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A Note on Recursive Models of Set Theories

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Abstract We construct two recursive models of fragments of set theory. We also show that the fragments of Kripke-Platek set theory that prove ε-induction for Σ_1-formulas have no recursive models but the standard model of the hereditarily finite sets.

1 Introduction

We ask which fragments of Kripke-Platek set theory have recursive models. Ideally, we would like to separate fragments that have (nontrivial) recursive models from those whose unique recursive model is the standard model of the hereditarily finite sets. In the context of models of arithmetic these questions have received considerable attention. Two classical results are well known: Tennenbaum's theorem [5] that says that every recursive model of $I\Sigma_1$ is isomorphic to the standard model and Shepherdson's theorem [4] that proves the existence of a recursive model of Open-Induction. Tennenbaum's theorem has been sharpened in Wilmers [6] where it is shown that IE_1 has no nonstandard recursive models. On the other side, Berarducci and Otero in [1] have shown that Open-Induction+'there are infinitely many primes' has a recursive model.

For fragments of set theory much less is known. Here we expose a few basic facts that can be obtained from classical techniques. We also present some open problems. We shall see that weak fragments of set theory have two ways of being nonstandard: they may be *simply* non-wellfounded (there is an infinite descending chain of sets) or *strongly* non-wellfounded (there is an infinite descending chain of ordinals). We construct recursive models for both the weak and the strong notion of non-wellfoundedness and show that some theories may have a recursive model of one sort but not of the other. We show that Tennenbaum's theorem in its strongest form (the unique recursive model is the standard model of the hereditarily finite sets)

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holds for the theory $\mathrm{KP}\Sigma_1$. This is the theory axiomatized by KP_- (extensionality, pair, union, foundation, Δ_0-comprehension, and Δ_0-collection) and the axiom of (ε)-induction

$$\forall a\,[(\forall x \in a)\varphi(x) \to \varphi(a)] \to \forall a\,\varphi(a)$$

restricted to Σ_1-formulas. The theory $\mathrm{KP}\Delta_0$ is defined by restricting induction to Δ_0-formulas.

2 Tennenbaum's Theorem for Set Theory

A *recursive model* of a fragment of KP is a domain $\mathcal{M}$, a binary a relation ε on $\mathcal{M}$, and a bijection of $\mathcal{M}$ onto the natural number that maps ε into a recursive set of pairs. We show that there is no recursive model of $\mathrm{KP}\Sigma_1$ but the standard model of the hereditarily finite sets V_ω (up to isomorphism, of course). As expected, the core of the argument lies in the idea of Tennenbaum's classical theorem [5], but before we can apply it, we must overcome a couple of difficulties. The first thing we need to show is that if the ordinals of a model of $\mathrm{KP}\Sigma_1$ are isomorphic to ω then the model itself is isomorphic to V_ω. (Throughout the paper ω is the set of the standard finite ordinals and V_ω is the standard model of the hereditarily finite sets.)

Lemma 2.1 *Let $\mathcal{M}$ be a model of $\mathrm{KP}\Sigma_1$. Then exactly one of the following occurs:*

1. *$\mathcal{M}$ contains a nonstandard finite ordinal;*
2. *$\mathcal{M}$ contains as an element a copy of the true ω;*
3. *$\mathcal{M}$ is isomorphic to V_ω.*

Proof From the Mostowski collapsing lemma we infer that if $\mathcal{M}$ is well-founded either (2) or (3) of Lemma 2.1 holds. When $\mathcal{M}$ is non-wellfounded the lemma follows from Lemmas 2.3 and 2.4 below. □

The next two facts show that in every non-wellfounded model of $\mathrm{KP}\Sigma_1$ there is an ordinal not in ω so that either (1) or (2) of Lemma 2.1 obtains. In the next section we shall see that this need not be true for other fragments of set theory. A model is said to be *non-wellfounded* if it has elements $\{c_i\}_{i\in\omega}$ forming an infinite descending ε-chain: $c_{i+1}\ \varepsilon\ c_i$ for all $i \in \omega$.

