inverse well-founded frames is

$$\mathbf{G}^+.t \oplus \Box^-(\Box^- p \to\ p) \to \Box^- p$$

(cf. Wolter [3]). Note however that $\mathbf{G}^+.t$ is complete, by (3) of Theorem 1 below. Wolter [3] and [4] deliver general positive results as concerns transfer properties of the map $\Lambda \mapsto \Lambda^+.t$. The following theorem summarizes some results of those papers. First recall the following definitions. All frames are assumed to be transitive. For a frame $\langle g, R\rangle$ we write $x\vec{R}y$ iff xRy and $x \neq y$ and $\neg(yRx)$. Let $x \in g$ and suppose that there is a longest finite chain $x = x_0\vec{R}\dots\vec{R}x_n$ in $\langle g, R\rangle$. Then the depth of x in $\langle g, R\rangle$ is $dp(x) = n$ and x is said to be of finite depth. We say that a modal logic Λ containing **K4** has finite depth if there exists an $n \in \omega$ such that all points in all frames validating Λ have depth $\leq n$.

Theorem 1 *Let Λ be a logic above* **K4**. *Then*

1. *if Λ has finite depth, then $\Lambda^+.t$ has fmp;*
2. *if Λ has finite width (in the sense of Fine [1]), then $\Lambda^+.t$ is complete. Especially, $\Lambda^+.t$ is complete whenever $\Lambda \supseteq$* **K4.3**;
3. *if Λ is a (cofinal) subframe logic (in the sense of Zakharyaschev [6] and [2], respectively), then $\Lambda^+.t$ is complete. $\Lambda^+.t$ has the fmp iff the frames validating Λ form a first order definable class.*

Given that this result covers all natural extensions of **K4**, it is clear that our example is (in some sense) similar to the construction of incomplete logics above **K4**. Let us start with the definition of the frames involved in the construction. Define $\langle g, R\rangle$ by putting:

$$g = \bigcup\{\omega \times \{i\} | 1 \leq i \leq 7\} \cup \{u\}$$

and R as the transitive closure of R_1 with

$$\begin{aligned} R_1 \quad = \quad & \{(x, y)|\ x \in \omega \times \{i\},\ y \in \omega \times \{j\},\ j < i \leq 5\} \cup \\ & \cup \{((m, i), (n, i))|\ m < n,\ i = 2, 5\} \cup \\ & \cup \{((m, i), (n, i))|\ m > n,\ i = 1, 3, 4\} \cup \\ & \cup \{(x, x)|\ x \in \omega \times \{6, 7\}\} \cup \{(u, u)\} \cup \\ & \cup \{((m, 5), (m, 6))|\ m \in \omega\} \cup \{((m, 4), (m, 7))|\ m \in \omega\} \cup \\ & \cup \{((m, 6), (m, 1))|\ m \in \omega\} \cup \{((m, 7), (m, 2))|\ m \in \omega\} \cup \\ & \cup \{(x, u)|\ x \in \omega \times \{3, 4, 5\}\}. \end{aligned}$$

See the figure below. We draw frames in such a way that $\bullet$ represents a reflexive point and x represents an irreflexive point.

Denote by $\mathcal{G}_n$ the subframe of $\langle g, R\rangle$ induced by

$$g_n = \{(m, i)|\ m \leq n,\ i = 1, 3, 5, 6\} \cup \{u\},$$

and denote by Λ the theory of the set of frames $\{\mathcal{G}_n | n \in \omega\}$. We will show the following.

Theorem 2 *Λ has the fmp and $\Lambda^+.t$ is incomplete.*

That Λ has the fmp follows from the definition. To prove that $\Lambda^+.t$ is incomplete we need a general tense frame validating $\Lambda^+.t$ and refuting a formula φ which holds in all Kripke frames validating $\Lambda^+.t$. (Consult, e.g., [3] for the definition of general frames). We first define a general monomodal frame $\mathcal{G} = \langle g, R, A\rangle$ by defining A as the boolean closure of $C \subseteq 2^g$, where $c \in C$ iff

