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Russell, Logicism, and the Choice of Logical Constants

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Abstract It is here argued that Russell's *Principles of Mathematics* contains an intriguing idea about how to demarcate logical concepts from nonlogical ones. On this view, implication and generality emerge as the two fundamental logical concepts. Russell's 1903 proposals for defining other logical concepts from these basic ones are examined and extended. Despite its attractiveness, the proposal is ultimately unsatisfactory because of problems about defining negation and existential quantification.

Introduction Traditional logicism holds that all mathematical concepts can be defined in terms of logical concepts and that all theorems of mathematics can be derived by logic from logical truths. Clearly, to assess the truth and philosophical significance of logicism, we must know what concepts are logical concepts and what truths are logical truths. Bertrand Russell held, for example, that the Axiom of Infinity, while perhaps true, was not a logical truth ([15], p. viii). Of equal importance is the question of what concepts are logical concepts. On this matter noted authorities such as Quine, Tarski, and Church have made pessimistic assessments. Quine standardly characterizes the logical truths as truths which involve only the logical words essentially. He enumerates the logical words (e.g., "not", "or", "all", but naturally not "necessarily"). But he refuses to go beyond enumeration: "Logical vocabulary is specified only I suppose by enumeration" ([12], p. 141). Tarski regarded it as quite possible that future investigations would compel us to hold that the division of terms into logical and extralogical was, to a greater or lesser degree, arbitrary ([19], p. 420; see also Wang [20], p. 54, and Church [2], p. 58, note 129). Naturally, such views challenge the philosophical importance of even the weaker logicist thesis that all mathematical concepts are definable in purely logical terms.

I shall, in what follows, examine a response to the problem of demarcating the logical constants which is implicit in the first English presentation of logi-

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cism, Bertrand Russell's *Principles of Mathematics* [15]. I argue that Russell's choice of primitive logical constants in Book I, Chapter II, is naturally motivated by a plausible and widely shared conception of what logic is. However, I then show that this choice of primitive logical constants fails to supply an adequate basis for defining the other logical constants – specifically, negation and existential quantification – needed to execute the logicist program.

I do not contend, of course, that Russell explicitly proposed the view which I elicit from *Principles*. For on the view I elicit there are just two primitive logical constants, and Russell's lists of primitive constants always contain more. However, the proposed view is, I shall argue, highly congenial to the foundational program found in *Principles*. For example, it fits nicely Russell's ideas concerning the nature of the propositions of pure mathematics.

Section 1 motivates Russell's choice of constants. In Sections 2 and 3 I outline the logical framework used by Russell in *Principles* and show how he proposed to define other logical constants from his chosen primitives. Section 4 concerns the primitive constants introduced in connection with the calculus of classes and relations. In Section 5 I explain the flaw in Russell's proposal.

1 According to Russell, "Symbolic Logic is essentially concerned with inference in general, and is distinguished from various special branches of mathematics mainly by its generality" ([15], p. 11). In a footnote to this passage, Russell states that he does not distinguish "inference and deduction". Thus, the key concepts in Russell's characterization of logic are generality and deduction. If we think, not of the application of symbolic logic to other areas, but rather of the principles of symbolic logic itself, it is to be expected that they should be the most general principles of deduction. What concepts should we expect to appear in principles of this sort? Quite clearly we would need a formal representative of the concept of generality, since we wish to express *general* principles. Also, we need a formal representative of the notion of deduction, for we wish to state general principles about *deduction*. We could certainly also permit the appearance in principles of logic of concepts definable solely in terms of these two basic ideas. But beyond this it is less clear what concepts must or should be given a place in the general principles of logic. A plausible, conservative course would be to attempt to frame our general logic using as primitive constants only the formal representatives of generality and deduction.[1] I shall argue below that Russell pursues precisely this course in his presentation of symbolic logic in the *Principles of Mathematics*.

The idea that formal representatives of the concepts of generality and deduction should be fundamental logical constants is especially compelling given the philosophical view on the nature of logic which Russell held in *Principles*. It is now widely agreed that Frege and the early Russell adhered to a conception of logic at odds with the standard, contemporary conception (see [8], [10], pp. 183–187; [7], pp. 122–135; and [9], pp. 4–6). Logic was the system of correct principles of reasoning, whose domain was the entire universe. Unlike theories framed within logic and possessing their own special primitives, logic does not admit alternative universes or alternative interpretations.[2] There is no external standpoint, no metalanguage or metalogic, from which to view logic.[3] On such a view, if logic is to involve claims about deducibility, these claims cannot be relegated to a metalanguage. Rather, these claims must be part of logic itself, and consequently it is plausible to think that the relation involved (i.e., deducibility) will be a fundamental logical constant. Similar remarks apply to the concept of generality.

In the presentation of quantificational logic in *Principles*, Russell treats material implication and formal implication as the sole primitive constants. From a contemporary point of view, formal implications are universally quantified material implications; e.g., "for all x, if x is human, then x is mortal". So, standardly, I shall treat Russell's choices of primitives as including just material implication and universal quantification.