Remark 2.2 We shall only consider fragments containing the axiom of foundation so every infinite decending chain $\{c_i\}_{i\in\omega}$ is necessarily external.

Let f be an element of $\mathcal{M}$. We say that f *is (or codes) a descending ε-chain* if f is a function, dom f is an ordinal, and $f(\alpha+1)\ \varepsilon\ f(\alpha)$ for every $\alpha+1\ \varepsilon$ dom f. The property of being a descending ε-chain is naturally expressed by a Δ_0-formula. Note that in a model of the axiom of foundation the domain of an ε-descending chain is (for the model) a finite ordinal. We write $f{\restriction}_\alpha$ for the restriction of f to α and $f(\alpha)\!\downarrow$ for $\alpha \in \mathrm{dom}\, f$.

Lemma 2.3 *The following is a theorem of $\mathrm{KP}\Sigma_1$. For every x there is a set $a \neq \varnothing$ such that for every $f \in a$*

1. *f codes a descending chain,*
2. *$f(0) = x$,*
3. *for every y and α such that $y \in f(\alpha)\!\downarrow$ there is a $g \in a$ such that $y = g(\alpha+1)\!\downarrow$ and $g{\restriction}_{(\alpha+1)} = f{\restriction}_{(\alpha+1)}$.*

Proof The reader can check that the conjunction of (1), (2), and (3) of Lemma 2.3 is naturally formalizable by a Δ_0-formula—we denote this formula with $\varphi(x, a)$. We show that if $(\forall y \in x)(\exists b)\, \varphi(y, b)$, then there is an a satisfying $\varphi(x, a)$. The lemma will follow applying ε-induction. We sketch the construction of a leaving details to the reader. Using collection, find a set B such that $(\forall y \in x)(\exists b \in B)\, \varphi(y, b)$. By Δ_0-comprehension, we can further require that $(\forall b \in B)(\exists y \in x)\, \varphi(y, b)$. Now observe that for every $f \in \bigcup B$ there exists some descending $\in$-chain f' such that $\operatorname{dom} f' = \operatorname{dom} f + 1$, $f'(0) = x$, and $f'(\alpha + 1) = f(\alpha)$ for every $\alpha \in \operatorname{dom} f$. Let a be a set containing all (and only) these functions. Check that a satisfies (1), (2), and (3) of Lemma 2.3. □

Lemma 2.4 *Let M be a model of* $\mathrm{KP}\Sigma_1$. *If in M there is an infinite descending chain, then M contains an ordinal not in ω.*

Proof Let $\{c_i\}_{i\in\omega}$ be an infinite descending chain. For every $n \in \omega$ there is an f in M that codes the descending chain $c_{n-1} \in \cdots \in c_0$. That is, $f(i) = c_i$ for $i = 0, \ldots, n-1$. Let a be the set given by Lemma 2.4 when we substitute c_0 for x. Using (3) of Lemma 2.3 it is easy to show, by external induction on n, that a contains some extension of f. Let D be the set of the domains of the functions in a. The set $\bigcup D$ is an ordinal and every $n \in \omega$ belongs to it. The lemma follows. □

Theorem 2.5 *Every recursive model of* $\mathrm{KP}\Sigma_1$ *is isomorphic to* V_ω.

Proof Let $\mathcal{M}$ be a recursive model of $\mathrm{KP}\Sigma_1$ nonisomorphic to V_ω. Let $\alpha \in \mathcal{M}$ be either ω or any nonstandard finite ordinal. The existence of α is guaranteed by Lemma 2.1. To apply Tennenbaum's trick in our setting we need show that the successor function $S : \beta \mapsto \beta+1$ becomes a recursive function when $\mathcal{M}$ is identified with the natural numbers. (Clearly, this problem does not occur for models in the language of arithmetic.) As a matter of fact, it suffices to restrict the domain of S to α. We use as parameters the following set s:

$$\mathcal{M} \models x\ \varepsilon\ s \leftrightarrow (\exists \beta\ \varepsilon\ \alpha)\ x = \{\beta, \{\beta, \beta + 1\}\}.$$