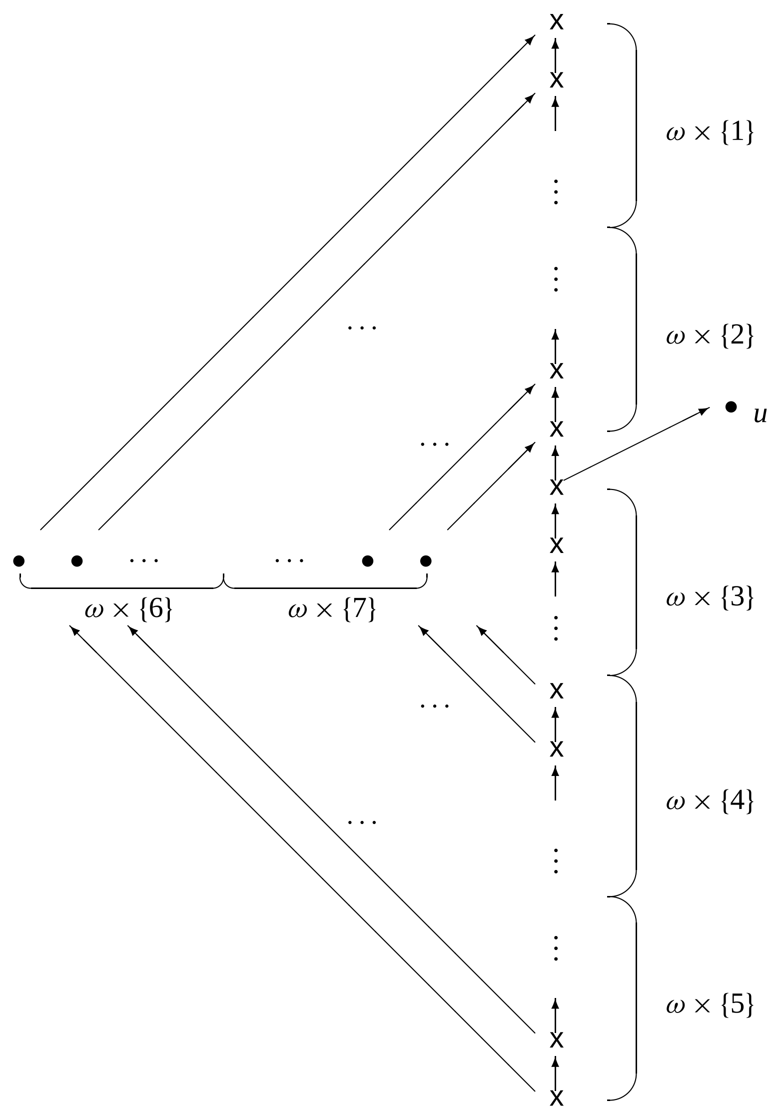

Figure: the frame $\mathcal{G}$

- $c \subseteq \omega \times \{3\}$ or
- $c \subseteq \omega \times \{i, j\}$ and c is finite or cofinite relative to $\omega \times \{i, j\}$ and $\{i, j\} = \{1, 2\}$, $\{4, 5\}$, $\{6, 7\}$.

It is readily checked that $\mathcal{G}$ is a general monomodal frame and also that

$$\mathcal{G}^t = \langle g, R, R^{-1}, A\rangle$$

is a general tense frame (i.e., that A is also closed under

$$\Box^- a := \{x \in g : (\forall y \in g)(yRx \Rightarrow y \in a)\}).$$

Lemma 3 $\mathcal{G} \models \Lambda$.

Proof: Suppose $\mathcal{G}$ refutes a formula $\neg\varphi$. We show that there is an $n \in \omega$ such that $\mathcal{G}_n$ refutes $\neg\varphi$. Take a valuation β so that $\langle \mathcal{G}, \beta\rangle \not\models \neg\varphi$. Call a point $x \in g$ φ-maximal iff there is a subformula ψ of φ such that $x \in \beta(\psi)$ but no proper R-successor of x is in $\beta(\psi)$. Denote by g^r the set of φ-maximal points which are in $\omega \times \{1, \dots, 5\}$. Now define an ordering $\preceq$ on $\omega \times \{6, 7\}$ by putting

$$\begin{array}{lll} (m, i) \preceq (n, j) & \text{iff} & i < j \\ & \text{or} & i = j = 6 \text{ and } m \leq n \\ & \text{or} & i = j = 7 \text{ and } m \geq n. \end{array}$$

Denote by h^r the set of φ-maximal points in $\omega \times \{6, 7\}$ relative to $\preceq$. (We say that $y \in \omega \times \{6, 7\}$ is φ-maximal in $\omega \times \{6, 7\}$ relative to $\preceq$ iff there exists a subformula ψ of φ such that $y \in \beta(\psi)$ and such that there does not exist a $z \in \beta(\psi) \cap (\omega \times \{6, 7\})$ with $y \neq z$ and $y \preceq z$.) Put

$$M := \max\{n \in \omega | (\exists i)(1 \leq i \leq 7 \text{ and } (n, i) \in g^r \cup h^r)\}.$$

Using the definition of A it is readily checked that $M \in \omega$. Put

$$h = \{u\} \cup \{(m, i) |\ m \leq M,\ i = 1, \dots, 7\} \cup \{(m, 3) | m \leq 2M + 1\}.$$

Define $\mathcal{H} = \langle h, R \cap (h \times h)\rangle$ and $\gamma(p) = \beta(p) \cap h$. A straightforward induction shows for all $x \in h$ and subformulas ψ of φ

$$\langle \mathcal{H}, \gamma, x\rangle \models \psi \Leftrightarrow \langle \mathcal{G}, \beta, x\rangle \models \psi.$$