It is worth noting that in *Principles* Russell did not accept this modern conception of formal implication. He viewed it as a genuine primitive, although he does offer an "analysis" of formal implication involving the notions of class, denoting, and *any* ([15], p. 92). Russell's insistence that formal implication is not reducible to material implication, plus a quantifier, appears to stem from his denial that "implies" in "x is human implies x is mortal" indicates a relation. If it did indicate a relation, its relata would apparently be propositional functions. But precisely this claim is repeatedly denied: "it is to be observed that "x is a man implies x is mortal" is not a relation of two propositional functions . . ." ([15], p. 38; see also pp. 84–85, 92). "Implies" cannot here relate propositional functions; for if it did, we would be unable, according to Russell, to distinguish "x is human implies x is mortal" from "x is human implies y is mortal". Although these views are clearly important in understanding Russell's early conception of quantification, they are secondary in this context. What is important is that formal implication embodies the notion of generality.[4]

Universal quantification is the formal representative of the concept of generality embodied in Russell's characterization of symbolic logic. Particularly noteworthy in this regard is that Russell's universal quantifier is unrestricted in its scope; it is not to be thought of as restricted to this or that domain. The variable has, in Russell's words, "an absolutely unrestricted field"; "any conceivable entity" is a value of the variable introduced by the universal quantifier ([15], pp. 7, 91). The unrestrictedness of the universal quantifier fits well with the idea that the principles of logic to be expressed through its employment are of complete generality. A general principle of logic will purportedly be expressed by a formal sentence beginning with one or more universal quantifiers. Given Russell's unrestricted quantifier, this means that the principle in question is true of absolutely every object. Indeed, I think that a primary appeal which logicism had for Russell was that it explained the generality of mathematics in terms of the generality of logic.[5]

The second constant in Russell's presentation is material implication. Russell clearly regards this constant as the formal representative of the concept of deduction, or more properly, of deducibility. He says of material implication that it is "the relation in virtue of which it is possible to validly infer" one proposition from another ([15], p. 33). He standardly paraphrases claims of the form "$A \supset B$" using locutions such as "A implies B" or "B can be deduced from A" ([15], p. 34). Naturally, in this day of ever more stringent implicational logics, Russell's view seems naive. But Russell was aware of, and emphasized, the "paradoxes" of material implication in 1903, and he realized that these properties would be at variance with what is "commonly maintained" ([15], p. 34).

I am unsure as to whether this view is ultimately defensible, but the richness of Russell's 1903 ontology gives it greater appeal than might be supposed. Russell attributes our dissatisfaction with the properties of material implication to our tendency to affirm material implications only when there is a suitable universally quantified conditional implying it ([15], p. 34). Now, even if this is so, it might be held that the sort of universal conditional involved is not a universally quantified *material* conditional. After all, might not a universally quantified material conditional be merely accidentally true? In such a case, it would certainly not make us inclined to affirm its implicational instances. But for Russell in 1903 a true universally quantified material condition was true of all terms whatsoever, and this includes possible nonactual objects such as chimeras ([15], p. 43). Thus, a true formal implication is, we might say, modal in character. It affirms (at least) of every possible object which has P that it also has Q. So true formal implications are not accidentally true; they are, in effect, true strict implications. As such, they are at least plausible formal representatives of quantified statements of entailment.

The view of logical constants proposed here coheres beautifully with the description of mathematical propositions in Chapter I of *Principles*. Two features of such propositions are mentioned: (1) that they assert implications, and (2) that they contain variables. Concerning the first feature, he writes: "We assert always in mathematics that if a certain assertion p is true of any entity x, or of any set of entities $x, y, z, \ldots$, then some other assertion q is true of these entities . . ." ([15], p. 5). As to the latter feature, he says: "So long as any term in our proposition can be turned into a variable, our proposition can be generalized; and so long as this is possible, it is the business of mathematics to do it" ([15], p. 7). On the present proposal, these features of mathematical propositions are precisely the fundamental logical concepts of implication and generality.

There are also interesting internal reasons, involving definition, which lead Russell to choose implication and universal quantification as primitive constants.[6] Russell notes Peano's view that the choice of indefinables is "largely arbitrary" ([15], p. 15). Russell apparently agrees, but also insists on some important limitations. In particular, he argues that implication is indefinable, and so must be among the primitive constants. The argument given concerns the proposal that material implication be defined using negation and disjunction[7]:

> the assertion that q is true or p false turns out to be strictly equivalent to "p implies q"; but as equivalence means mutual implication, this still leaves implication fundamental, and not definable in terms of disjunction. ([15], p. 15)

The proposed definition of material implication apparently has the form of a universally quantified biconditional: for all p,q, p implies q if and only if not-p or q. But the "if" in the biconditional is the same concept as that occurring in the definiendum. Thus, the definition amounts to this: for all p,q, p implies q implies and is implied by not-p or q.[8] Here, the concept to be defined occurs outside the definiendum. This, of course, is not a straightforwardly circular definition, which would make the defined expression ineliminable, but it does conflict with the idea that we are, by the definition, introducing a new concept into the language. For "implies" already occurs in the language in which the definition is stated. Quite clearly, an argument of this same type could be mounted to show that universal quantification is indefinable. Further, it seems that only implication and universal quantification could be argued to be indefinable on these grounds though no attempt is made, on Russell's part, to show this.[9]

