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Pseudo Treealgebras

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Abstract A pseudotree $\langle T, \leq\rangle$ is a partially ordered set for which $\{u \in T : u \leq t\}$ is a linear ordering for each $t \in T$. Define $\mathcal{B}(T)$, the *pseudo treealgebra* over T, as the subalgebra of the power set of T generated by $\{b_t : t \in T\}$ where $b_t = \{u \in T : t \leq u\}$. It is shown that every pseudo treealgebra is embeddable into an interval algebra; thus it is a retractive Boolean algebra. Moreover, superatomicity of $\mathcal{B}(T)$ is described using conditions on $\langle T, \leq\rangle$.

1 Elementary Material

A pseudotree T is a poset in which the set of predecessors of any element is a linearly ordered set. For $t \in T$, put $b_t = \{u \in T : t \leq u\}$. The subalgebra of the power set of T generated by $\langle b_t : t \in T\rangle$ is called the pseudo treealgebra generated by T. Almost all properties of treealgebras remain valid in the case of pseudo treealgebras (see Brenner and Monk [1], Koppelberg [2], and Koppelberg and Monk [3]). Thus we can write a nonzero element of $\mathcal{B}(T)$ in its normal form (see [1]) and for a pseudotree with a least element, the Stone space $\mathcal{U}lt(\mathcal{B}(T))$ of a pseudo treealgebra $\mathcal{B}(T)$ is homeomorphic to $I_c(T)$ = the set of all initial chains endowed with Tychonoff's topology inherited from the catersian product ${}^T2$.

Throughout this note each pseudotree is assumed to have a single root as is shown by the following proposition.

Proposition 1.1 *Any pseudo treealgebra is isomorphic to a pseudo treealgebra over a pseudotree with a single root.*

Proof Let $\mathcal{B}(T)$ be a pseudo treealgebra.

Case 1 T has finitely many roots $t_1, \ldots, t_n$ and no rootless elements. Define $s \leq^* t$ if and only if ($s \leq_T t$ or ($s = t_1$ and $t \neq t_1$)). Let T^* be T under $\leq^*$. Note that $\langle b_t^{T^*} : t \neq t_1\rangle = \mathcal{B}(T^*)$. Define $f(b_t^{T^*}) = b_t^T$ for all $t \neq t_1$. Then f extends to an

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isomorphism of $\mathcal{B}(T^*)$ into $\mathcal{B}(T)$ by Sikorski's Criterion (see Theorem 5.5, p. 67 in [2]).

Case 2 T has infinitely many roots or has a rootless element. Let $x \notin T$ and put $T^* = T \cup \{x\}$. Define $\leq^*$ on T^* as follows:

$$s \leq^* t \text{ iff } (s, t \in T \text{ and } s \leq_T t) \text{ or } (s = x \text{ and } t \in T).$$

Now put $f(b_t^{T^*}) = b_t^T$ for all $t \neq x$. Then f extends, again, to an isomorphism of $\mathcal{B}(T^*)$ onto $\mathcal{B}(T)$ by Sikorski's Criterion. □

Notice that chains are pseudotrees. Hence if C is a chain, $\mathcal{B}(C)$ is called the interval algebra over C. The Stone space $\mathcal{U}lt(\mathcal{B}(C))$ is homeomorphic to the set of initial chains of C, denoted by $I(C)$, whenever C has a least element. Superatomic interval algebras are characterized by the following theorem.

Theorem 1.2 *The following are equivalent for any chain C with a least element.*

1. *η, the chain of rational numbers with its natural ordering, does not embed into C;*
2. *η does not embed into $I(C)$;*
3. *$(I(C), \subset)$ is a scattered topological space;*
4. *$\mathcal{B}(C)$ is a superatomic interval algebra.*

First, we give a definition.

