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The Categoricity Spectrum of Pseudo-elementary Classes

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Abstract Given a pseudo-elementary class $\mathcal{K}$ we investigate the associated class of cardinals where $\mathcal{K}$ is categorical. We show that any such class must be closed and if it is nonempty then there is an ordinal $\delta \neq 0$ so that $\{\kappa : \kappa < \beth_{\delta\cdot\alpha}$ and $\mathcal{K}$ is κ-categorical$\}$ is closed and unbounded in $\beth_{\delta\cdot\alpha}$ for all $\alpha > 0$. Also, assuming the consistency of a huge cardinal, we show that the statement "$\mathcal{K}$ $\aleph_2$-categorical implies $\mathcal{K}$ $\aleph_3$-categorical" is independent of ZFC.

1 Introduction In [8] Morley proved that if an elementary class in a countable language is categorical in some uncountable power, then it is categorical in all uncountable powers. The result was extended to elementary classes in uncountable languages by Shelah. The aim of this paper is to explore possible generalizations of these results to pseudo-elementary (PC_Δ) classes (i.e., reducts of an elementary class to a smaller language).

In [1] Keisler proved that one direction of Morley's theorem extends to pseudo-elementary classes. He showed that if a PC_Δ class in a countable language is $\aleph_1$-categorical, then it is categorical in every uncountable power. However, Silver gave an example of a PC_Δ class that is κ-categorical if and only if κ is a strong limit cardinal. The other known positive result was proved independently by Keisler [4], Čudnovskiǐ [2], and Shelah [11]. Suppose $\mathcal{K}$ is a PC_Δ class whose underlying language has power λ and $\beth_\delta$ is the Hanf number for omitting a type in a first order language of power λ (e.g., if $\lambda = \aleph_0$ then $\beth_\delta = \beth_{\omega_1}$). They showed that if $\mathcal{K}$ is categorical in some power $> \lambda$, then $\mathcal{K}$ is categorical in all powers $\beth_{\delta\cdot\alpha}$ for $\alpha \in \mathrm{ORD}$, $\alpha > 0$.

In Section 2, we obtain two new positive results. First, we show that for any pseudo-elementary class $\mathcal{K}$, the class of cardinals κ where $\mathcal{K}$ is κ-categorical is closed in the order topology. Next we extend the result above by showing that if $\mathcal{K}$ is a pseudo-elementary class whose underlying language has power λ and $\mathcal{K}$ is categorical in some power $> \lambda$ then $\{\beta < \delta\cdot\alpha : \mathcal{K}$ is $\beth_\beta$-categorical$\}$ is closed and unbounded in $\delta\cdot\alpha$ for all $\alpha > 0$. In particular, if the underlying language of $\mathcal{K}$ is countable and $\mathcal{K}$ is categorical in some uncountable power, then $|\{\mu <$

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$\beth_{\omega_1} : \mathcal{K}$ is μ-categorical$\}| \geq \aleph_1$. Silver's example demonstrates that this result is the best possible.

In Section 3 we obtain a number of negative results about specific cardinals. We first obtain a converse to the theorem of Keisler, Čudnovskiĭ, and Shelah. If, for all PC_Δ classes having a countable underlying language, categoricity in some uncountable cardinal implies κ-categoricity, then $\kappa = \beth_{\omega_1 \cdot \alpha}$ for some $\alpha > 0$. Next we ask whether $\mathcal{K}$ being κ-categorical implies $\mathcal{K}$ being λ-categorical for specific values of κ and λ. There are a couple of cases that are still open, but our results indicate that very few (if any) transfer results are provable in ZFC other than the two mentioned above. As so little can be proved even in the case where $\mathcal{K}$ has a countable underlying language, we make this assumption throughout this section.

Concerning the question of whether $\mathcal{K}$ being $\aleph_{\alpha+2}$-categorical implies $\mathcal{K}$ $\aleph_{\alpha+3}$-categorical (i.e., $\aleph_{\alpha+2} \to \aleph_{\alpha+3}$ in the notation of Definition 3.1), we find that, assuming the consistency of a huge cardinal above $\aleph_\alpha$, the question is independent of ZFC. One direction simply amounts to translating results of Mitchell [7] on special Aronszajn trees to our context. The other direction comes by showing that the relation $\kappa^+ \to \kappa^{++}$ follows from a generalization of Keisler's two-cardinal theorem. Then, given a huge cardinal above κ, we employ a construction of Kunen [5] to establish the consistency of this generalization for regular κ.

Next we explore other instances of $\kappa \to \lambda$ and find that most of these are either refutable or independent of ZFC. (A few are still open.) For example, most instances of $\kappa \to \lambda$ with $\lambda < \kappa$ are refuted by employing Morley's notion of a cardinal being characterizable. We also show that many instances of $\kappa \not\to \lambda$ follow from certain properties of cardinal exponentiation that are at least relatively consistent with ZFC. We conclude this survey by investigating the consistency of $\forall \kappa < 2^{\aleph_0} (\kappa \to 2^{\aleph_0})$ for various choices of the continuum.

Our final topic of this section is to show by a routine Hanf number argument that there is a cardinal κ_0 large enough to determine the full categoricity spectrum of a PC_Δ class from its initial segment below κ_0. However, we show that there is no provable upper bound on κ_0 in the $\beth$-hierarchy. In Section 4 we state a few problems that are still open.

As for notation, $\kappa, \lambda, \mu, \rho$ always denote infinite cardinals. In Section 3 we insist that they be uncountable as well. $\alpha, \beta, \gamma, \delta, \eta$ denote ordinals. If $\varphi(\bar{x})$ is a formula, φ^M denotes $\{\bar{a} \in |M|^n : M \models \varphi[\bar{a}]\}$.

If a first order theory has a distinguished unary predicate U, we say M is a (κ,λ) model if M has power κ and $|U^M| = \lambda$. M is $(\kappa, < \kappa)$ if M is a (κ,λ) model for some $\lambda < \kappa$ and in this case, we call M a two-cardinal model.

$S_L(B)$ denotes the set of complete one-types in the language L with parameters from the set B. (B will always be a subset of the universe of an L-structure.) When L is understood we simply write $S(B)$. If $a \in |M|$, $p \in S(B)$, we write $a \models p$ to denote $M \models p[a]$.

$S_L^*(B)$ denotes the set of complete strong types in one variable in the language L with parameters from B. If $p \in S^*(A)$ and p is based on B, then $p|B$ denotes the unique $q \in S^*(B)$ parallel to p.

Formally, for $p \in S^*(B)$, $\langle a_\alpha : \alpha < \lambda \rangle$ is a Morley sequence over B built from p if for all $\alpha < \lambda$, $a_\alpha \models p | B \cup \{a_\beta : \beta < \alpha\}$. However, by symmetry, the order of the a_α is irrelevant so we will call the *set* $\bigcup\{a_\alpha : \alpha < \lambda\}$ a Morley sequence of length λ over B built from p if the above property holds.

2 Positive results

Definition 2.1 For L a first order language, a class $\mathcal{K}$ of L-structures is *pseudo-elementary* (PC_Δ) if there is a language $L_1 \supset L$ with $\|L_1\| = \|L\|$ and an L_1-theory T_1 such that

$$\mathcal{K} = \{L\text{-structures } M : M = M_1 \upharpoonright L \text{ for some } M_1 \models T_1\}.$$

Note that L_1 and T_1 are not determined by $\mathcal{K}$. When L_1 and T_1 are known, we write $\mathcal{K} = \mathrm{Mod}(T_1) \upharpoonright L$. If $\|L\| = \aleph_0$, we call $\mathcal{K}$ a $\mathrm{PC}_{\aleph_0}$ class.