Two considerations detract from the force of this argument. First, at least as I presented it, Russell's argument treats definitions as ordinary equivalences in the formal system itself. But, following Russell's own example in *Principia Mathematica*, definitions in logistic systems are most commonly regarded as statements which introduce a new symbol or expression as an abbreviation for a certain expression of the system. Statements of definition are clearly not statements of the logistic system itself, since the abbreviation does not occur in the formal system. They are statements, probably of the metametalanguage, which in certain contexts state the interchangeability of certain sequences of symbols (see [2], pp. 76–78). This view of definition was attractive to Russell. He endorses it at least once in *Principles* and it is the offical conception of definition to be found in *Principia* (see [15], p. 429, and [17], pp, 11–12). But Russell also purports in *Principles* to define nonlinguistic items, such as terms and relations ([15], pp. 15, 27, 111). Further, Russell apparently regards the definitions offered of numerical terms as true or false, and hence not as mere stipulations for the sake of convenience. Moreover, the object language/metalanguage distinction itself is alien to Russell's 1903 conception of logic, wherein the logistic system itself is meant to be all-embracing.

Even granting that definitions are to be expressed in the logistic system itself, the argument may still be faulted for its insistence that definitions of concepts take a specific form: universally quantified biconditionals. Why would it not suffice to express definitions in a logically equivalent form? Instead of defining concepts using the form "for all x, $P(x)$ if and only if . . . x. . . ," we could express definitions in the form "it is not the case that there is some x for which it is not the case that ($P(x)$ if and only if . . . x . . .)". Expressing definitions in this form would allow us to use definitions in just the same ways as are allowed by the more conventional form of definition. For example, definitions framed in the second way would allow us, in the presence of a reasonable set of axioms, to establish the interchangeability of definiens and definiendum in all extensional contexts. And clearly, if definitions were framed in terms of the existential quantifier, it would be possible to define the universal quantifier. Given that there are many nonoverlapping ways of framing definitions, all of which serve the purposes of definition equally well, no concept is indispensable for the framing of definitions. And so no concept can be shown to be indefinable in the way Russell proposes.

I cannot here assess the merit of this criticism, for in my view it involves large issues. I wish to make just two points about it, one concerning its role in current philosophical discussion and one concerning what Russell's attitude toward it would have been at the time of *Principles*. First, arguments which have the form taken by the second criticism are employed widely in current discussions of ontology in the philosophy of mathematics. In the case of arithmetic, they give rise to what has come to be known as Benacerraf's problem, which has been stated forcefully and succinctly by W. V. Quine:

> Numbers in turn are known only by their laws, the laws of arithmetic, so that any constructs obeying those laws—certain sets, for instance—are eligible in turn as explications of number. . . . Arithmetic is, in this sense, all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic. ([11], pp. 44–45)

Now even as early as *Principles* Russell was sensitive to this sort of issue. Having in Book II set out his definition of particular numbers as classes of similar classes, Russell admits that there may be other adequate definitions of numbers in terms of general logic, although he is not aware of any. Moreover, in response to the suggestion that numbers are not classes, but rather properties (or predicates, as Russell says), he responds that *philosophically* we may admit this. However, he says that all mathematical purposes are completely served by the classes he has used ([15], p. 116). While aware of the possibility of alternative definitions, Russell seems unclear as to what conclusion to draw from this. Russell's repeated insistence in *Principles* that philosophy is largely a matter of a kind of intellectual perception suggests that even if such classes served all the relevant purposes, still they might not be the numbers themselves, a fact which could be revealed to us by a sort of direct intellectual inspection ([15], pp. xv, 129–130).[10]

Against the entire proposal of this section, it may be objected that Russell himself held that the logical constants could only be enumerated. Early in *Principles*, he writes: "The logical constants themselves are to be defined only by enumeration, for they are so fundamental that all the properties by which the class of them might be defined presuppose some terms of the class" ([15], pp. 8–9). Several points need to be made in response. First, nothing fundamental to Russell's logicism turns on the indefinability of the concept *logical constant*. In fact, in the penultimate draft of *Principles*, written in May, 1901, Russell himself proposed a definition of the concept.[11] Second, the reason given isn't persuasive even on Russell's conception of logic. Perhaps any definition of the concept *logical constant* will employ logical constants. Precisely why is this unsatisfactory? There is no circularity here, direct or indirect, unless we assume the implausible thesis that to deploy a specific logical constant such as implication requires deployment of the concept *logical constant*. Third, and most important, the proposal under consideration need not be a definition to serve its purpose. What is wanted is a well-motivated characterization of the logical constants which brings out their importance and distinctiveness. There is no reason to think that such a characterization has to take the form of a set of necessary and sufficient conditions for being a logical constant.

To sum up, Russell's choice of primitive logical constants—universal quantification and material implication—is naturally motivated by the characterization of logic as expressing the most general principles of deduction. This choice fits with Russell's conception of logic and the nature of mathematics, and is further reinforced by his views on the nature of definition.

2 In this section, I will sketch the ontological framework within which Russell articulates the logical system presented in *Principles*. As stated earlier, this framework departs significantly both from the one implicit in Frege's work and the one implicit in most current work in logic.