Definition 1.3 Let X be a topological space. We say that $a \in A \subset X$ is an isolated point in A whenever there exists an open set U, in X, containing a so that $U \cap A = \{a\}$. Isol(A) shall denote the set of isolated points of A in X. Also, $\bar{A}$ denotes the topological closure of A in X. A topological space X is a scattered space whenever Isol(F) is not empty for every nonempty closed subspace F of X. Finally, a poset $(P, <)$ is scattered whenever the chain of rational numbers, under its natural ordering, does not embed in $(P, <)$.

Lemma 1.4 *Let C be a complete chain. If C is a scattered topological space, then η does not embed into $(C, <)$.*

Proof First we note the following:

1. If $S \subseteq C$ is infinite, then $\bar{S} \setminus \text{isol}(\bar{S}) \neq \varnothing$. This follows since C is a compact. Now suppose that S is a chain in C of type η; we shall get a contradiction. Choose $x \in S' =_{\text{def}} \bar{S} \setminus \text{isol}(\bar{S})$, x isolated in C. Say, $u < x < v$; $(u, v) \cap C = \{x\}$.
2. There are $s, t \in S$ so that $u < s < t < v$, and $x \notin [s, t]$. In fact, since $x \notin \text{isol}(\bar{S})$, the set $(u, s) \cap \bar{S}$ is infinite. Hence there clearly exist $u < w_1 < w_2 < w_3 < v$ such that $(w_1, w_2) \neq \varnothing \neq (w_2, w_3)$ and $x \notin (w_1, w_2)$. Choose $s \in (w_1, w_2) \cap S$, $t \in (w_2, w_3) \cap S$; this proves (2).

Taking s and t as in (2), put $S' = (x, t) \cap S$. So S' has type η. Clearly $\bar{S'} \setminus \text{isol}(\bar{S'}) \subseteq C$. Picking w in $\bar{S'} \setminus \text{isol}(\bar{S'})$ by (1), we obtain $w \in (u, v) \cap C \setminus \{x\}$, contradiction. □

Remark 1.5 The hypothesis that C is complete in Lemma 1.4 is really needed. This is seen by the example $\omega \cdot \eta$ which is a scattered space.

Proof of Theorem 1.2 (3) and (4) are equivalent by the duality theory. (2) implies (1) since C embeds in $I(C)$. (1) implies (4) since a quotient of $\mathcal{B}(C)$ is isomorphic to $\mathcal{B}(C')$ for some subchain C' of C (see Theorem 15.22, p. 253 in [2]). Finally, (3) implies (2) by Lemma 1.4. □

2 Retractiveness of Pseudo Treealgebras

Our approach to proving that every pseudo treealgebra is in fact a subalgebra of an interval algebra, and hence is a retractive algebra by Rubin's Theorem (see Theorem 15.22, p. 253 in [2]), is done in a very *canonical* and *constructive* way compared to Theorem 16.12, p. 262 in [2]. In this fashion one will have a link between superatomicity of a pseudo treealgebra $\mathcal{B}(T)$ and the superatomicity of the canonical interval algebra in which $\mathcal{B}(T)$ embeds.

Let T be a pseudotree. For each initial chain p of T set

$$T_p =_{\text{def}} \{t \in T : s <_T t \text{ for all } s \in p\}.$$

Next we define $\equiv_p$ on T_p by the following rule:

$$t \equiv_p t' \text{ iff there is } s \in T \setminus p \text{ such that } s \leq_T t, t'.$$

Note then that $s \in T_p$; for if $u \in p$, then $u \leq t, t'$. So u and s are comparable, and $s \leq u$ is ruled out. So $u < s$. Thus $s \in T_p$.

Lemma 2.1 *$\equiv_p$ is an equivalence relation on T_p.*

Proof Suppose $t \equiv_p t' \equiv_p t''$. Say $s, s' \in T \setminus p$ and $s \leq t, t'$ and $s' \leq t', t''$. So s, s' are comparable. Say $s \leq s'$. Thus $s \leq t, t''$ and so $t \equiv_p t''$.