Definition 2.2 For $\mathcal{K}$ a PC_Δ class, let $L(\mathcal{K})$ denote the underlying language, let $|\mathcal{K}| = \|L(\mathcal{K})\|$ and let

$$T(\mathcal{K}) = \bigcap\{\mathrm{Th}_L(M) : M \in \mathcal{K} \text{ and } M \text{ infinite}\}.$$

$\mathcal{K}$ is *κ-categorical* if $\mathcal{K}$ contains exactly one model of power κ up to isomorphism. $\mathrm{Spec}(\mathcal{K}) = \{\kappa > |\mathcal{K}| : \mathcal{K} \text{ is } \kappa\text{-categorical}\}$.

The definition of $T(\mathcal{K})$ is chosen to satisfy completeness in Lemma 2.3. Our first goal is to achieve a workable characterization of $\kappa \in \mathrm{Spec}(\mathcal{K})$ for $\mathcal{K}$ a PC_Δ class with a nonempty spectrum.

Lemma 2.3 *If* $\mathrm{Spec}(\mathcal{K}) \neq \emptyset$ *then* $T(\mathcal{K})$ *is complete, superstable and stable in all* $\lambda \geq |\mathcal{K}|$.

Proof: If $T(\mathcal{K})$ is not complete then there are infinite $M_1, M_2 \in \mathcal{K}$ with $M_1 \not\equiv M_2$ so by Löwenheim-Skolem there are nonelementarily equivalent models in $\mathcal{K}$ of every cardinal $\geq |\mathcal{K}|$, contradicting $\mathrm{Spec}(\mathcal{K}) \neq \emptyset$. $T(\mathcal{K})$ is superstable and stable in each $\lambda \geq |\mathcal{K}|$ by VIII, 4.1(2) of [11].

Lemma 2.4 *If* $\mathrm{Spec}(\mathcal{K}) \neq \emptyset$ *then for all* $\lambda \geq |\mathcal{K}|$, *there is a saturated model* $M_\lambda \in \mathcal{K}$ *of power* λ.

Proof: Suppose $\mathcal{K} = \mathrm{Mod}(T_1) \upharpoonright L$. $T(\mathcal{K})$ is λ-stable, so by III, 3.12 of [11] there is a saturated model $M_\lambda \models T(\mathcal{K})$ of power λ. Now T_1 is consistent and M_λ is saturated so by I, 1.13 of [11] there is an expansion $N_\lambda \models T_1$ with $N_\lambda \upharpoonright L = M_\lambda$. Hence $M_\lambda \in \mathcal{K}$.

Note that the above lemma asserts the existence of a saturated model in the restricted language. However, it should be noted that the L_1-models of which $\mathcal{K}$ are the reducts can be quite wild.

Conclusion 2.5 *If* $\mathrm{Spec}(\mathcal{K}) \neq \emptyset$ *then for all* $\lambda > |\mathcal{K}|$,

$$\lambda \in \mathrm{Spec}(\mathcal{K}) \textit{ iff every model in } \mathcal{K} \textit{ of power } \lambda \textit{ is saturated.}$$

Proposition 2.6 *Suppose* $\lambda \in \mathrm{Spec}(\mathcal{K})$, $\kappa \geq \lambda$. *Then every* $M \in \mathcal{K}$ *of power* κ *is* λ*-saturated.*

Proof: Suppose $\mathcal{K} = \mathrm{Mod}(T_1) \upharpoonright L$. Let $M \in \mathcal{K}$ be of power κ. Let $N \models T_1$ be such that $M = N \upharpoonright L$. Let $A \subseteq |M|$, $|A| < \lambda$, and let $p \in S_L(A)$. Choose $N' \leq$

N, $\|N'\| = \lambda$, $A \subseteq |N'|$ and let $M' = N' \upharpoonright L$. As $M' \in \mathcal{K}$ is of power λ, M' is saturated. Pick $a \in |M'|$ realizing p. By elementarity, a realizes p in M.

The following corollary says that any spectrum is closed in the order topology.

Corollary 2.7 *Let I be a nonempty index set. Assume $\{\lambda_i : i \in I\} \subseteq \mathrm{Spec}(\mathcal{K})$, and $\lambda = \sup\{\lambda_i : i \in I\}$. Then $\lambda \in \mathrm{Spec}(\mathcal{K})$.*

Proof: Let $M \in \mathcal{K}$ be of power λ. From above, M is λ_i-saturated for each $i \in I$. As $\lambda = \sup\{\lambda_i : i \in I\}$, M is saturated. Thus $\lambda \in \mathrm{Spec}(\mathcal{K})$ by Conclusion 2.5.

Thus the categoricity spectrum of any PC_Δ class is closed. Our next goal is to see that if it is nonempty then it forms a rather large class of cardinals. The following definition is classical.

Definition 2.8 For $\lambda \geq \aleph_0$, the Hanf number of omitting types $h(\lambda)$ is the least cardinal κ satisfying: for all first order theories T in a language of power λ and all types p,

if there is a model $M_\kappa \models T$ of power κ omitting p,
then for all $\mu \geq \kappa$ there is a model $N \models T$ of power μ omitting p.

The crucial facts about $h(\lambda)$ and the corresponding ordinal $\delta(\lambda)$ are summarized in the following theorem of Morley [9]. Schmerl and Shelah [10] independently proved (a) for $\lambda > \aleph_0$. The proofs of these facts can be found in [1] for $\lambda = \aleph_0$ and in VII, 5.4 and 5.5 of [11] for arbitrary λ.

Theorem 2.9
(a) *For all $\lambda \geq \aleph_0$, $h(\lambda) = \beth_{\delta(\lambda)}$ for some ordinal $\delta(\lambda)$.*
(b) *For all $\lambda \geq \aleph_0$, $\delta(\lambda)$ is a limit ordinal of cofinality $> \lambda$.*
(c) $\delta(\aleph_0) = \omega_1$.

In fact, the ordinal $\delta(\lambda)$ can be characterized as the least ordinal δ such that for all first order theories T in a language of power λ that interpret a linear ordering and all one-types $p(x)$, if for all $\alpha < \delta$ there is a well-ordered model M_α of T omitting p of order type $\geq \alpha$, then there is a non-well-ordered model $N \models T$ omitting p. We will not use this fact.

The following theorem (when $\lambda = \aleph_0$) is a combination of Morley's omitting types theorem [9] and Vaught's two-cardinal theorem for cardinals far apart [13]. The hypotheses simply ensure (via the Erdös–Rado theorem) that there is a two-cardinal model omitting p containing an infinite set of order-indiscernibles. A proof can be found in VII, 5.3 of [11].

Theorem 2.10 *Let $\|L\| \leq \lambda$ contain a distinguished unary predicate U, T an L-theory and $p(x)$ a one-type.*

Assume that for all $\alpha < \delta(\lambda)$ there is a cardinal χ_α and a model $M_\alpha \models T$ omitting p with

$$\|M_\alpha\| \geq \beth_\alpha(\chi_\alpha) \text{ and } |U^{M_\alpha}| \leq \chi_\alpha.$$

Then for all $\kappa \geq \lambda$ there is an $M \models T$ of power κ omitting p, but $|U^M| \leq \lambda$.

Fix $\mathcal{K}$ a PC_Δ class with $|\mathcal{K}| = \lambda$, let $M \in \mathcal{K}$, $A \subseteq |M|$ finite, and let $p \in S^*_{L(\mathcal{K})}(A)$ be nonalgebraic. Define

$$f_p : \mathrm{ORD} \to \mathrm{ORD}$$

by $f_p(\alpha) =$ the least ordinal β such that there is $M_\alpha \in \mathcal{K}$, $A \subseteq |M_\alpha|$, $\|M_\alpha\| \geq \beth_\alpha$ and every Morley sequence $J \subseteq |M_\alpha|$ built from p over A has length $\leq \beth_\beta$.