For the purposes of this paper, I want to emphasize three features of Russell's philosophy of logic in 1903. First, there is a type of variable which has an absolutely unrestricted field. Whatever there is—humans, points, chimeras, relations—is a value of this variable. Russell calls this variable the true or formal variable ([15], pp. 43, 91). Correlative to the unrestricted variable is the idea that there is a category to which absolutely everything belongs: the category of terms. To this category belongs anything which may be an object of thought, occur in a true or false proposition, or be counted as one ([15], p. 43). All terms have being, although not all have a spatial or temporal location ([15], p. 449). Terms include relations, properties, and propositions themselves.

Russell was willing to employ variables ranging over some, but not all, terms. In *Principles* he employs a variety of restricted quantifiers—over classes, numbers, and points, for example. Restricted quantifiers are permitted but they are regarded as a convenience only. They can be avoided by a hypothesis which states the restriction itself. Of course, this way of avoiding restricted quantification is unavailable generally in Frege's philosophic logic and in type theory. Under the pressure of the paradoxes, Russell admitted that propositional functions might constitute an exception to the policy of eliminating restricted quantifiers ([15], pp. 88, 91). The views on propositional functions to be found in *Principles* contain severe tensions. Quantifiers over propositional functions are needed in the logicist program ([15], p. 104); yet if propositional functions are treated as terms the paradoxes ensue ([15], p. 88). So Russell in *Principles* assigns to propositional functions a status which threatens to undermine the universal status of the category of terms. The "ϕ in ϕx is not a separate and distinguishable entity: it lives in propositions of the form ϕx, and cannot survive analysis" ([15], p. 88).

A second distinctive feature of Russell's 1903 logic is that properties and relations may occupy the subject position in a proposition. Properties and relations are terms; for they are certainly possible objects of thought. He thus permits self-predication for properties and relations: it is true that termhood is a term and false that humanity is human. Generally, for any predicate P and term t, either it is true that t is P or it is false that t is P ([15], pp. 43–45). Russell considers views, like Frege's, according to which some entities can occur only predicatively in a proposition, and rejects them on the grounds that they cannot be consistently stated.[12] Naturally, one is inclined to think that if this is a sound criticism of Frege's views, it applies equally to Russell's incipient account of propositional functions.

A third feature important for the concerns of this paper is that propositions are terms. For Russell, at the time of *Principles*, propositions are complex, mind-independent unities ([15], p. 139). As such, they meet the criteria for termhood. Given the termhood of propositions and the applicability of the Law of Excluded Middle to all terms and relations, material implication emerges as a relation which holds or fails to hold between any pair of terms. As Russell's first axiom for propositional logic states: "whatever p and q might be, 'p implies q'

is a proposition" ([15], p. 16). For the relation to hold between terms p and q, both p and q must be propositions; Russell's second and third axioms for propositional logic state precisely this ([15], p. 16). Moreover, the relation holds only if either the first is a false proposition or the second is a true proposition. So, "Socrates implies Plato" is a proposition, but it is a false one. Given that the terms of a true implication must be propositions, Russell defines 'proposition' ([15], p. 15) as follows:

D1 $(p)(p \text{ is a proposition } .\equiv. \ p \supset p)$.

It should be noted that, on this understanding of material implication, many sentences which standardly express laws of logic no longer express truths. Thus, "$(p)(p \supset p)$" is false, because not everything is a proposition. To express the reflexivity of implication, we have to relativize the quantifier to propositions. Taking D1 into account this yields: $(p)(p \supset p \ .\supset. \ p \supset p)$ as the law of reflexivity. I should note that not all logical laws need to be so relativized. Transitivity can be correctly stated in its usual form: $(p)(q)(r)(p \supset q \ .\supset: q \supset r \ .\supset. \ p \supset r)$.[13]

Russell's logic, in the early chapters of *Principles* and in the late pre-Peano manuscripts on the philosophy of mathematics, embodies the metaphysics of what Peter Hylton has called Platonic atomism (see [10], pp. 65–87). This view, set forth by Moore in his paper "The Nature of Judgment," maintains that the objective world consists entirely of certain basic entities, concepts, and complexes having these basic entities as constituents. The complexes are all propositions, and it is proposed that ordinary objects are to be identified with propositions.[14] Crucially, concepts and the propositions which they compose are mind-independent and real, in precisely the same sense.

Russell incorporates the egalitarianism of Moore's metaphysics: Every term is a value of the unrestricted variable, and predicates and relations are defined for all terms. Aside from the difficulties caused by the paradoxes, there is at least one further important departure from Moore's metaphysics. Russell distinguishes two kinds of terms: things and concepts. In propositions, things are always said to have certain properties or to stand in certain relations, but they are never predicated of other terms. As examples of things, Russell uses ordinary physical objects, points of space, and instants of time. Concepts, however, can occur both substantively and predicatively. In some propositions a concept is predicated of a thing, while in others that very concept may have another concept predicated of it. Moreover, the same concept may be predicated, in the same sense, of both a concept and a thing. Concepts include both properties (e.g., humanity) and relations (e.g., difference). Although I shall not argue the point here, it appears that this distinction between things and concepts is motivated by Russell's desire to give some account of the structure and unity of propositions (see [10], pp. 155–158, and Section 54 of [15]).