Next, put $s \wedge t =_{\text{def}} \{u \in T : u < s, t\}$ and fix a well-ordering $\preceq_p$ on $T_p/\equiv_p$. Define $\leq_{\text{lin}}$ on T as follows:

$$s \leq_{\text{lin}} t \text{ iff } \begin{cases} s \leq t \text{ in } T, \text{ or} \\ s, t \text{ are incomparable in T and } [s]_{\equiv_{s\wedge t}} \preceq_{s\wedge t} [t]_{\equiv_{s\wedge t}} \end{cases}$$

where $[s]_{\equiv_{s\wedge t}}, [t]_{\equiv_{s\wedge t}}$ denote the equivalence classes of s, t with respect to $\equiv_{s\wedge t}$. □

Lemma 2.2 *$\leq_{\text{lin}}$ is a linear ordering on T.*

Proof Clearly $\leq_{\text{lin}}$ is irreflexive and for all s, t in T, $s \leq_{\text{lin}} t$ or $t \leq_{\text{lin}} s$. Now suppose $x \leq_{\text{lin}} y \leq_{\text{lin}} z$.

Case 1 $x < y < z$. So $x < z$. Thus $x <_{\text{lin}} z$.

Case 2 $x < y$; y, z are incomparable in T, $[y]_{\equiv_{y\wedge z}} <_{y\wedge z} [z]_{\equiv_{y\wedge z}}$. If $x < z$, we are done. Thus, assume $x \nleq z$. Clearly $z \nleq x$. We claim now that $x \wedge z = y \wedge z$. Clearly $x \wedge z \subseteq y \wedge z$. Suppose $w \in y \wedge z$. Thus $w < y$, so w, x are comparable. If $x \leq w$, then $x < z$, contradiction. So $w < x$. Thus $x \wedge z = y \wedge z$. Clearly $[x]_{\equiv_{y\wedge z}} = [y]_{\equiv_{y\wedge z}}$. So $x <_{\text{lin}} z$.

Case 3 x, y incomparable in T. $[x]_{\equiv_{x\wedge y}} <_{x\wedge y} [y]_{\equiv_{x\wedge y}}$; $y < z$. This case is similar to Case 2.

Case 4 x, y incomparable in T. $[x]_{\equiv_{x\wedge y}} <_{x\wedge y} [y]_{\equiv_{x\wedge y}}$; y, z are incomparable in T, $[y]_{\equiv_{y\wedge z}} <_{y\wedge z} [z]_{\equiv_{y\wedge z}}$.

Subcase 4.1 $x \wedge y = y \wedge z$.

1. $x < z$. Thus $x <_{\text{lin}} z$.

2. $x \equiv_{x\wedge y} z$. For $z \le z$, $z < x$, and $z \notin x \wedge y\ (= y \wedge z)$. So $[x]_{\equiv_{x\wedge y}} = [z]_{\equiv_{x\wedge y}}$. Therefore, by the assumption in this case $[x]_{\equiv_{x\wedge y}} = [z]_{\equiv_{x\wedge y}} \preceq_{x\wedge y} [y]_{\equiv_{x\wedge y}} \preceq_{x\wedge y} [z]_{\equiv_{x\wedge y}}$ since $x \wedge y = y \wedge z$. Hence, $[z]_{\equiv_{x\wedge y}} \preceq_{x\wedge y} [z]_{\equiv_{x\wedge y}}$, contradiction.
3. x, z incomparable in T. Since $x \wedge y = y \wedge z$, we have $[x]_{\equiv_{x\wedge y}} \preceq_{x\wedge y} [y]_{\equiv_{x\wedge y}} \preceq_{x\wedge y} [z]_{\equiv_{x\wedge y}}$. Now $x \wedge y = x \wedge z$. For $x \wedge y \subseteq x \wedge z$ is clear, and if $w \in x \wedge z \backslash x \wedge y$, then $[x]_{\equiv_{x\wedge y}} = [z]_{\equiv_{x\wedge y}}$, contradiction. So $[x]_{\equiv_{x\wedge z}} \preceq_{x\wedge z} [z]_{\equiv_{x\wedge z}}$ follows.