(Shavrukov drew our attention to this algorithm). On input β the algorithm to compute $\beta + 1$ is as follows. List all elements of $\mathcal{M}$ until one x is found such that $x\ \varepsilon\ s$ and $\beta\ \varepsilon\ x$. List again the elements of $\mathcal{M}$ to find one $y \neq \beta$ such that $y\ \varepsilon\ x$. Look for $z \neq \beta$ such that $z\ \varepsilon\ y$. The definition of s guarantees that such a z exists. Output z. It is immediate that when $\beta\ \varepsilon\ \alpha$ then $S(\beta){\downarrow} = \beta + 1$. (Clearly $S(\beta){\uparrow}$ when $\beta \not\varepsilon\ \alpha$.)

At this point the theorem proceeds as in the arithmetical case. The reader should convince him/herself that formalization of recursive computations is possible in $\mathrm{KP}\Sigma_1$. □

3 A Recursive Model with Only Standard Ordinals

In this section we consider a weak form of non-wellfoundedness. We construct a recursive non-wellfounded model of KP_- where all the ordinals are standard finite. The model constructed contains only sets with a (standard) finite number of elements, so it is a model of the whole of ZF up to the axiom of infinity. We do not know precisely how much ε-induction holds in it. All we know is that, by Theorem 2.5, Σ_1-induction fails whereas from [7] we know that open-induction holds.

Theorem 3.1 *There is a recursive model of* ZF *minus the axiom of infinity.*

Proof The proof is taken from [3]. It uses the so-called Fraenkel-Mostowski "permutation model" (see, e.g., [2]). The domain of the model is V_ω, the set of hereditarily finite sets. A new membership relation ε^f is defined on it. Let $f : V_\omega \to V_\omega$ be a bijection. We define ε^f as follows:

$$x \ \varepsilon^f \ y \Leftrightarrow x \in f(y).$$

Clearly, if f is recursive, then the model $\langle V_\omega, \varepsilon^f \rangle$ is recursive. In $\langle V_\omega, \varepsilon^f \rangle$ the cardinality of any $a \in V_\omega$ cannot exceed the cardinality that $f(a)$ has in the standard model. Therefore, every set in $\langle V_\omega, \varepsilon^f \rangle$ is standard finite. It is well known that $\langle V_\omega, \varepsilon^f \rangle$ models all the axioms of ZF but the axiom of infinity and the axiom of foundation. (In general, it is sufficient that the relation $x \in f(y)$ is definable in V_ω.) We check that a careful choice of f makes $\langle V_\omega, \varepsilon^f \rangle$ a model of the axiom of foundation.

Definition 3.2 Let $\omega^* = \{\{n+1\} : n \in \omega\}$. Define the bijection f on V_ω as follows: $f(n) = \{n+1\}$, $f(\{n+1\}) = n$, and $f(a) = a$ if $a \notin \omega \cup \omega^*$.

From $f(n) = \{n+1\}$ follows that $n+1 \ \varepsilon^f \ n$, so there is an infinite descending ε^f-chain $\cdots \varepsilon^f \ n+1 \ \varepsilon^f \ n \ \varepsilon^f \ \cdots \varepsilon^f \ 1 \ \varepsilon^f \ 0$. It remains to prove that $\langle V_\omega, \varepsilon^f \rangle$ is a model of the axiom of foundation. Since $x \in f(y)$ is definable in $\langle V_\omega, \in \rangle$, to show that $\langle V_\omega, \varepsilon^f \rangle$ is a model of ZF$^-$, it suffices to prove that $\langle V_\omega, \varepsilon^f \rangle$ models the axiom of foundation. Let a be an arbitrary element of V_ω. We consider three cases. First suppose that $n \ \varepsilon^f \ a$ for some $n \in \omega$. Let n be the largest (as standard ordinal) n such that $n \ \varepsilon^f \ a$ (recall a is finite in the standard sense). By the definition of f we have that $x \ \varepsilon^f \ n$ if and only if $x \in \{n+1\}$ if and only if $x = n+1$. So, n being the largest ε^f-element of a, $\langle V_\omega, \varepsilon^f \rangle$ models $n \cap a = \varnothing$. Now suppose the first case does not obtain and that a contains some element of the form $\{n+1\}$ for $n \in \omega$. Observe that $x \ \varepsilon^f \ \{n+1\}$ if and only if $x \in n$, so since the first case as been excluded, $x \not\varepsilon^f a$. Again we conclude that $\langle V_\omega, \varepsilon^f \rangle$ models $\{n+1\} \cap a = \varnothing$. Finally, we are left to consider the case when a contains no elements in $\omega \cup \omega^*$. Observe that a itself is not in $\omega \cup \omega^*$. Let b be such that $b \in a \wedge b \cap a = \varnothing$ holds in the standard model $\langle V_\omega, \in \rangle$. We claim that $\langle V_\omega, \varepsilon^f \rangle$ also models $b \cap a = \varnothing$. Since f is the identity on a, $b \ \varepsilon^f$ is clear. Since f is the identity also on b, if $x \ \varepsilon^f \ a$ and $x \ \varepsilon^f \ b$, then $x \in b \wedge x \in a$. $\square$