It is easy to see that $f_p(\alpha) \leq \alpha$ for all α and that f_p is monotone increasing.

Proposition 2.11 *Suppose $|\mathcal{K}| = \lambda$, $\mathrm{Spec}(\mathcal{K}) \neq \varnothing$, and $\gamma > 0$. Let $M \in \mathcal{K}$, let $A \subseteq |M|$ be finite, and let $p \in S^*_{L(\mathcal{K})}(A)$ be nonalgebraic. Then $f_p \restriction \delta(\lambda) \cdot \gamma$ is cofinal in $\delta(\lambda) \cdot \gamma$.*

Proof: By absorbing A into $L(\mathcal{K})$, we may assume that $A = \varnothing$. Let $\eta = \delta(\lambda) \cdot \gamma$ and assume that $\mathcal{K}$ is κ-categorical where $\kappa > \lambda$. Assume by way of contradiction that $f_p \restriction \eta$ is bounded by $\beta < \eta$. Then for each $\alpha < \eta$ with $\beth_\alpha \geq \lambda$, there is $M_\alpha \in \mathcal{K}$, $\|M_\alpha\| = \beth_\alpha$ and $J_\alpha \subseteq |M_\alpha|$, J_α a maximal Morley sequence built from p over $\varnothing$ with $|J_\alpha| \leq \beth_\beta$. Now suppose $\mathcal{K} = \mathrm{Mod}(T_1) \restriction L$. For each such α choose $N_\alpha \models T_1$ with $N_\alpha \restriction L = M_\alpha$. Let

$$FE = \{E(x,y) : E \in L,\ T(\mathcal{K}) \vdash \text{“}E \text{ is an equivalence relation with finitely many classes”}\}.$$

For each $E \in FE$, let $n(E)$ denote the number of classes, and let $C(E) = \{c_0^E, \ldots, c_{n(E)-1}^E\}$, be a set of $n(E)$ new constant symbols. Let $C = \bigcup\{C(E) : E \in FE\}$.

Let $L_1^* = L_1 \cup \{U\} \cup C$, where U is a new unary predicate symbol. For each $\alpha < \delta(\lambda)$ with $\beth_\alpha \geq \lambda$, expand N_α into an L_1^*-structure N_α^* as follows:

$$U^{N_\alpha^*} = \{e \in |N_\alpha| : e \in J_\alpha\},$$

and for each $E \in FE$, choose (arbitrarily) a set $\{d_0^E, \ldots, d_{n(E)-1}^E\}$ of representatives of E's equivalence classes in N_α and assign

$$(c_i^E)^{N_\alpha^*} = d_i^E$$

for all $i < n(E)$.

For each α, let $\hat{p}_\alpha$ denote the unique nonforking extension of p to $S_L(|M_\alpha|)$. Now, by the Finite Equivalence Relation theorem, $\hat{p}_\alpha$ is stationary over $\{(c_i^E)^{N_\alpha^*} : E \in FE,\ i < n(E)\}$, so by the Definability of Types theorem, for each formula $\varphi(x, \bar{y}) \in L$ there is a formula $d_p\varphi(\bar{y}) \in L \cup C$ such that for all $\bar{e} \in |M_\alpha|$,

$$N_\alpha^* \models d_p\varphi[\bar{e}] \quad \text{iff } \varphi(x, \bar{e}) \in \hat{p}_\alpha.$$

Let $T_1^* = \bigcap\{\mathrm{Th}(N_\alpha^*) : \alpha < \delta(\lambda), \beth_\alpha \geq \lambda\}$ and let

$$q(x) = \left\{\forall \bar{y}\left[\bigwedge_{i<m} U(y_i) \to (d_p\varphi(\bar{y}) \leftrightarrow \varphi(x, \bar{y}))\right] : \varphi(x, \bar{y}) \in L\right\}$$

where $\bar{y} = y_0, \ldots, y_{m-1}$ (i.e., q is the nonforking extension of p to the set of realizations of U).

So, for each α satisfying $\beta \leq \alpha < \delta(\lambda)$, the maximality of each J_α implies that N_α^* is a $(\beth_\alpha, \leq \beth_\beta)$ model of T_1^* omitting q.

Thus, by Theorem 2.10, there is a $(\kappa, \leq \lambda)$ model N^* of T_1^* omitting q. Let $M^* = N^* \restriction L$ and let $J = U^{N^*}$. Now $M^* \in \mathcal{K}$, $\|M^*\| = \kappa$, and $\mathcal{K}$ is κ-categorical, so M^* must be saturated by Conclusion 2.5. We will contradict this by showing that M^* omits

$$r(x) = \{\varphi(x, \bar{b}) : \varphi(x, \bar{y}) \in L, \bar{b} \in J, N^* \models d_p \varphi[\bar{b}]\}.$$

Let us first show that r is consistent with $\mathrm{Th}(N^*)$. Certainly since

$$T_1^* \vdash \forall \bar{y}(d_p(\varphi_1(\bar{y}) \wedge \varphi_2(\bar{y})) \leftrightarrow (d_p\varphi_1(\bar{y}) \wedge d_p\varphi_2(\bar{y}))),$$

r is closed under finite conjunctions. So it suffices to show that

$$N^* \models \forall \bar{y}\left(\bigwedge U(\bar{y}) \wedge d_p\varphi(\bar{y}) \rightarrow \exists x \varphi(x, \bar{y})\right) \tag{1}$$

for all $\varphi(x, \bar{y}) \in L$. To show this, fix $\alpha < \eta$ with $\beth_\alpha \geq \lambda$, $\varphi(x, \bar{y}) \in L$ and choose $\bar{b}$ from J_α such that $N_\alpha^* \models d_p\varphi[\bar{b}]$. Then, by definition of $d_p(\bar{y})$, $\varphi(x, \bar{b}) \in p \mid J_\alpha$ which is consistent as p is nonalgebraic. So

$$T_1^* \vdash \forall \bar{y}\left(\bigwedge U(\bar{y}) \wedge d_p\varphi(\bar{y}) \rightarrow \exists x \varphi(x, \bar{y})\right)$$

and (1) holds.

Finally, to show that M^* omits r, assume by way of contradiction that $a \in |N^*|$ realized r. Then fix $\varphi(x, \bar{y}) \in L$ and $\bar{b}$ from $|N^*|$ such that $N^* \models \bigwedge U(b)$. Now if $N^* \models d_p\varphi[\bar{b}]$, then $N^* \models \varphi[a, \bar{b}]$ by definition of r. But if $N^* \models \neg d_p\varphi[\bar{b}]$, then $N^* \models d_p\neg\varphi[\bar{b}]$ (as $T_1^* \vdash \forall \bar{y}(d_p\neg\varphi(\bar{y}) \leftrightarrow \neg d_p\varphi(\bar{y})))$, so $N^* \models \neg\varphi[a, \bar{b}]$. Thus, a realizes q, which is a contradiction.