Subcase 4.2 $x \wedge y \neq y \wedge z$.

1. There is $w \in x \wedge y \setminus y \wedge z$. Thus $w < x$, $x < y$, and $w \nleq z$.
2. x, z are incomparable. In fact $x \nleq z$. Otherwise $w < z$, and if $z < x$, then w, z are comparable. Hence $z \le w < y$, contradiction.
3. $x \wedge z = y \wedge z$. Let $t \in x \wedge z$. Then $t < x$, so t, w are comparable. If $w \le t$, then $w < z$, contradiction. So $t < w$. Hence $w < z$, contradiction. So $t \le w$, hence $t < x$ as desired. □

Theorem 2.3 *Any pseudo treealgebra embeds into an interval algebra and thus it is a retractive Boolean algebra.*

Proof Let $\mathcal{B}(T)$ be a pseudo treealgebra. First of all we may assume that T has no maximal element. To this end, define $\check{T}$ to be T, and add a well-ordered chain C_t of type ω above each maximal element t in T. Hence $\check{T}$ has no maximal element and by copying the proof of Theorem 16.7, p. 260 in [2], $\mathcal{B}(T)$ embeds in $\mathcal{B}(\check{T})$.

So suppose T has no maximal element and denote by L the completion of $\langle T, \le_{\text{lin}}\rangle$. For each $t \in T$, let $y_t = \sup_L(b_t)$. Note that $y_t \in L \setminus T$. Let 0_T be the root of T and define f from $\mathcal{B}(T)$ into the interval algebra over $L \setminus \{y_{0_T}\}$ by

$$f(b_t) = [t, y_t).$$

Notice that $f(b_t) = 0$ if and only if $t = y_t$ if and only if t is maximal in T; but this never happens.

Next f extends to an isomorphism of $\mathcal{B}(T)$ into $\text{Int}(L \setminus \{y_{0_T}\})$. Indeed, look at

$$(*) \qquad b_{t(1)}, \dots, b_{t(m)} - b_{s(1)} - \cdots - b_{s(n)}.$$

If $(*)$ is zero, we get then three cases.

Case 1 There are i, j so that $t(i), t(j)$ are incomparable. Then either every element of $b_{t(i)}$ is $\le_{\text{lin}}$-less than every element of $b_{t(j)}$ or conversely. In any case we get

$$f(b_{t(i)}) \cap f(b_{t(j)}) = \varnothing.$$

Case 2 There are i, j such that $s_i \le t_j$. Thus $b_{t(j)} \subseteq b_{s(i)}$. So $y_{t(j)} \le y_{s(i)}$, $f(b_{s(i)}) \supseteq f(b_{t(j)})$ as desired.

Case 3 There is an $i_0 \in [1, n] : s(i_0) = 0_T$. So $f(b_{s(i_0)}) = f(1_{\mathcal{B}(T)}) = [0_T, y_{0_T}) = L \setminus \{y_{0_T}\} = 1$. Thus f extends by Sikorski's Criterion to a homomorphism from $\mathcal{B}(T)$ into $\text{Int}(L \setminus \{y_{0_T}\})$. Suppose that $(*)$ is not zero. Without loss of generality $m \neq 0$. If $t(i)$ is maximal among $t(1), \dots, t(m)$, clearly $t(i)$ is in the image of $(*)$. This finishes up the proof of Theorem 2.3. □

3 Characterization of Superatamic Pseudo Treealgebras

Theorem 3.1 *Let T be a pseudotree. The following statements are equivalent.*

1. *$\mathcal{B}(T)$ is a superatomic Boolean algebra.*
2. *η and the binary tree ${}^{<\omega}2$ do not embed in $\langle T, \leq\rangle$.*