Hence $\mathcal{H}$ refutes $\neg\varphi$ and $\mathcal{H} \simeq \mathcal{G}_{2M+1}$. It follows that $\mathcal{G}^t \models \Lambda^+.t$. $\Box$

We are now going to write down some important formulas belonging to Λ. In what follows we shall assume that Λ is formulated in the monomodal language with $\Box$. Put $\Box^{(1)}\psi = \psi \wedge \Box\psi$. With each finite and rooted frame $\langle h, S\rangle$ we can associate the formula

$$\begin{array}{rcl} W(\langle h, S\rangle) & = & \bigwedge\langle p_x \to \Diamond p_y | xSy\rangle \wedge \\ & & \wedge \bigwedge\langle p_x \to \neg\Diamond p_y | x \neq y, \neg(xSy)\rangle \wedge \\ & & \wedge \bigwedge\langle p_x \to \neg p_y | x \neq y\rangle \end{array}$$

(Here p_x denotes a propositional variable attached to a point $x \in h$). Put $\mathcal{D}_m := \langle\{0, \ldots, m\}, <\rangle$ and

$$dp_m^{\geq} = p_0 \wedge \Box^{(1)} W(\mathcal{D}_m.)$$

Clearly $dp_m^{\geq}$ is satisfiable in a point x in a frame f iff x has depth $\geq m$ in f. By extending the formula $W(\langle h, S\rangle)$ to

$$\Delta(\langle h, S\rangle) = W(\langle h, S\rangle) \wedge \bigwedge\langle p_x \to \neg\Diamond p_y | \neg(xSy)\rangle,$$

we get the well-known subframe formula $\alpha(\langle h, S\rangle) = \Box^{(1)}\Delta(\langle h, S\rangle) \to \neg p_r$, where r denotes a root of $\langle h, S\rangle$ (cf. [2]). The following axioms belong to Λ. (In the frames below 0 is intended to be the root r.)

$$\begin{aligned}
\varphi_1 &= \alpha(\langle\{0, 1, 2\}, \{(0, 1), (0, 2)\}\rangle) \\
\varphi_2 &= \alpha(\langle\{0, 1, 2\}, \{(0, 1), (0, 2), (0, 0)\}\rangle) \\
\varphi_3 &= \alpha(\langle\{0, 1\}, \{0, 1\} \times \{0, 1\}\rangle) \\
\varphi_4 &= \alpha(\langle\{0, 1\}, \{(0, 1), (0, 0), (1, 1)\}\rangle)
\end{aligned}$$

$\varphi_1 \wedge \varphi_2$ says that there are no two incomparable irreflexive points with a common ancestor. The meaning of $\varphi_3 \wedge \varphi_4$ is that there is no infinite strictly ascending chain, no cluster with more than one element and no reflexive point which sees a reflexive point. We now come to the axioms which force the incompleteness of $\Lambda^+.t$. Define, for $i \in \omega$,

$$\begin{aligned}
\alpha_0 &= \Box\bot;\ \alpha_{i+1} = \Box^{i+2}\bot \wedge \neg\Box^{i+1}\bot; \\
\beta_i &= \Diamond\Diamond\alpha_i \wedge \neg\Diamond\alpha_{i+1}; \\
\gamma_0 &= \neg\beta_0 \wedge \Diamond\beta_0;\ \gamma_{i+1} = \neg\beta_{i+1} \wedge \Diamond\beta_{i+1} \wedge \neg\gamma_i \wedge \neg\Diamond\gamma_i.
\end{aligned}$$

In $\mathcal{G}_m$ the formulas α_i hold precisely in $(i, 1)$, $i \leq m$, the formulas β_i hold precisely in $(i, 6)$, $i \leq m$, and the formulas γ_i hold precisely in $(i, 5)$, $i \leq m$. So we have, for all $m \in \omega$,

$$d_m := dp_{3m+2}^{\geq} \wedge \gamma_0 \to \Box^{(1)} \bigwedge\langle\gamma_i \to \Diamond\gamma_{i+1} | i < m\rangle \in \Lambda.$$

For a monomodal formula ψ formulated in the language with $\Box$, let ψ^+ and ψ^- denote the translation of ψ into the language with $\Box^+$ and $\Box^-$ respectively. Put

$$\begin{aligned}
\varphi \;=\; & \Diamond^-\neg\alpha^-(\langle\langle\{0, 1\}, \{0, 1\} \times \{0, 1\}\rangle) \wedge \\
& \wedge \Box^-((p_0 \vee p_1) \to \Diamond^-\gamma_0^+ \wedge \Diamond^+(\Diamond^+\top \wedge \Box^+\Diamond^+\top))
\end{aligned}$$

Lemma 4 $\neg\varphi \notin \Lambda^+.t$.