Proposition 2.12 *Let* $|\mathcal{K}| = \lambda$, $\mathrm{Spec}(\mathcal{K}) \neq \emptyset$, *and* $\beth_\alpha > \lambda$. *Let* $\overline{M} \in \mathcal{K}$ *be any saturated model and assume that* $f_p(\alpha) = \alpha$ *for every finite* $A \subseteq |\overline{M}|$ *and every* $p \in S^*_{L(\mathcal{K})}(A)$. *Then* $\mathcal{K}$ *is* $\beth_\alpha$*-categorical.*

Proof: It follows from the saturation of $\overline{M}$ that $f_p(\alpha) = \alpha$ for any $L(\mathcal{K})$-type over any finite set of parameters. Now let $M \in \mathcal{K}$, $\|M\| = \beth_\alpha$, $p \in S_{L(\mathcal{K})}(B)$, with $B \subseteq |M|$, $|B| < \beth_\alpha$. We must show that p is realized in M. First, if p is algebraic then it is trivially realized in any model containing B, so assume that p is nonalgebraic. Let $q \in S_{L(\mathcal{K})}(|M|)$ be any nonforking extension of p to $|M|$. As $T(\mathcal{K})$ is superstable, we can choose $A \subseteq |M|$ finite so that q is based on A. Now as $f_{q|A}(\alpha) = \alpha$, there is $J \subseteq |M|$, $|J| = \beth_\alpha$, J a Morley sequence over A built from $q|A$.

But now, as $T(\mathcal{K})$ is superstable, there is $J_0 \subseteq J$, $|J_0| \leq |B| + \aleph_0$ such that $(J \setminus J_0)$ is a Morley sequence over B built from $q|B$. Thus every element of $J \setminus J_0$ realizes p, as desired.

Theorem 2.13 *Suppose* $\mathrm{Spec}(\mathcal{K}) \neq \emptyset$, $|\mathcal{K}| = \lambda$ *and* $\gamma > 0$. *Then* $\{\alpha < \delta(\lambda) \cdot \gamma : \mathcal{K}$ is $\beth_\alpha$-categorical$\}$ *is a closed, unbounded subset of* $\delta(\lambda) \cdot \gamma$.

Proof: First of all, $\{\alpha < \delta(\lambda) \cdot \gamma : \mathcal{K}$ is $\beth_\alpha$-categorical$\}$ being closed in $\delta(\lambda) \cdot \gamma$ follows immediately from $\mathrm{Spec}(\mathcal{K})$ being closed. Let $\eta = \delta(\lambda) \cdot \gamma$. Note that it certainly suffices to prove the theorem for γ a successor ordinal, so we may assume that $\mathrm{cf}(\eta) > \lambda$ by Theorem 2.9(2).

It is immediate from the definition of f_p that if $p \in S^*(A)$, $p' \in S^*(A')$, $b \models p$, $b' \models p'$ and $\mathrm{stp}(Ab/\emptyset) = \mathrm{stp}(A'b'/\emptyset)$ then $f_p = f_{p'}$. With this in mind, let $M \in \mathcal{K}$ be any saturated model and define $g: \mathrm{ORD} \to \mathrm{ORD}$ by

$$g(\alpha) = \min\{f_p(\alpha): p \in S^*(A), A \subseteq |M|, A \text{ finite}\}.$$

As $T(\mathcal{K})$ is λ-stable g is the minimum of at most λ distinct functions, each of which is monotone increasing and satisfies $f_p(\alpha) \leq \alpha$ for all α. Thus, g inherits each of these properties. Further, as each $f_p \restriction \eta$ is unbounded in η by Proposition 2.11, and as $\mathrm{cf}(\eta) > \lambda$ by Theorem 2.9(2), $g \restriction \eta$ is also unbounded in η.

But now, as $\mathrm{cf}(\eta) > \lambda \geq \aleph_0$, if $\{\alpha < \eta : g(\alpha) = \alpha\}$ were bounded in η then by Fodor's lemma there would be an element $\beta \in \eta$ and an unbounded subset $S \subseteq \eta$ such that $g \restriction S = \{\beta\}$. However, this together with g monotone increasing implies that g is bounded, which is a contradiction. Thus $\{\alpha < \eta : g(\alpha) = \alpha\}$ is unbounded in η. Therefore, $\{\alpha < \eta : \mathcal{K} \text{ is } \beth_\alpha\text{-categorical}\}$ is unbounded in η by Proposition 2.12.

The following theorem of Keisler, Čudnovskiĭ, and Shelah now follows as an immediate corollary.

Corollary 2.14 *Let* $|\mathcal{K}| = \lambda$, $\mathrm{Spec}(\mathcal{K}) \neq \emptyset$ *and let* $\gamma > 0$. *Then* $\mathcal{K}$ *is* $\beth_{\delta(\lambda)\cdot\gamma}$-*categorical.*

Proof: By Theorem 2.13 and Corollary 2.7.

3 Negative results Whereas the theorems in the previous section indicate that a nonempty spectrum is rather large, the theorems are not very specific about *which* cardinals must be included in a nonempty spectrum. Our first result will show that Corollary 2.14 is the best possible. That is, the only cardinals included in any nonempty spectrum of a $\mathrm{PC}_{\aleph_0}$ class are $\beth_{\omega_1\cdot\gamma}$ for $\gamma > 0$.

A related question one can ask is for which pairs of cardinals does categoricity in one cardinal imply categoricity in the second cardinal? This suggests the following definition.

Definition 3.1 $\kappa \to \lambda$ denotes the following statement: for all $\mathrm{PC}_{\aleph_0}$ classes $\mathcal{K}$, $\kappa \in \mathrm{Spec}(\mathcal{K})$ implies $\lambda \in \mathrm{Spec}(\mathcal{K})$.

An example of such a transfer is due to Keisler. He showed $\forall\lambda > \aleph_0(\aleph_1 \to \lambda)$. We will prove a slight generalization of this in Proposition 3.14.

Unfortunately, the results of the subsequent subsections indicate that almost all instances of this relation are either independent of or refutable in ZFC. As there is so little that is provable in ZFC, *we assume throughout this section that all classes* $\mathcal{K}$ *have a countable underlying language.*

The starting point of our investigations is a nice characterization of the relation $\kappa \to \lambda$ due to Shelah. A proof appears in VIII, 4.3 of [11].

Theorem 3.2 *For* $\kappa, \lambda > \aleph_0$ *the following are equivalent:*

1. $\kappa \to \lambda$
2. *For all first order theories T in a countable language containing a distinguished unary predicate U and all one-types p, if there is a* $(\lambda, < \lambda)$ *model* $M \models T$ *omitting p then there is a* $(\kappa, < \kappa)$ *model* $N \models T$ *omitting p.*

3.1 Computing $\bigcap\{\mathrm{Spec}(\mathcal{K}) : \mathrm{Spec}(\mathcal{K}) \neq \emptyset\}$

Definition 3.3 For $\alpha > 0$, define $\mathrm{res}(\alpha)$ to be the least ordinal γ such that $\exists\delta < \alpha(\delta + \gamma \geq \alpha)$.

Intuitively, $\mathrm{res}(\alpha)$ is the last term of the Cantor normal form of α. It is easy to see that $\mathrm{res}(\alpha) = 1$ for any successor ordinal. The following lemma is also easy.

Lemma 3.4 *If* $\mathrm{res}(\alpha) \geq \beta$ *then* $\alpha = \beta\cdot\gamma$ *for some ordinal* γ.

Proof: Choose γ least such that $\alpha < \beta\cdot\gamma$. γ must be a successor ordinal, so assume $\gamma = \tau + 1$. By hypothesis, $\beta\cdot\tau < \alpha$ and $\beta\cdot\tau + \beta > \alpha$, so $\mathrm{res}(\alpha) < \beta$ as desired.

The theories and the types used in the following proposition are a slight modification of those used by Morley in showing that the Hanf number of omitting types is at least $\beth_{\omega_1}$.

Proposition 3.5 *Let* α *and* β *be nonzero ordinals satisfying* $\mathrm{res}(\alpha) < \min\{\omega_1, \mathrm{res}(\beta)\}$. *Then* $\beth_\beta \not\to \beth_\alpha$.