The main step in proving this theorem is Lemma 3.4 below. So denote by E the set $T \cup \{y_t : t \in T\} \setminus \{y_{0_T}\}$, where T is a pseudotree without maximal elements, and recall that y_t denotes $\sup(b_t)$ in the completion of $(T, \leq_{\text{lin}})$. Notice that this assumption on T does not restrict the generality as shown by the following two facts. Recall that $\breve{T}$ is constructed as in the beginning of the proof of Theorem 2.3.

Fact 3.2 For any pseudotree, the following statements are equivalent.

1. η or ${}^{<\omega}2$ embeds into T.
2. η or ${}^{<\omega}2$ embeds into $\breve{T}$.

Fact 3.3 $\mathcal{B}(T)$ is superatomic if and only if $\mathcal{B}(\breve{T})$ is.

Lemma 3.4 *The following statements are equivalent.*

1. *E contains η.*
2. *Either η or ${}^{<\omega}2$ embeds in T.*

Assuming Lemma 3.4 we give the proof of Theorem 3.1.

Proof of Theorem 3.1

$\neg$(2) implies $\neg$(1) If η or ${}^{<\omega}2$ embeds in T, then $\text{Int}(\eta)$ or $\mathcal{B}(T_\omega)$ embeds in $\mathcal{B}(T)$, where T_ω is the tree of height ω so that any node in T_ω has ω immediate successors. Hence (1) implies (2) follows.

$\neg$(1) implies $\neg$(2) If $\mathcal{B}(T)$ is not superatomic, then by Fact 3.3 neither is $\mathcal{B}(\breve{T})$. Forming E as we stated previously, it follows that $\text{Int}(E \setminus \{y_{0_T}\})$ is not superatomic. So $\eta \leq E$. So by Lemma 3.4, η or ${}^{<\omega}2$ embeds in $\breve{T}$. Hence by Fact 3.2 η or ${}^{<\omega}2$ embeds into T. This finishes up the proof of Theorem 3.1. □

Proof of Lemma 3.4

(2) implies (1) If η embeds in $\langle T, \leq_T\rangle$ then it embeds into E by the above. Suppose that ${}^{<\omega}2$ embeds into $\langle T, \leq_T\rangle$. Then so does T_ω, where T_ω is of height ω, has one root, and each element has ω immediate successors. Hence $\mathcal{B}(T_\omega)$ (which is atomless) embeds into $\text{Int}(E)$. Hence (1) follows.

(1) implies (2) Suppose that η does not embed in $\langle T, \leq_T\rangle$. Let F be a subset of E of type η. Because of the following fact, we may assume that $F \subseteq T$.

Fact 3.5 If a linear ordering L is scattered, so is its completion.

Proof Since L is scattered, so is $I(L)$ (by Theorem 1.2). Next, since the completion of L is order embeddable in $I(L)$, it follows that the completion of L is scattered as well. □

Now back to the proof of Lemma 3.4. F cannot be a chain in T since $\eta \not\leq T$. Choose $u_0, v_0 \in F$ such that u_0, v_0 are incomparable; say $u_0 <_{\text{lin}} v_0$. Pick $w_0 \in u_0 \wedge v_0$. It suffices now to prove the following.

$$(\theta)\quad \left\{\begin{array}{l}\text{There exist } u_1, u_2, v_1, v_2, w_1, w_2 \text{ so that}\\ \quad 1.\quad u_i, v_i \in F \text{ for } i = 1, 2,\\ \quad 2.\quad u_i \text{ and } v_i \text{ are incomparable for } i = 1, 2,\\ \quad 3.\quad u_i <_{\text{lin}} v_i \text{ for } i = 1, 2,\\ \quad 4.\quad w_1, w_2 \text{ are upper bounds of } u_0, v_0 \text{ in } (T, <),\\ \quad\quad\ \ w_i \text{ is in } u_i \wedge v_i \text{ for } i = 1, 2 \text{ and } w_1, w_2 \text{ are incomparable.}\end{array}\right.$$

First set

$(u_0, v_0)_T =_{\text{def}} \{u \in T : u_0 <_{\text{lin}} u <_{\text{lin}} v_0\}$,
$\Omega = \{s \wedge t : s, t \in (u_0, v_0)_T \cap F; s, t$ are incomparable elements of T$\}$.