3 I now turn to Russell's definitions of the familiar connectives of propositional and quantifier logic. Since I argue later that some of these definitions are unsatisfactory and since, as we have seen, Russell's logic is not a standard one, I will present and discuss these definitions in more detail than is customary.

In Chapter II of *Principles* Russell first defines conjunction: "If p implies p, then if q implies q, pq (the logical product of p and q) means that if p implies that q implies r then r is true". Symbolically, this is represented:

$$(p)(q)((p \supset p) \supset ((q \supset q) \supset ((pq) \equiv (r)((p \supset (q \supset r)) \supset r)))).$$

This definition only characterizes conjunction for cases in which p and q are propositions. This limitation is at variance with Russell's treatment of implication, and indeed with his use of conjunction in the statement of axioms. For instance, Russell's fifth axiom, Simplification, states: if p implies p and q implies q, then pq implies p ([15], p. 16). Here, as usual, the variables range over all terms. So, in the consequent of the axiom, the component "pq" must have an interpretation when p and q are not propositions. There are several natural ways to handle this problem, while retaining the idea, found in Russell's treatment, that "p implies q" expresses a proposition even when "p" and "q" denote nonpropositions.[15] The simplest way, I think, is this:

D2 $(p)(q)((pq) \equiv (r)((r \supset r) \supset ((p \supset (q \supset r)) \supset r)))$.

The antecedent of the definiens reveals an additional dissatisfaction with the original proposal. By Russell's definition, if p and q are true propositions, (pq) is false, because $(p \supset (q \supset r)) \supset r$ is false for values of r which are not propositions. Thus, D2 defines conjunction in all cases, and without the defect just noted, since $(r \supset r) \supset ((p \supset (q \supset r)) \supset r)$ is automatically true for nonpropositional values of r.

While D2 defines conjunction for nonpropositional terms, it is not entirely consonant with Russell's treatment of implication, for an implication having a nonpropositional term is false. But inspection shows that according to D2 a conjunction with a nonpropositional term is true, regardless of the character of the other term. This feature of D2 is unusual, but, I think, not objectionable. We cannot, of course, state laws of conjunction in their simplest form; e.g., $(p)(q)(pq$ implies $p)$, or we will infer the truth of $2 + 2 = 5$ from the truth of the proposition that Plato and $2 + 2 = 5$. But relativizing the laws to propositions, as Russell does, suffices to remove the difficulties: $(p)(q)(((p \supset p)(q \supset q)) \supset ((pq) \supset p))$. In the case in question the relevant instance of the antecedent is false, because the proposition that Plato implies Plato is false.[16]

Russell defines disjunction in the following well-known way:

D3 $(p)(q)((p \vee q) \equiv ((p \supset q) \supset q))$.

As for implication and conjunction, the definition is intended to apply to propositional and nonpropositional terms. If q is a nonpropositional term, then, by D3, $p \vee q$ is a false proposition. However, if q is a propositional term and p is a nonpropositional term, then $p \vee q$ is a true proposition. This asymmetry apparently has no objectionable consequences, since the relevant propositional laws are relativized to propositions, as noted above.

The final definitions concern negation and existential quantification, and they will figure prominently in my critical remarks. The crucial feature of these definitions is that they are universally quantified conditionals. Negation is defined as follows:

D4 $(p)(\sim p \equiv (r)((r \supset r) \supset (p \supset r)))$.

Again the definition applies to all terms, so that "it is not the case that Plato" is a proposition. It is a false proposition, however, given that there are some propositional terms.

Russell uses the existential quantifier informally in *Principles* to state primitive laws about classes and relations ([15], p. 25). He never actually states a general definition of existentially quantified propositions. Still, we can assume that Russell intended to treat existential quantification as defined, for he nowhere includes it in his list of primitive ideas.

From earlier work and letters, one can glean two ideas as to how he might have defined existentially quantified propositions. In Russell's first post-Peano work in logic "The logic of relations", Russell follows Peano's *Formulaire* (1897, p. 13) and introduces "$\exists$" as a predicate, which is to be read "exists". This predicate attaches to terms purporting to denote classes and relations. Thus, Russell writes "$\exists y \ni (xRy)$", which is to be read "The class of y's such that xRy exists". Now, contrary to what this reading of "$\exists$" suggests to us, Russell and Peano do not mean by "$\exists a$" to assert that there is such a thing as the class a; rather they mean that a is a nonempty class.[17] Russell explains this terminology in *Principles*: "A class is said to exist when it has at least one term" ([15], p. 21). He then supplies a definition:

$$(a)(\exists a \equiv (p)((p \supset p) \supset ((x)(x \in a \supset p) \supset p))).$$

If we are willing to apply this idea to propositional functions, a general definition of existential quantification emerges. For, by analogy with "$\exists a$", we would have "$\exists \phi$", meaning that something x is such that ϕx. And paralleling the formal definition for "$\exists a$" we would have:

D5A $(\phi)(\exists x \phi x \equiv (p)((p \supset p) \supset ((x)(\phi x \supset p) \supset p)))$.

Again, this is intended to hold for *all* terms, assuming uneasily that propositional functions are terms. How to interpret instances of D5A, with values of ϕ that are not propositional functions, will emerge from discussions in Section 4.