Proof: First note that the hypotheses require β to be a limit ordinal. Let $\eta = \mathrm{res}(\alpha)$. Let

$$L = \{U, W\} \cup \{<, V\} \cup \{\epsilon\} \cup \{c_\gamma : \gamma \leq \eta\}$$

where U and W are unary predicates, $<$ and V are binary relations, ϵ is a ternary relation, and $\{c_\gamma : \gamma \leq \eta\}$ are constant symbols. Note that L is countable as $\eta < \omega_1$.

Let $p(x) = \{W(x)\} \cup \{x \neq c_\gamma : \gamma \leq \eta\}$ and let T consist of the following axioms:

For notation let $V_x = \{y : V(x, y)\}$.

1. $W(c_\gamma)$ for all $\gamma \leq \eta$.
2. $c_\gamma < c_\delta$ for all $\gamma < \delta \leq \eta$.
3. $<$ is a linear ordering on W.
4. $V_x \subseteq V_y$ for all $x < y$ in W.
5. $\forall y(V(c_0, y) \leftrightarrow U(y))$.
6. $\forall y(V(c_\eta, y))$.
7. $\forall x(W(x) \wedge (x \text{ a limit}) \to V_x = \bigcup\{V_y : y < x\})$.
8. $\forall xy(W(x) \wedge (x \text{ a successor of } y) \to$
   $\forall uv[V(x,u) \wedge V(x,v) \wedge \{\forall w(V(y,w) \to (\epsilon(x,w,u) \leftrightarrow \epsilon(x,w,v)))\} \to u = v])$.

Models of T omitting p are subsets of the cumulative hierarchy of sets up to $\eta + 1$ over the set of urelements U^M. The elements of W^M are the 'ordinals' and Axiom 8 is 'comprehension'.

It is easy to prove by induction on $\gamma \leq \eta$ that for any $M \models T$ omitting p, that $|V_{c_\gamma}| \leq \beth_\gamma(|U^M|)$. As $|M| = |V_{c_\eta}|$ it follows that for any cardinal κ,

there is a $(\kappa, < \kappa)$ model $M \models T$ omitting p iff $(\exists\lambda < \kappa)(\beth_\eta(\lambda) \geq \kappa)$.

Thus, taking $\lambda = \beth_\delta$ for any $\delta < \alpha$ satisfying $\delta + \mathrm{res}(\alpha) \geq \alpha$, there is a $(\beth_\alpha, < \beth_\alpha)$ model of T omitting p. However, there is no $(\beth_\beta, < \beth_\beta)$ model of T

omitting p. To see this, suppose there were a $\lambda < \beth_\beta$ with $\beth_{\mathrm{res}(\alpha)}(\lambda) \geq \beth_\beta$. Now as β is a limit ordinal, there would be a $\tau < \beta$ such that $\lambda \leq \beth_\tau < \beth_\beta$. So

$$\beth_{\mathrm{res}(\alpha)}(\beth_\tau) \geq \beth_{\mathrm{res}(\alpha)}(\lambda) \geq \beth_\beta$$

which implies $\tau + \mathrm{res}(\alpha) \geq \beta$, contradicting $\mathrm{res}(\alpha) < \mathrm{res}(\beta)$. Thus $\beth_\beta \not\rightarrow \beth_\alpha$ by Theorem 3.2.

The following corollary provides a converse to Corollary 2.14.

Corollary 3.6 *For all cardinals κ, $\kappa \in \mathrm{Spec}(\mathcal{K})$ for all $\mathrm{PC}_{\aleph_0}$ classes $\mathcal{K}$ having nonempty spectra if and only if $\kappa = \beth_{\omega_1\cdot\alpha}$ for some $\alpha > 0$.*

Proof: First of all, it suffices to consider only strong limit cardinals, as we will see in Subsection 3.5 an example of a $\mathrm{PC}_{\aleph_0}$ class whose spectrum is exactly the class of strong limit cardinals. So assume that $\kappa = \beth_\beta$ for some ordinal $\beta > 0$. Now if $\beta = \omega_1\cdot\alpha$ for some $\alpha > 0$ then κ is in every nonempty spectrum by Corollary 2.14. On the other hand, if κ were in every nonempty spectrum then certainly $\beth_{\omega_1} \rightarrow \kappa$, so by the proposition above $\mathrm{res}(\beta) \geq \omega_1$. Thus, $\beta = \omega_1\cdot\alpha$ for some $\alpha > 0$ by Lemma 3.4.

3.2 The consistency of $\aleph_{\alpha+2} \not\rightarrow \aleph_{\alpha+3}$ In [4], Keisler proved that $\aleph_1 \rightarrow \lambda$ for all uncountable λ. It is natural to ask whether the cardinal $\aleph_1$ can be replaced by any other cardinal. Via Shelah's characterization given above, it follows immediately that $\lambda \not\rightarrow \aleph_1$ for any $\lambda > \aleph_1$. (Just take T and p to have a model of T of power $\aleph_1$ omitting p, but all models of T of larger powers realize p, see e.g., [9].) However, the question of whether $\forall\lambda > \aleph_2(\aleph_2 \rightarrow \lambda)$ is consistent is more delicate. In this section, we show that consistency is the best we can hope for.

Specifically, we show that if there is a Mahlo cardinal $> \aleph_\alpha$, then it is consistent that $\aleph_{\alpha+2} \not\rightarrow \aleph_{\alpha+3}$. By way of contrast, in the next subsection we will see that $\aleph_{\alpha+2} \rightarrow \aleph_{\alpha+3}$ is consistent with the existence of a huge cardinal above $\aleph_\alpha$.

In this section we show also that among the cardinals $\{\aleph_n : 2 \leq n < \omega\}$, given any $X \subseteq \omega \setminus \{0,1\}$ it is consistent with the existence of $\aleph_0$ Mahlo cardinals that there be a $\mathrm{PC}_{\aleph_0}$ class $\mathcal{K}_X$ such that

$$n \in X \quad \text{iff } \aleph_n \in \mathrm{Spec}(\mathcal{K}_X)$$

for all $2 \leq n < \omega$.

Our results are simply restatements of Mitchell's work on special Aronszajn trees into our context.

Definition 3.7 A *κ-Aronszajn tree* is a κ-tree having no branch of length κ. A κ^+-Aronszajn tree is *special* if it is embeddable in the tree

$$T = \bigcup_{\alpha<\kappa^+} \{f : \alpha \rightarrow \kappa, f \text{ is one-one}\},$$

with $f \leq_T g$ iff $f \subseteq g$ as functions.

Aronszajn showed that there is an ω_1-special Aronszajn tree. Later Specker proved that if κ is regular and $2^{<\kappa} = \kappa$ then there is a κ^+-special Aronszajn tree. The following two theorems are due to Mitchell and can be found in [7].

Theorem 3.8 *Assume $\kappa = \aleph_{\alpha+1}$, $V \models$ GCH and there is a Mahlo cardinal $> \kappa$. Then there is a forcing notion $\mathcal{P}$ preserving cardinals $\leq \kappa$ with $\mathcal{P} \Vdash \kappa^+ = 2^\kappa$ and*

$$\mathcal{P} \Vdash \text{'There are no special } \kappa^+ \text{ Aronszajn trees.'}$$

Combining this with Specker's theorem yields the consistency of ZFC plus "There is no $\aleph_{\alpha+2}$ special Aronszajn tree but there are special $\aleph_{\alpha+3}$ Aronszajn trees" from the consistency of ZFC plus the existence of a Mahlo cardinal $> \aleph_\alpha$.

The following theorem, also due to Mitchell, follows by iterating the forcing used above ω times.