Second, $\Omega \neq \varnothing$ since $(u_0, v_0)_T \cap F$ cannot be a chain.

Lemma 3.6 *$(\Omega, \supseteq)$ is not a chain.*

Proof The proof of this lemma uses the following claims. Indeed, suppose the contrary, and let $\mathcal{D}$ be the union of all members of Ω.

Claim 3.7 *If $t, t' \in T$ are incomparable, $s \in T$, and $t <_{\text{lin}} s <_{\text{lin}} t'$; then $t \wedge t' < s$, that is, for all $w \in t \wedge t' (w < s)$.*

Proof For if $t < s$, obviously $t \wedge t' < s$. So assume t, s are incomparable. If $s < t'$, take any $w \in t \wedge t'$. So w, s are comparable. If $s \leq w$, then $s < t$, contradiction. So $w < s$. So $t \wedge t' < s$. Hence we may assume s, t' are incomparable. Now all elements of $(t \wedge t') \cup (t \wedge s)$ are comparable since all are less than t. Hence $t \wedge s \subseteq t \wedge t'$ or $t \wedge t' \subseteq t \wedge s$. Suppose $t \wedge s \subset t \wedge t'$. Pick $w \in (t \wedge t') \setminus (t \wedge s)$. So $w > s$. Thus $t \equiv_{t \wedge s} t'$. We claim that $t \wedge s = t' \wedge s$. One needs only show $t' \wedge s \subseteq t \wedge s$ since the other inclusion is clear by supposition. Suppose $u \in t' \wedge s$. Thus w, u are comparable since both are less than t'. If $w \leq u$, then $w \leq s$, contradiction. So $u < w$. Hence $u < t$. This proves our assertion. Now $[t]_{\equiv_{t \wedge s}} \preceq [s]_{\equiv_{t \wedge s}}$. So $[t']_{\equiv_{t' \wedge s}} \preceq [s]_{\equiv_{t' \wedge s}}$. Hence $t' <_{\text{lin}} s$, contradiction. This shows that $t \wedge t' \subseteq t \wedge s$, and Claim 3.7 holds. □

For each $t \in T$, put $T \downarrow t = \{u \in T : u \leq_T t\}$ and for each $G \subseteq \mathcal{D}$ set

$$T(G) = \{t \in (u_0, v_0)_T \cap (F \setminus \mathcal{D}) : (T \downarrow t) \cap \mathcal{D} = G\}.$$

Claim 3.8 *If t, t' are members of $T(G)$ and are incomparable, then $t \wedge t' = G$.*

Proof Assume the hypothesis. Then $t \wedge t' \in \Omega$, so $t \wedge t' \subseteq \mathcal{D}$. If $u \in t \wedge t'$, then u is in $(T \downarrow t) \cap \mathcal{D} = G$; if $u \in G$, then $u \in (T \downarrow t) \cap (T \downarrow t') = t \wedge t'$. So Claim 3.8 holds. □

Claim 3.9 *If $t \in T(G)$ and $a \in [t]_{\equiv_G} \cap (u_0, v_0)_T \cap (F \setminus \mathcal{D})$, then $a \in T(G)$.*

Proof Say that $G < x$ (i.e., x is above all members of G) and $x \leq a$, $x \leq t$. Since $(T \downarrow t) \cap \mathcal{D} = G$, we have $x \notin \mathcal{D}$. Hence $(T \downarrow a) \cap \mathcal{D} = G$. So $a \in T(G)$. □