A second and better known way of handling existential quantification appears frequently in work just subsequent to *Principles*. This is the definition using negation and universal quantification:

D5B $(\phi)(\exists x \phi x \equiv \sim(x) \sim \phi x)$.

This definition does not, to my knowledge, appear in *Principles* or in earlier publications on logic. Shortly thereafter, however, it becomes dominant. In a letter to Frege on May 5, 1903 (almost exactly contemporaneous with the publication of *Principles*), Russell proposes D5B as a part of a new idea to resolve the paradoxes (see [5], p. 159). In this letter Russell retains the idea of using only universal quantification and the conditional as primitive. In subsequent work, such as "On denoting", the definition recurs, but in a context in which Russell seems to treat negation (or falsity) as a further primitive (see [13], pp. 104–106).

It is worth noting that the extensionality of the material conditional is essential for the success of these definitions. For instance, the analogue of D4 using strict implication (or relevant implication) isn't correct, since "$\sim p$" can be true even when the proposed definiens is not.

4 In *Principles* Russell lists indefinables other than implication and universal quantification. I will argue that, with one exception, no additional indefinables are necessary in Russell's enterprise. I consider this important, since, as argued in Section 1, implication and universal quantification have a special claim on the status of being *logical* constants; it is less clear why Russell's other primitives should be so regarded. The standard list of indefinables in *Principles* is the following: formal implication, material implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and truth ([15], pp. 3, 11). Of these, the last is not a concept which actually occurs in the propositions of mathematics, so we will not consider it further here.

The notion of relation occurs as a predicate in primitive propositions in two ways. First, to effect quantification over relations only, it is used in the antecedents of conditionals: "For all R, if R is a relation, then . . ." ([15], p. 87). Second, it occurs in propositions asserting that certain specific relations (e.g., material implication and class membership) are relations ([15], p. 26). These uses of "relation" are eliminable by the same device that Russell used to define "proposition". The point is that "aRb" does not express a proposition unless R is a binary relation. If "aRb" expresses a proposition, then "aRb" is true, and otherwise it isn't. Thus, being a relation, or more precisely, being a binary relation, is definable:

D6 $(R)(R \text{ is a binary relation} \equiv (x)(y)(xRy \supset xRy))$.

One apparent problem with D6 is that it is not clear how to interpret "xRy" if R is not a binary relation. How, for example, is "xRy" to be interpreted for the value Socrates? This problem is not one introduced into the logic of *Principles* by D6; because of the unrestrictedness of Russell's quantifiers, this problem is there from the beginning. Russell recognized this, and in a footnote concerning a proposition like D6, he wrote: "It is necessary to assign some meaning (other than a proposition) to aRb when R is not a relation" ([15], p. 87). Several options are open here; a very natural one would be to let the relevant meaning be the class $\{a,R,b\}$ when R is not a binary relation. So, "relation" seems to be satisfactorily definable. *Of course*, to say that "relation" is definable is not to deny that Russell must make substantial existence assumptions about relations to carry out his program.

In the section in *Principles* which concerns the calculus of classes, Russell finds three indefinables. They are the relation of class membership, the notion of *such that*, and the notion of a propositional function ([15], p. 19). The *such that* operator functions in Russell as a device for forming classes from propositional functions. Applied to the propositional function "Fx" it yields '$x \ni Fx$', which is to be read "the x's such that Fx". In "The logic of relations" Russell describes the role of this operator as follows: "I regard $\ni$ as a primitive idea which permits me to put this sign before propositions which are not reducible without its aid to the form $x \in \alpha$" ([14], p. 5). So, for a relation R and term x, the terms to which x bears the R are $y \ni (xRy)$. Clearly then '$y \ni (xRy)$' plays the role played by the familiar brackets notation, '$\{y:xRy\}$'. As such, it is well known that the device is contextually eliminable (e.g. as in Quine, *Mathematical Logic*, Sections 24, 26). The relation of class membership is also contextually definable as Russell himself showed in *Principia* ([17], *20).

Here it might be objected that the proposed eliminations by paraphrase are incompatible with the naive theory of meaning held by Russell when he wrote *Principles*. There Russell writes "Words all have meaning, in the simple sense that they are symbols for something other than themselves" ([15], pp. 42, 47).[18] But on the current proposal the symbol "$\in$" in "$a \in p$" does not have meaning in this simple sense; it is not a symbol for something other than itself. As Russell later says, it has no meaning in itself, but every proposition in whose verbal expression it occurs has a meaning.