Theorem 3.9 *For $Y \subseteq \omega \setminus \{0,1\}$,*

$$\begin{aligned}&\mathrm{Con}(\mathrm{ZFC} + \exists \aleph_0 \text{ Mahlo cardinals})\\ &\quad \rightarrow \mathrm{Con}(\mathrm{ZFC} + \forall n \in \omega \setminus \{0,1\}\,[\exists \text{ an } \aleph_n\text{-special Aronszajn tree iff } n \in Y]).\end{aligned}$$

To obtain results about $\mathrm{PC}_{\aleph_0}$ classes, we need some way of describing a special Aronszajn tree as an element in a $\mathrm{PC}_{\aleph_0}$ class. Fortunately, this is given to us by Silver and Rowbottom. They independently found a sentence σ in a language with a distinguished predicate U so that for all κ,

There is a κ^+ special Aronszajn tree iff there is a (κ^+, κ) model $\mathfrak{A} \models \sigma$.

A suitable definition of σ can be found in [7]. Now, just let $L = \{U\}$, $T_1 = \{\sigma\}$, and $\mathcal{K} = \mathrm{Mod}(T_1) \upharpoonright L$. Thus for all cardinals κ,

$$\kappa^+ \in \mathrm{Spec}(\mathcal{K}) \quad \text{iff there is no } \kappa^+\text{-special Aronszajn tree.}$$

So by Theorem 3.8,

$$\mathrm{Con}(\mathrm{ZFC} + \exists \text{ Mahlo} > \aleph_\alpha) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \aleph_{\alpha+2} \not\rightarrow \aleph_{\alpha+3}).$$

Also, by taking $Y = (\omega \setminus \{0,1\}) \setminus X$ in Theorem 3.9 we have the following proposition.

Proposition 3.10 *Let $X \subseteq \omega \setminus \{0,1\}$. There is a $\mathrm{PC}_{\aleph_0}$ class $\mathcal{K}_X$ such that*

$$\begin{aligned}&\mathrm{Con}(\mathrm{ZFC} + \exists \aleph_0 \text{ Mahlo cardinals})\\ &\quad \rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{Spec}(\mathcal{K}_X) \cap \aleph_\omega = \{\aleph_n : n \in X\}).\end{aligned}$$

3.3 The consistency of $\aleph_{\alpha+2} \rightarrow \aleph_{\alpha+3}$

The major result of this subsection is that if there is a huge cardinal above $\aleph_\alpha$ then it is consistent that $\aleph_{\alpha+2} \rightarrow \aleph_{\alpha+3}$. This theorem can be proved directly from $\mathrm{CC}(\aleph_{\alpha+3}, \aleph_{\alpha+2})$, but it is of interest that it also follows from the weaker assumption of $\mathrm{KT}(\aleph_{\alpha+3}, \aleph_{\alpha+2})$. (CC and KT are defined below.) It follows from Proposition 3.14 that all of the negative transfer results of the preceding and following subsections can be viewed as counterexamples to generalizations of Keisler's two-cardinal theorem to larger cardinals.

Each of the next two definitions assumes that the underlying language is countable and contains a distinguished unary predicate U.

Definition 3.11 For κ an infinite cardinal, $\mathrm{CC}(\kappa^{++}, \kappa^+)$ denotes the following statement:

For every (κ^{++}, κ^+) model there is a (κ^+, κ) elementary submodel.

Definition 3.12 For all infinite $\lambda \geq \kappa^+$, $\mathrm{KT}(\lambda, \kappa^+)$ denotes the following statement:

For every (λ, μ) model $\mathfrak{A}$ with $\kappa^+ \leq \mu < \lambda$
there is a (κ, κ) elementary submodel $\mathfrak{B}$ and
a (κ^+, κ) model $\mathfrak{C}$ with $\mathfrak{B} \leq \mathfrak{C}$ and $U^{\mathfrak{B}} = U^{\mathfrak{C}}$.

$\mathrm{CC}(\kappa^{++}, \kappa^+)$ states that a particular instance of Chang's conjecture holds. As for the second definition, note that $\forall\lambda \geq \aleph_1 (\mathrm{KT}(\lambda, \aleph_1))$ is a restatement of Keisler's two-cardinal theorem. That $\mathrm{CC}(\kappa^{++}, \kappa^+)$ implies $\mathrm{KT}(\kappa^{++}, \kappa^+)$ is immediate by the Downward Löwenheim–Skolem theorem. However, the converse fails as $L \models \mathrm{KT}(\aleph_2, \aleph_1) \wedge \neg\mathrm{CC}(\aleph_2, \aleph_1)$ as is witnessed by the existence in L of a Kurepa family of subsets of $\aleph_2$.

We need one lemma that is a slight strengthening of Theorem 3.2.

Lemma 3.13 *For uncountable cardinals κ and λ, the following are equivalent*:

(a) $\kappa \to \lambda$

(b) *For every theory T in a language containing a distinguished unary predicate U and for every one-type p with $U \in p$,*

if there is a $(\lambda, < \lambda)$ model $M \models T$ omitting p
then there is a $(\kappa, < \kappa)$ model $N \models T$ omitting p.

The only difference between this result and Theorem 3.2 is the extra assumption of $U \in p$. Its proof simply translates an arbitrary T and p into T^* and q in an expanded language L^* with $U \in q$.

Proof: (a) ⇒ (b) is immediate via 3.2. For the converse, assume that cardinals κ and λ satisfy (2). Let T be an L-theory, $p(x) = \{\varphi_n(x) : n \in \omega\}$ (with $U \notin p$ else we are done), and let $M \models T$, M a $(\lambda, < \lambda)$ model omitting p. By Theorem 3.2 we must produce a $(\kappa, < \kappa)$ model omitting p.

Let $L^* = L \cup \{c_n : n \in \omega\} \cup \{V, R\}$, where V is a new unary predicate and R is a new binary relation. Let

$$T^* = T \cup \{V(c_n) : n \in \omega\} \cup \{\forall x(V(x) \to U(x))\} \cup \{\forall x \forall y R(x,y) \to V(x)\}$$
$$\cup \{\forall y(R(c_n, y) \leftrightarrow \varphi_n(y)) : n \in \omega\} \cup \{\forall y \exists x(\neg R(x,y))\}.$$

Let $q(x) = \{V(x)\} \cup \{x \neq c_n : n \in \omega\}$.

As M omits p, there is a natural expansion of M to $M^* \models T^*$, M^* omitting q. As $U \in q$, by (b) there is a $(\kappa, < \kappa)$ model $N^* \models T^*$ omitting q. Now $N^* \upharpoonright L \models T$ and omits p, as desired.

The following proposition follows immediately.

Proposition 3.14 *$KT(\lambda, \kappa^+)$ implies $\kappa^+ \to \lambda$.*

Proof: We use the lemma above. Let $\mathfrak{A}$ be a $(\lambda, < \lambda)$ model of T omitting p with $U \in p$. There are two cases. If $|U^{\mathfrak{A}}| \leq \kappa$ then simply by the Downward Löwenheim–Skolem theorem there is a $(\kappa^+, \leq \kappa)$ elementary submodel that surely omits p. If $|U^{\mathfrak{A}}| > \kappa$ then choose $\mathfrak{B}$ and $\mathfrak{C}$ as in the definition of KT. Now $\mathfrak{C}$ is (κ^+, κ), $\mathfrak{C} \models T$ and $\mathfrak{C}$ omits p as desired.

As a corollary, we get the following theorem of Keisler.

Corollary 3.15 *For $\mathcal{K}$ a $\mathrm{PC}_{\aleph_0}$, if $\aleph_1 \in \mathrm{Spec}(\mathcal{K})$ then $\mathrm{Spec}(\mathcal{K})$ consists of all uncountable cardinals.*

Proof: Immediate, as $\forall\lambda \geq \aleph_1(\mathrm{KT}(\lambda, \aleph_1))$ is simply Keisler's two-cardinal theorem.