Claim 3.10 *If $t \in T(G)$, then $[t]_{\equiv_G} \cap (u_0, v_0)_T \cap (F \setminus \mathcal{D})$ is a chain in T.*

Proof Let $a, b \in [t]_{\equiv_G} \cap (u_0, v_0)_T \cap (F \setminus \mathcal{D})$, and suppose that they are incomparable. Claim 3.8 and Claim 3.9 hold. By Claim 3.9, $a, b \in T(G)$ and so by Claim 3.8 $a \wedge b = G$, contradicting $a \equiv_G b$. □

Claim 3.11 *If $t \in T(G)$, then $[t]_{\equiv_G} \cap (u_0, v_0)_T \cap (F \setminus \mathcal{D}) = \{t\}$.*

Proof Suppose the left-hand side has more than two elements. By Claim 3.10 and $\eta \not\leq T$, let $a < b$ be in the left-hand side, and no member of the left-hand side between them. Say $a <_{\text{lin}} c <_{\text{lin}} b, c \in F$. Suppose a, c are incomparable in $(T, <)$. Now $a \wedge c = b \wedge c$. For $a \wedge c \subseteq b \wedge c$ is clear. Suppose $x \in b \wedge c$. Now $a < b, x < b$, so x, a are comparable. Note that b, c are incomparable ($c < b$ implies a, c are comparable, which is a contradiction). So $b \wedge c \in \Omega, b \wedge c \subseteq \mathcal{D}$. If $a < x$, then $a \in \mathcal{D}$, contradiction. So $x < a$. Thus $a \wedge c = b \wedge c$. $[a]_{\equiv_{a \wedge c}} < [c]_{\equiv_{a \wedge c}}, a \equiv_{a \wedge c} b$. So $[b]_{\equiv_{b \wedge c}} < [c]_{\equiv_{b \wedge c}}, b <_{\text{lin}} c$, contradiction. It follows that $a < c$. Hence $c \in [t]_{\equiv_G} \cap (u_0, v_0)_T \cap (F \setminus \mathcal{D})$, so by Claim 3.10, b and c are comparable, hence $c < b$, contradicting the choice of a and b. □

Claim 3.12 *If $u \in T(G), t \in (u_0, v_0)_T \cap F$, and $t < u$, then $t \in \mathcal{D}$.*

Proof For otherwise Claim 3.11 is contradicted. □

An element $t \in (u_0, v_0)_T \cap (F \setminus \mathcal{D})$ is left of $\mathcal{D}$ whenever it is less than u in $(E, <_{\text{lin}})$ for some $u \in \mathcal{D}$. Suppose there exist such t, u. For $G \subseteq \mathcal{D}$ let $T'(G)$ be defined by

$$T'(G) = \{s \in T(G) : s <_{\text{lin}} t\}.$$

Suppose $| T'(G) | \geq 2$ for some G. By Claim 3.8 through Claim 3.11, $(T'(G), <_{\text{lin}})$ cannot be in itself and thus choose $s <_{\text{lin}} s'$ both in $T'(G)$, with no member of $T'(G)$ between them. Choose $v \in F$ such that $s <_{\text{lin}} v <_{\text{lin}} s'$. Note that s, s' are incomparable by Claim 3.11 and hence by Claim 3.7, $s \wedge s' < v$. If $b \in T(G)$, then since $v <_{\text{lin}} s' <_{\text{lin}} t$, $v \in T'(G)$, contradicting the choice of s, s'. So there is an $x \in \mathcal{D}$, with $G < x \leq v$. Since $s <_{\text{lin}} t <_{\text{lin}} u$, we have $u \notin G$. So $s \wedge s' = s \wedge u = s' \wedge u = G$. Also, $s' \wedge v = G$. In fact, $s' \wedge v \supseteq G$ is true since $s \wedge s' < v$, and to see that $s' \wedge v \subseteq G$, assume that $r < s', r < v$. So r and x are comparable. If $r \leq x$, then $r \in \mathcal{D}$ and hence $r \in G$, as desired. If $x < r$, then $x < s'$, hence $x \in G$, contradiction. Now $[s]_{\equiv_G} < [s']_{\equiv_G} < [u]_{\equiv_G} = [v]_{\equiv_G}$, so $s' <_{\text{lin}} v$, contradiction. So $| T'(G) | \leq 1$, for all G. Let