An immediate corollary of the construction above is that ZF minus the axiom of infinity does not prove that every set is contained in a transitive set. In fact, the transitive closure of 0 does not exists in $\langle V_\omega, \varepsilon^f \rangle$—it should be infinite. We do not know if there are nontrivial recursive models of KPΔ_0. (Note that, as far as we know, KPΔ_0 could coincide with KP$_-$.)

4 A Recursive Model with Nonstandard Ordinals

We conjecture the existence of recursive models of KPΔ_0 having nonstandard finite ordinals. However, at the moment we cannot exhibit any of such models even for KP$_-$. Here we present a much weaker result. For convenience, we denote by C the theory axiomatized as KPΔ_0 without the schema of collection.

Theorem 4.1 *Let $\mathbb{L}$ be a discrete linear order with a first but no last element. Then there is a model of C whose finite ordinals are isomorphic to $\mathbb{L}$.*

Proof Let $\mathbb{L}$ be as above; let 0 be the first element of $\mathbb{L}$. An *interval* of $\mathbb{L}$ is a set of the form $[a, b) = \{x \in \mathbb{L} : a \leq x < b\}$ for some $a, b \in \mathbb{L}$. Let $\mathcal{I}$ be the set of all intervals and let $\mathcal{O} \subseteq \mathcal{I}$ be the set of intervals of the form $[0, a)$ (these will turn out to be the ordinals of the model). The domain of the model $\mathcal{M}$ is a subset of $\bigcup_{i\in\omega} \mathcal{P}^{i+1}(\mathbb{L})$.

(a) Define $\langle x\rangle$ to be the singleton $\{x\}$ when x is a set not in $\mathcal{O}$—that is, when $x = [0, a) \in \mathcal{O}$ for some $a \in \mathbb{L}$—else, we let $\langle x\rangle = \{a\}$ (i.e., the interval $[a, a + 1)$).
(b) The domain of $\mathcal{M}$ is the closure of $\mathcal{I}$ under the operations $\cup$ (binary union) and $\langle\cdot\rangle$.
(c) We define $t \ \varepsilon\ s$ if and only if $\langle t\rangle \subseteq s$. That is, $t \ \varepsilon\ s$ when either $t \in s$ or when $t = [0, a)$ for some $a \in \mathbb{L}$ and $a \in s$.

We claim that $\langle\mathcal{M}, \varepsilon\rangle$ is a model of C. Below we show in turn that the axioms of exensionality, pairing, and union hold in $\langle\mathcal{M}, \varepsilon\rangle$; thereafter we consider the axioms of comprehension and induction (these require more details). First we list without proof some easy facts that are needed below.