As far as the consistency of these notions is concerned, we recall Kunen's method for producing κ^{++}-saturated ideals on κ^+ for regular κ. (In [5] he gives the construction for $\kappa = \aleph_0$, but the generalization is straightforward. The details of the generalization are discussed in Donder and Koepka [3].) He starts with a model of ZFC + GCH + $\exists\lambda(\lambda$ a huge cardinal $> \kappa)$ and then collapses λ down to κ^{++}. (The forcing preserves all cardinals $\leq \kappa^{++}$.) In the process, Kunen observes that in this model, $\mathrm{CC}(\kappa^{++}, \kappa^+)$ holds. The reader is referred to [5] and [3] for a thorough description of the forcing. Thus for all $\alpha \in \mathrm{ORD}$,

$$\mathrm{Con}(\mathrm{ZFC} + \exists\kappa(\kappa \text{ huge and } \kappa > \aleph_\alpha)) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{CC}(\aleph_{\alpha+3}, \aleph_{\alpha+2})).$$

So, by Proposition 3.14,

$$\mathrm{Con}(\mathrm{ZFC} + \exists\kappa(\kappa \text{ huge and } \kappa > \aleph_\alpha)) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \aleph_{\alpha+2} \rightarrow \aleph_{\alpha+3}).$$

3.4 Other (non-) transfer results

In this section we survey a number of instances of the relation $\kappa \rightarrow \lambda$ and find that either they are outright refutable in ZFC or they are refutable using some extra set-theoretic assumptions. Our first goal is to show that most instances of $\kappa \rightarrow \lambda$ where $\lambda < \kappa$ are refuted in ZFC. We recall the following definition of Morley.

Definition 3.16 μ is *characterizable* if there is a theory T in a countable language and a type p such that for each $\eta < \mu$, T has a model of power η omitting p but T has no model of power μ omitting p.

It is shown by examples in [9] that $\aleph_0$ and $\aleph_1$ are characterizable. Further, if κ^+ is characterizable, then so are κ^{++} and $(2^\kappa)^+$. Also, if $\kappa = \sup\{\kappa_n : n \in \omega\}$ and κ_n is $< \kappa$ and is characterizable for each $n \in \omega$, then both κ and κ^+ are characterizable as well. It follows that the set of characterizable cardinals is cofinal in $\beth_{\omega_1}$ and under GCH, every infinite cardinal below $\beth_{\omega_1}$ is characterizable.

However, it follows immediately from Theorem 3.2 that if $\lambda < \mu \leq \kappa$ with μ characterizable, then $\kappa \not\rightarrow \lambda$. (Simply take the theory T witnessing μ characterizable and affix a "dummy" predicate U.) These easy observations yield the following proposition.

Proposition 3.17
(a) *If $\lambda < \beth_{\omega_1} \leq \kappa$ then $\kappa \not\rightarrow \lambda$.*
(b) (GCH) *If $\lambda < \kappa \leq \beth_{\omega_1}$, then $\kappa \not\rightarrow \lambda$.*

Our discussion now splits into a number of cases depending on whether κ and λ are limit cardinals, successors of limits, or successors of successors. We will see an example in Subsection 3.5 of a $\mathrm{PC}_{\aleph_0}$ class $\mathcal{K}_{\mathrm{LIM}}$ whose spectrum is exactly the class of uncountable limit cardinals. So trivially $\kappa \not\rightarrow \lambda$ when κ is a limit cardinal and λ is a successor. Also, in Subsection 3.5 there is an example of a $\mathrm{PC}_{\aleph_0}$ class whose spectrum is exactly the class of strong limit cardinals. So if κ

and λ are both limit cardinals with $\kappa < \lambda$, then it is relatively consistent with ZFC that $\kappa \not\to \lambda$, as it is relatively consistent for the GCH to hold for all cardinals below κ^+ but $2^{\kappa^+} > \lambda$.

If $\alpha < \beta$, then by modifying Mitchell's construction so as to kill all special $\aleph_{\alpha+2}$-Aronszajn trees but requiring that $2^{\aleph_\beta} = \aleph_{\beta+1}$ we obtain the consistency of $\aleph_{\alpha+2} \not\to \aleph_{\beta+2}$ from the consistency of a Mahlo cardinal above $\aleph_\alpha$. To consider the case where κ and λ are both successors of limits we begin with the following lemma.

Lemma 3.18 *Suppose μ^+ is characterizable and $\kappa > \mu$. Then there is a theory T and a type p such that T has a $(\kappa, < \kappa)$ model omitting p iff* $(\exists\lambda)(\aleph_0 \leq \lambda < \kappa \wedge \lambda^\mu \geq \kappa)$.

Proof: Let T_0 be an L_0-theory and let p_0 be an L_0-type such that there exists a model of T_0 of power μ omitting p_0, but every model of T_0 of power μ^+ realizes p_0.

Let $L = L_0 \cup \{U, V, f\}$ where U and V are new unary predicates and f is a binary function symbol. Let T consist of the axioms of T_0 relativized to the predicate V, together with the axioms

$$\forall x(f(x,\cdot): V \to U)$$

and

$$\forall x \forall y[\forall z\{V(z) \to (f(x,z) = f(y,z))\} \to x = y].$$

Let p be p_0 relativized to V.

If $\mathfrak{A} \models T$ then associated to every element $a \in |\mathfrak{A}|$ there is a unique function $f(a,\cdot): V \to U$. So if $\|\mathfrak{A}\| = \kappa$ then certainly $|U^{\mathfrak{A}}|^{|V^{\mathfrak{A}}|} \geq \kappa$. However, if $\mathfrak{A}$ omits p, then surely $|V^{\mathfrak{A}}| \leq \mu$, so $|U^{\mathfrak{A}}|^\mu \geq \kappa$. The converse is clear.

The following corollaries are now immediate.

Corollary 3.19
(a) (CH) *For* $2 \leq n < \omega$, $\aleph_n \not\to \aleph_{\omega+1}$.
(b) (GCH) *If* $\mathrm{cf}(\lambda) < \min\{\mathrm{cf}(\kappa), \beth_{\omega_1}\}$ *then* $\kappa^+ \not\to \lambda^+$.

Proof: Under CH, for $0 < n < \omega$, $\aleph_n^\omega = \aleph_n$ by Hausdorff's lemma. It follows from the lemma above by taking $\mu = \omega$ that there is an $(\aleph_{\omega+1}, \aleph_\omega)$ model of T omitting p, while every $(\aleph_n, < \aleph_n)$ model of T realizes p. So $\aleph_n \not\to \aleph_{\omega+1}$ by Theorem 3.2.

To prove (b), take $\mu = (\mathrm{cf}(\lambda))^+$. As $\mu < \beth_{\omega_1}$ it follows from the GCH that μ is characterizable. So from the lemma above, for all $\rho \geq \mu$, there is a $(\rho, < \rho)$ model of T omitting p if and only if $\rho = \eta^+$ and $\mathrm{cf}(\rho) \leq \mathrm{cf}(\lambda)$.

Thus there is a (λ^+, λ) model omitting p, but no (κ^+, κ) model omitting p, so $\kappa^+ \not\to \lambda^+$ again by Theorem 3.2.

As for the case when κ is the successor of a limit cardinal and λ is the successor of a regular cardinal we offer the following. Litman and Shelah [6] have announced the consistency of ZFC + GCH plus "The existence of a first order theory T that has two-cardinal models but no $(\aleph_{\omega+1}, \aleph_\omega)$ model", assuming the consistency of ZFC + the existence of a supercompact cardinal. This example can be easily modified to have no $(\aleph_{\omega+1}, < \aleph_{\omega+1})$ models. Then, by Chang's two-cardinal theorem, T has two-cardinal models in all cardinals λ^+ for λ regular. So, by Theorem 3.2, we have

$$\mathrm{Con}(\mathrm{ZFC} + \exists \textit{ a supercompact cardinal}) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \aleph_{\omega+1} \not\rightarrow \lambda^+)$$

for any regular λ.