$$\Omega' = \{s \wedge s' : s, s' \text{ are incomparable members of } (u_0, t)_T \cap F\}.$$

Notice that $\Omega' \subset \Omega$ and by our assumption $(\Omega, \supset)$ is assumed to be a chain. Thus $\Omega' \neq \varnothing$. Now $| \Omega' | \geq 2$. Suppose $\Omega' = \{G\}$. Pick incomparable elements s, s' in $(u_0, t)_T \cap F$. So $s \wedge s' = G$. Say $s \notin \mathcal{D}$. Pick incomparable $w, w' \in (u_0, s)_T \cap F$. Say $w \notin \mathcal{D}$. Then $w \in T'(G), s \in T'(G), w \neq s$, but this contradicts $| T'(G) | \leq 1$. So $| \Omega' | \geq 2$.

The next fact follows easily.

Fact 3.13 If D is a chain and $E = I(D)$ is the set of all initial segments of D, then D is scattered if and only if $(E, \supseteq)$ is scattered.

Hence, notice that $(\Omega, \supset)$ is scattered since $\eta \not\leq (T, <)$ and thus choose $G, H \in \Omega'$ with $G \subseteq H$, so that no member of Ω' is between them. Pick $s \in T(G), s' \in T(H)$. So $s \in T'(G), s' \in T'(H)$. Note that $s \wedge s' = G = s \wedge u$, so any $h \in H \setminus G$ shows that $s' \equiv_G u$ and so $s <_{\text{lin}} s'$. Pick incomparable w, w' in $(s, s')_T \cap F$. Say $w \notin \mathcal{D}$. Say $w \in T(K)$. So $w \in T'(K)$. Now s and s' are incomparable by Claim 3.11. So $s \wedge s' < w$ by Claim 3.7, that is, $G < w$. Hence $H \subset K$ by the choice of G and H plus $T'(G) = \{s\}, T'(H) = \{s'\}$. But then $s' <_{\text{lin}} u$ implies

$s' <_{\text{lin}} w$ (by considering an element of $K \setminus H$), contradiction. Thus, no element of $(u_0, v_0)_T \cap (F \setminus \mathcal{D})$ is left of $\mathcal{D}$. Suppose $|\ T(G)\ | \geq 2$, for some G. Let $t <_{\text{lin}} t'$ both in $T(G)$ with no element of $T(G)$ between them. We easily reach a contradiction as in the case $|\ T'(G)\ | \geq 2$ above. So $|\ T(G)\ | \leq 1$ for all G. Then we reach a contradiction as above. This finishes up the proof of Lemma 3.6. □

So (θ) is finally established. Choose $G, H \in \Omega$, incomparable. Take $w_1 \in G \setminus H$, $w_2 \in H \setminus G$. Without loss of generality, $w_1 <_{\text{lin}} w_2$. Clearly, w_1, w_2 are incomparable. Say $u_1 <_{\text{lin}} v_1$, u_1 and v_1 are incomparable, $u_1, v_1 \in F \cap (u_0, v_0)_T$, and $u_1 \wedge v_1 = G$. Similarly, we get u_2, v_2 in H. $u_0 <_{\text{lin}} w_1 <_{\text{lin}} v_0$; so $u_0 \wedge v_0 < w_1$ by Claim 3.7. Hence we are through with the proof of Lemma 3.4. □

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