Proof: Define a valuation β of $\mathcal{G}^t$ so that $\beta(p_0), \beta(p_1) \subseteq \omega \times \{3\}$ are disjoint and so that both sets are cofinal in $\omega \times \{3\}$ with respect to R^{-1} (i.e., $\forall x \in \beta(p_i)\ \exists y \in \beta(p_i)(yRx)$). Clearly $\langle\mathcal{G}^t, \beta, (0, 1)\rangle \models \varphi$. $\Box$

Lemma 5 *$\neg\varphi$ holds in all Kripke-frames for $\Lambda^+.t$.*

Proof: Suppose there is a Kripke-frame $\mathcal{H} = \langle h, S^+, S^- \rangle$ for $\Lambda^+.t$ such that $\mathcal{M}, x \models \varphi$ for a model $\mathcal{M} = \langle \mathcal{H}, \beta \rangle$. By $\varphi_3 \in \Lambda$ and $\mathcal{M}, x \models \varphi$, there is an infinite S^--chain $\langle y_i | i \in \omega \rangle$ with $xS^- y_0$ and $\mathcal{M}, y_i \models p_0 \vee p_1$, for $i \in \omega$. Furthermore, $\mathcal{M}, y_0 \models \Diamond^+(\Diamond^+\top \wedge \Box^+\Diamond^+\top)$. We may assume, by $\varphi_3, \varphi_4 \in \Lambda$, that all $y_i, i \in \omega$, are irreflexive. There are points $z_i, i \in \omega$, with $z_i S^+ y_i$ and $\mathcal{M}, z_i \models \gamma_0^+$.

Claim 6 *There is a $z_i, i \in \omega$, of infinite S^+-depth.*

Assume there is no z_i of infinite depth. Then y_0 has finite depth, say $m \in \omega$. There is a $z_i, i \in \omega$, of depth $\geq 3m + 2$, since the depth of y_i is increasing. Hence there exists y with $z_i S^+ y$ and $\mathcal{M}, y \models \alpha_m^+$. y has depth m, y is irreflexive and y is incomparable with y_0 since $\mathcal{M}, y_0 \models \Diamond^+(\Diamond^+\top \wedge \Box^+\Diamond^+\top)$. But this contradicts $\varphi_1 \wedge \varphi_2 \in \Lambda$.

Take a $z_i, i \in \omega$, of infinite depth. Then $\mathcal{M}, z_i \models (\Box^+)^{(1)}(\gamma_j \rightarrow \Diamond\gamma_{j+1})^+$, for all $j \in \omega$, since $D_m \in \Lambda$. $\mathcal{H}$ contains an infinite strictly ascending S^+-chain which contradicts to $\varphi_3 \in \Lambda$. $\square$

By Lemmas 4 and 5 the logic $\Lambda^+.t$ is incomplete and the Theorem is shown.

One can prove that Λ is not finitely axiomatizable. Hence the following remains open.

Problem 7 Is there a finitely axiomatizable complete logic whose minimal tense extension is incomplete?

Let us finally note another question about transfer from Λ to $\Lambda^+.t$.

Problem 8 Does decidability transfer from Λ to $\Lambda^+.t$?

Although we believe that there is a counterexample, the construction of such an example seems to be quite difficult. Again for all standard systems, decidability transfers, as follows from the results of Wolter [4] and [5].

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REFERENCES

[1] Fine, K., "Logics containing K4, Part I," *The Journal of Symbolic Logic*, vol. 39 (1974), pp. 229–237. Zbl 0287.02010 MR 49:8814 2

[2] Fine, K., "Logics containing K4, Part II," *The Journal of Symbolic Logic*, vol. 50 (1985), pp. 619–651. Zbl 0574.03008 0, 3, 0

[3] Wolter, F., "The finite model property in tense logic," *The Journal of Symbolic Logic*, vol. 60 (1995), pp. 757–774. Zbl 0836.03015 MR 96j:03037 0, 0, 0

[4] Wolter, F., "Completeness and decidability of tense logics closely related to logics above K4," forthcoming in *The Journal of Symbolic Logic*. Zbl 0893.03005 MR 98c:03054 0, 0

[5] Wolter, F., "Tense Logic without tense operators," *Mathematical Logic Quarterly*, vol. 42 (1996), pp. 145–171. Zbl 0858.03019 MR 97c:03074 0

[6] Zakharyaschev, M., "Canonical formulas for K4, Part I," *The Journal of Symbolic Logic*, vol. 57 (1992), pp. 377–402. Zbl 0774.03005 MR 94b:03040 3

Department of Information Science
JAIST
Tatsunokuchi
Ishikawa, 923–12
Japan
email: wolter@jaist.ac.jp