1. $t \cap \mathcal{O} = \varnothing$ for every $t \in \mathcal{M}$ (intersection is in the sense of the true membership);
2. if $x \in t \in \mathcal{M}$ then either $x \in \mathcal{M}$ (hence $x \ \varepsilon\ t$) or $x \in \mathbb{L}$ and $[0, x) \ \varepsilon\ t$;
3. $y \ \varepsilon\ \langle x\rangle$ if and only if $y = x$ holds for every $x, y \in \mathcal{M}$ (i.e., $\langle x\rangle$ is the singleton of x in the model $\langle\mathcal{M}\ \varepsilon\rangle$);
4. $x \ \varepsilon\ t \cup s$ if and only if $x \ \varepsilon\ t \vee x \ \varepsilon\ s$ for every $x \in \mathcal{M}$, (i.e., binary union in $\langle\mathcal{M}, \varepsilon\rangle$ and in the real world coincide).

To show that extensionality holds in $\mathcal{M}$ we have to prove that if $\forall x\, [x \ \varepsilon\ t \leftrightarrow x \ \varepsilon\ s]$ then $t = s$. It suffices to show that $\forall x\, [x \in t \leftrightarrow x \in s]$ and apply extensionality in the real world. Let $x \in t$. First, assume that $x \in \mathcal{M}$. By (2) above, $x \ \varepsilon\ t$ from which it follows that $x \ \varepsilon\ s$, and since by (1) above $x \notin \mathcal{O}$, that $x \in s$. Second, suppose that $x \notin \mathcal{M}$. Then from $x \in t$ it follows that $x \in \mathbb{L}$, which implies $[0, x) \ \varepsilon\ t$ and hence $[0, x) \ \varepsilon\ s$. So $x \in s$. The converse is symmetric.

The pairing axiom holds in $\mathcal{M}$ by (3) and (4) above. Indeed, the pair of t and s is given by $\langle s\rangle \cup \langle t\rangle$.

The axiom of union is proved by induction on the construction of $\mathcal{M}$. Given $t \in \mathcal{M}$ we must show that for some $t^* \in \mathcal{M}$ we have $(\forall x \ \varepsilon\ t)(\forall y \ \varepsilon\ x)\ y \ \varepsilon\ t^*$. When $t = \varnothing$ then $t^* := \varnothing$ suffices. If t is a nonempty interval—say $t = [a, b)$ with $a < b$—then we let $t^* := [0, b)$. If $t_1 \cup t_2$ then let $t^* = t_1^* \cup t_2^*$. Property 4 above guarantees that this is a correct choice. Finally when $t = \langle s\rangle$, let $t^* = s$.

Now we prove that the schemata of Δ_0-comprehension and induction hold in $\mathcal{M}$. We need to prove a quantifier elimination lemma for Δ_0-formulas. For this it is convenient to consider formulas in the language expanded with functions symbols for $\langle\cdot\rangle$ and $\cup$. We write Δ_0^* for the class of bounded formulas in this expanded language. Terms may appear in the bound of the quantifiers.

Claim 4.2 *Every Δ_0^*-formula with parameters in $\mathcal{M}$ is equivalent to a Δ_0-formula without symbols of equality and with parameters occurring only on the right-hand side of ε. The required formula is obtained by applying in turn the following three procedures.*

1. *Eliminate atomic formulas of the form $t \ \varepsilon\ s$, when t is not a variable, by replacing $t \ \varepsilon\ s$ with $(\exists y \ \varepsilon\ s)\ t = y$.*

2. *Eliminate equalities by replacing $s = t$ with $(\forall x \;\varepsilon\; t)x \;\varepsilon\; s \wedge (\forall x \;\varepsilon\; s)x \;\varepsilon\; t$ (this does not spoil (1)).*
3. *Eliminate all complex terms (and leave only variables and parameters). This is possible because every atomic formula $x \;\varepsilon\; t$ where t has complexity $n+1$ is equivalent to a formula where all terms have complexity n; this formula does not contain equalities and only variables occur on the left-hand side of ε (so we do not spoil (1) and (2) above). More precisely, $x \;\varepsilon\; t_1 \cup t_2$ is equivalent to $x \;\varepsilon\; t_1 \wedge x \;\varepsilon\; t_2$ and $x \in \langle t\rangle$ is equivalent to $(\forall y \;\varepsilon\; t)y \;\varepsilon\; x \wedge (\forall y \;\varepsilon\; x)y \;\varepsilon\; t$. So the procedure is clear.*