The theory actually presented in Part I of *Principles* is less naive than Russell's general remarks suggest. First, even at an abstract level, Russell is prepared to admit what, in retrospect, we would call a divergence between grammatical and logical form. He describes the relation between grammar and logic in a way clearly tolerating such divergence: "Grammar seems to me to bring us nearer to a correct logic than the current opinions of philosophers, and in what follows, grammar, though not our master, will yet be taken as our guide" ([15], p. 42). There are also several cases in which Russell self-consciously departs from the idea that each meaningful word indicates something. For instance, in Chapter 6, Russell discusses the analysis of "all u's" where u is a class concept. His conclusion is: "It would seem, then, that "all u's" is not validly analyzable into *all* and u, and that language, in this case as in some others, is a misleading guide" ([15], pp. 72–73). We might say that "all" has no meaning in itself but every denoting phrase in which it occurs has a meaning. A second example, prominent in *Principles*, concerns the word "implication". In certain contexts, such as "Socrates is human implies Socrates is mortal", the word indicates a relation, but when it occurs in contexts involving variables Russell is more cautious. In the formal implication "x is human implies x is mortal", "implies" does not denote a relation between propositional functions ([15], pp. 38, 92), nor, evidently, does it denote a relation between propositions. Indeed, the attempted analyses of formal implication in *Principles* are contextual; that is, they seek to explain the meaning of the entire context "x is human implies x is mortal". For example, Russell writes: "We may say, if we choose, that the whole formula expresses a relation of *any* term of 'x is an a' to *some* term of 'x is a b'" ([15], p. 92).

Finally, the proposal dispenses with classes in favor of propositional functions. This idea had long been appealing to Russell. Writing to Jourdain in March 1906, Russell recalls that in May 1903 he had concocted a plan that would have supplanted classes by propositional functions (see [6], p. 78). Although the plan did not succeed at that juncture, it shows clearly that Russell was never wedded to classes.

Of the three indefinables in the calculus of classes, only the notion of a propositional function remains. I noted earlier that Russell's notion of propositional function in *Principles* is a point at which the tension between Russell's conception of logic and the paradoxes is especially clear. This also comes out clearly if an attempt is made to define the notion of a propositional function. Suppose we wish to define the notion of a unary propositional function. The natural idea is to parallel the definition of binary relation:

D7 $(\phi)(\phi$ is a unary propositional function $.\equiv.$ $(x)(\phi x \supset \phi x))$.

D7 is, however, unsatisfactory for two reasons, both connected with the paradoxes. First, in the definiendum ϕ appears as a term. If propositional functions are terms, then, according to the logical theory of *Principles*, $\sim\phi(\phi)$ would be a propositional function, which could take itself as argument. Second, if we use D7 to define the notion of a unary propositional function, then how is the notion of a unary relation, i.e. a predicate or property, to be defined? If we proceed as suggested by D6, the definitions of unary propositional function and property will coincide. That is an unsatisfactory result: Russell permits self-predication of properties but not of propositional functions. So the course followed so far is not open in the case of propositional functions.[19]

Of course in *Principles* Russell is on the verge of recognizing the true extent of this problem. For it will not do to take the notion of a propositional function as *indefinable* either. If propositional functions are acceptable values of the primitive predicate "x is a propositional function", then, according to the logical theory of *Principles*, propositional functions are terms, and the paradoxes ensue. There is thus a clear pressure here to move toward a type theory in which differences of type cannot be expressed in the material mode.[20]

To summarize, we see that, with one exception, the indefinables listed are definable from implication and universal quantification alone. It should be noted that the relation of identity also occurs in the logic of *Principles*, but is not included in Russell's list of primitives. It seems then that Russell assumed, even at the time of *Principles*, that identity was definable. In Chapter II Russell offers a piecemeal definition of identity for the case of classes and the case of relations ([15], pp. 20, 24). But in his May 1903 letter to Frege we find the general and well-known definition of $x = y$ by $(\phi)(\phi x \supset \phi y)$.

5 I now turn to a critical evaluation of the project of using implication and universal quantification as sole primitives. There is, I think, a fatal flaw in this idea. It also seems clear from Russell's 1906 paper "The theory of implication" that he recognized this flaw and indeed had abandoned the project because of it. The difficulty occurs at a fundamental level and concerns the definitions of negation and existential quantification. The correctness of these definitions depends on assumptions about existence (or, in Russell's terminology, being). To state these assumptions explicitly requires use of existential quantification (or negation) as a primitive notion; if the attempt is made to state the relevant existence assumptions using only implication and universal quantification the result fails to express adequately these assumptions.

It isn't surprising that a basis using only universal quantifiers and implication has this failing. Russell himself emphasized the fact that the truth of a universally quantified conditional does not require that there be entities satisfying the antecedent or the consequent. Indeed, this feature is central to Russell's implicational treatment of geometries ([15], pp. 5, 429). This certainly suggests that universally quantified conditionals are not a satisfactory medium for the expression of existence assumptions.

Specifically, Russell's definitions D4, D5A, and D5B are correct only on the assumption that there are false propositions. This is evident in the case of D4. Unless r is a false proposition, "$(r \supset r) \supset (p \supset r)$" is true, regardless of the value of p. So, unless there are some false propositions in the domain of the quantifier, "$\sim p$" will be true when p is true. The definitions of existential quantification face the same difficulties. In D5A for example, "$\exists x\phi x$" is defined as "$(p)((p \supset p) \supset ((x)(\phi x \supset p) \supset p))$". Again, unless there are false propositions, the definition renders "$\exists x\phi x$" true regardless of what propositional function "ϕx" is; for instance, let "ϕx" be "x is a false proposition".