Finally, as an illustration of the dependence of these notions on set-theoretic assumptions beyond ZFC, we investigate the consistency of the statement $\forall\kappa < 2^{\aleph_0}(\kappa \rightarrow 2^{\aleph_0})$. Not surprisingly, the truth of this statement depends heavily on the value of the continuum. If $2^{\aleph_0} = \aleph_1$ it holds vacuously. If $2^{\aleph_0} = \aleph_2$ then it follows by Corollary 3.15. If $2^{\aleph_0} > \aleph_\omega$ and is a successor, then the statement is refuted by $\mathcal{K}_{\mathrm{LIM}}$ (see the proof of Proposition 3.22 for a definition). However, using a theorem of Solovay, we note that it is consistent with the existence of a measurable cardinal that $2^{\aleph_0}$ be weakly inaccessible and $(\forall\kappa < 2^{\aleph_0})(\kappa \rightarrow 2^{\aleph_0})$. To see this, we need the notion of a Rowbottom cardinal.

A cardinal λ is *Rowbottom* if every theory T in a countable language and every $(\lambda, < \lambda)$ model of T has a $(\lambda, \aleph_0)$-elementary submodel. The following lemma is almost immediate.

Lemma 3.20 *If λ is Rowbottom then $\forall\kappa < \lambda(\kappa \rightarrow \lambda)$.*

Proof: Fix $\kappa < \lambda$. Assume T has a $(\lambda, < \lambda)$ model $\mathfrak{A}$ omitting p. Then λ Rowbottom implies T has a $(\lambda, \aleph_0)$ model $\mathfrak{B}$ omitting p. So by Downward Löwenheim–Skolem, there is a $(\kappa, \aleph_0)$ model $\mathfrak{C}$ omitting p, so $\kappa \rightarrow \lambda$ by 3.2.

But now, Solovay [12] has shown that it is consistent with the existence of a measurable cardinal that $2^{\aleph_0}$ be weakly inaccessible and Rowbottom.

3.5 Determining Spec($\mathcal{K}$) *from an initial segment*

Definition 3.21 κ_0 is the least cardinal κ such that for all $\mathrm{PC}_{\aleph_0}$ classes $\mathcal{K}_1$ and $\mathcal{K}_2$,

$$\text{if } \mathrm{Spec}(\mathcal{K}_1) \cap \kappa = \mathrm{Spec}(\mathcal{K}_2) \cap \kappa, \text{ then } \mathrm{Spec}(\mathcal{K}_1) = \mathrm{Spec}(\mathcal{K}_2).$$

As there are only $2^{\aleph_0}$ such spectra, κ_0 exists. However, the following proposition, suggested by Hrushovski, shows that the value of κ_0 is nonabsolute in a very strong sense.

Proposition 3.22 *Working in* ZFC + GCH, *for each $\beta \in$* ORD *there is a forcing notion $\mathcal{P}$ that preserves cardinals and $\mathcal{P} \Vdash \kappa_0 > \beth_\beta$.*

Proof: Let $\mathcal{K}_{SL}$ be the $\mathrm{PC}_{\aleph_0}$ class suggested by Silver having

$$\mathrm{Spec}(\mathcal{K}_{SL}) = \{\text{all uncountable strong limit cardinals}\}$$

and let $\mathcal{K}_{\mathrm{LIM}}$ be a $\mathrm{PC}_{\aleph_0}$ having

$$\mathrm{Spec}(\mathcal{K}_{\mathrm{LIM}}) = \{\text{all uncountable limit cardinals}\}.$$

Specifically, $L_{SL} = \{U\}$, $L'_{SL} = \{U, \epsilon, f\}$ and

$$\begin{aligned} T'_{SL} = {} & \{f \text{ is a one-one function}, f\colon U \rightarrow \neg U\} \\ & \cup \{\forall x \forall y (x \in y \rightarrow Ux)\} \\ & \cup \{\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]\} \end{aligned}$$

and $L_{\mathrm{LIM}} = \{U\}$, $L'_{\mathrm{LIM}} = \{U, f, g, \leq\}$ and

$$\begin{aligned} T'_{\mathrm{LIM}} = &\{f \text{ is a one-one function}, f\colon U \to \neg U\} \\ &\cup \{\leq \text{ is a linear ordering}\} \\ &\cup \{\forall x\, g(x, \cdot) : \{y : y \leq x\} \to^{1-1} U\}. \end{aligned}$$

To compute $\mathrm{Spec}(\mathcal{K}_{\mathrm{LIM}})$ note that for all cardinals κ, $\kappa \in \mathrm{Spec}(\mathcal{K}_{\mathrm{LIM}})$ iff ($\mathfrak{A} \models T'_{\mathrm{LIM}}$ implies $|U^{\mathfrak{A}}| = \kappa$) iff (for every linear ordering on κ and every $\lambda < \kappa$, there is an element having more than λ predecessors) iff κ is a limit cardinal.

Now, given $\beta \in \mathrm{ORD}$, let $\lambda = \beth_\beta^+$, $\mu = \aleph_\omega(\lambda)$, and $\eta = \mu^+$. Let $\mathcal{P}$ denote the partial order of functions from subsets of $\lambda \times \mu$ of power $< \lambda$ to 2. Now $\mathcal{P} \Vdash 2^\lambda = \eta$, so

$$\mathcal{P} \Vdash \mu \text{ is a limit cardinal but not a strong limit.}$$

That is, $\mathcal{P} \Vdash \mu \in \mathcal{K}_{\mathrm{LIM}} \setminus \mathcal{K}_{SL}$. However, $\mathrm{Spec}(\mathcal{K}_{SL}) \cap \beth_\beta = \mathrm{Spec}(\mathcal{K}_{\mathrm{LIM}}) \cap \beth_\beta$, so $\mathcal{P} \Vdash \kappa_0 > \beth_\beta$, as desired.

4 Open problems In Section 3 we considered a great many instances of the relation $\kappa \to \lambda$ and obtained a number of partial results about their consistency. The major remaining cases are those where κ is a successor and λ is a limit cardinal with $\kappa < \lambda$. Lemma 3.18 is a source of negative information in the case where a cardinal of cofinality ω is strictly between κ and λ and $0^\#$ exists. (By Jensen's Covering Lemma, it is easy to see that at least the existence of $0^\#$ is necessary for what follows.) For example, we might have $\kappa = \rho^+$ with $\rho^\omega = \rho$ while $\mu^\omega \geq \lambda$ for some $\mu < \lambda$, so $\kappa \not\to \lambda$ by Lemma 3.18 and Theorem 3.2. Arguments such as this establish the consistency of, say, $\aleph_2 \not\to \aleph_{\omega+\omega}$. However, nothing of this sort can work when $\lambda = \aleph_\omega(\kappa)$. In particular, whether $\aleph_2 \to \aleph_\omega$ is open.

Our second open question is whether $\kappa \to \beth_\omega(\kappa)$. If we ignore the necessity of omitting a type in Theorem 3.2 then this transfer would follow immediately from Vaught's theorem on two cardinals far apart. However, as $\beth_\omega(\kappa)$ is less than the Hanf number of omitting types, we cannot hope to produce a two-cardinal model containing an infinite set of order-indiscernibles that omits a given type as well.

Our final question is to look at all of these transfer questions in L. In particular, whether $\aleph_2 \to \aleph_3$ holds in L is open.

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