Claim 4.3 *For every Δ_0^*-formula $\varphi(x_1, \ldots, x_n)$ with parameters in $\mathcal{M}$ there is a Δ_0-formula $\psi(x_1, \ldots, x_n)$ with parameters in $\mathcal{O}$ that is equivalent to φ for all $x_1, \ldots, x_n$ in $\mathcal{O}$.*

First write an equivalent formula with parameters in $\mathcal{I}$ (recall that $\mathcal{I}$ generates $\mathcal{M}$). Now eliminate in $\varphi(x_1, \ldots, x_n)$ equalities and functions as in Claim 4.2 and assume that no parameters occur on the right-hand side of ε. If the formula $x \;\varepsilon\; t$ with $t = [a, b)$ and $a > 0$ occurs in $\varphi(x_1, \ldots, x_n)$, substitute it with $x \;\varepsilon\; [0, b) \wedge x \not\varepsilon\; [0, a-1)$. This proves Claim 4.3.

Claim 4.4 *For every Δ_0^*-formula $\varphi(x_1, \ldots, x_n)$ with parameters in $\mathcal{M}$ there is a quantifier-free formula $\theta(x_1, \ldots, x_n)$ with parameters in $\mathcal{O}$ that is equivalent to $\varphi(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n$ in $\mathcal{O}$.*

Apply Claim 4.3 above to obtain a formula $\psi(x_1, \ldots, x_n)$. Observe that when $x_1, \ldots, x_n$ range over $\mathcal{O}$, we can restrict the evaluation of $\psi(x_1, \ldots, x_n)$ to $\mathcal{O}$. The order ε in $\mathcal{O}$ is a discrete linear order, so quantifiers can be eliminated. This proves Claim 4.4.

We can now prove comprehension for Δ_0-formulas. Let $\varphi(x)$ be a Δ_0-formula with parameters in $\mathcal{M}$ and let t be an element of $\mathcal{M}$. We need to show that $\{x \in t : \varphi(x)\}$ exists. Clearly, we can assume that t is a closed term depending on parameters in $\mathcal{I}$ and that $\varphi(x)$ is a Δ_0^*-formula with all parameters in $\mathcal{I}$. Proceed by induction on the complexity of t. If t is atomic, then we can restrict x to range over the $\mathcal{O}$. Apply Claim 4.4 to find an open formula $\theta(x)$ equivalent to $\varphi(x)$. Clearly, $x \in t \wedge \theta(x)$ defines an interval of $\mathbb{L}$ and hence is in $\mathcal{I}$. The induction steps are straightforward.

It remains to prove induction for Δ_0-formulas. Let $\varphi(x)$ be a Δ_0-formula and suppose $\exists x\varphi(x)$ holds in $\mathcal{M}$. So $\varphi(t)$ holds for some closed term t depending on parameters in $\mathcal{I}$. We prove that there exists an ε-least witness of $\varphi(x)$. Let n be the complexity of t and suppose $t_0 \;\varepsilon \cdots \varepsilon\; t_n \;\varepsilon\; t$ are terms such that $\varphi(t_i)$. We claim that t_0 is in $\mathcal{O}$. The claim is immediately proved by induction on n. So $\varphi(a)$ holds for some $a \in \mathcal{O}$. Now the existence of an ε-least element satisfying $\varphi(x)$ follows from Claim 4.4. This completes the theorem. □

Theorem 4.5 *There is a recursive model of C.*

Proof Fix $\mathbb{L} := \mathbb{N} \cup \mathbb{Z}$ where $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is a copy of the positive integers disjoint of $\mathbb{Z}$. The order relations of $\mathbb{N}$ and $\mathbb{Z}$ are extended to $\mathbb{L}$ by stipulating that the elements of $\mathbb{N}$ precede any element of $\mathbb{Z}$. The reader can verify that the model defined in the proof of the theorem above is recursive. □

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