The difficulties with D4 and D5 do not derive entirely from their specific features; any definition that employs only implication and universal quantification is defective in much the same way. To see this in the case of D4, let A be an open sentence, formed from propositional variables, material implication, and universal quantification, in which the only free variable is p. This is the form a definition of "$\sim p$" will have in Russell's framework. Now, suppose that the only values of the bound variables in A are true propositions. Then A has the same truth value, relative to any assignment of values, as A^*, which is obtained from A by deleting universal quantifiers. A^* has the form $A_1 \supset (A_2 \supset \ldots (A_{n-1} \supset A_n) \ldots)$, where A_n is a single letter. A_n is either p or some other letter. If A_n is not the letter p, then, as in the previous paragraph, A^* has the value true, even when p is true. If A_n is the letter p, then A^* will once again be true when p is true, since a material implication with a true consequent is true.

In all these cases the definitions have unacceptable consequences when certain existence assumptions are not met; for example, D4 is unacceptable if there are no false propositions. This necessary background condition is symbolically rendered "$(\exists q)((q \supset q)\ \&\ \sim q)$"; so it might seem that the problem can be solved by stating the assumption explicitly as an additional axiom. But, of course, the axiom is not stated in primitive notation.

When expanded according to D5A it becomes: $(p)((p \supset p) \supset ((q)(((q \supset q)\ \&\ \sim q) \supset p) \supset p))$. This fails to express what we want, for this sentence would be true if there were *no* false propositions. Since the background assumptions for the correctness of the definitions of negation and crucially existential quantification essentially involve existence, and so, formally, existential quantification, the attempt to state these conditions explicitly within the limited original framework necessarily fails.

To the preceding argument it might be replied that far from showing the definition, D4, of negation to be defective, I have in fact shown its correctness. For example, the difficulties about D4 arise for cases where there are no false propositions among the values of the variables. But if there are no false propositions among the values, then "$\sim p$" should not be assigned the value false. Rather, since "$\sim p$" is to denote a proposition, whatever p is, "$\sim p$" should clearly be true in such a case. So it is certainly not an *objection* to the definition that in such a case "$\sim p$" is true when p is true.

Even if one finds this point of view persuasive, the problem with D4 and D5 survives. Suppose we defined conjunction in such a way that in some circumstances a conjunctive proposition was false when one of its conjuncts was false, but in other circumstances a conjunctive proposition could be true even when both components were false. Clearly, then, to know what was meant by and inferable from "p and q" we need to know which set of circumstances prevails. If we propose to infer "q" from "p and q", we need to assume that the current circumstances are of the kind covered by the first part of the definition. Similarly, if in our definition of negation there are cases where $\sim p$ is true when p is true and cases where $\sim p$ is false when p is true, it will be important to know which of the cases we suppose to be in force. Generally, of course, it will be the second. What characterizes the second case is the assumption that there are false propositions. But, as argued before, we can't state this assumption successfully within a framework which employs only implication and universal quantification.

Russell himself abandoned the primitive basis used in *Principles* for reasons very much like those suggested here. In his little known 1906 article "The theory of implication", Russell treats negation as primitive in addition to the conditional and the universal quantifier. At the end of the paper, Russell notes that negation might be regarded as defined and gives a variant of the definition found in *Principles*: namely, $p \supset (s)s$. He notes that introducing negation in this way is artificial and more difficult, but claims that this is outweighed by the reduction in primitive ideas. But then Russell rejects this plan on philosophical grounds. I quote his argument in full:

> My reason for not adopting this method is not its artificiality or its difficulty, but the fact that it never enables us to know that anything whatever is *false*. It enables us to prove the *truth* of whatever can be proved true by the method adopted here, and it does not enable us to prove the truth of anything which in fact is false. It even enables us to prove, concerning all propositions which can be proved false by the above method, that, if they are true, then everything is true; but if any man is so credulous as to believe that everything is true, then the method in question is powerless to refute him. For example, we get the law of contradiction in the form
>
> $$\vdash: p \cdot \sim p \;.\supset.\; (s) \cdot s;$$
>
> but this does not show that $p \cdot \sim p$ is false, unless $(s) \cdot s$ is false. Now in the system considered, *falsehood* is not among the ideas that occur in our apparatus: hence we cannot assume that $(s) \cdot s$ *is false* without introducing a new primitive idea. ([16], pp. 200–201)

In closing, I wish to make three comments on this passage in order to connect it with my own contentions. First, the initial sentences in the passage may be somewhat misleading concerning the exact location of the problem. They may suggest that the difficulties occur at the level of *proof*, but the fundamental issue concerns the *expressibility* of negative and existential propositions. Of course, since such propositions cannot even be adequately expressed within the *Principles* framework, *a fortiori* no such proposition can be proved.

Second, the end of the third sentence may be wrongly taken to minimize the significance of the difficulties. But, of course, Russell himself is taking the objection seriously, and I think he is right to do so. One of Russell's aims in *Principles* is to show, in opposition to Kant, that the reasonings in mathematics are no different from those of formal logic ([15], p. 457). To accomplish this aim successfully, it is important to make fully explicit the assumptions used in formal logic and in various branches of mathematics. Naturally, some of these assumptions may be evident and universally believed. Still, it is essential to the enterprise that they be explicitly stated.

Finally, Russell emphasizes the inexpressibility of falsehood (or negation) whereas I have emphasized the inexpressibility of *existence* assumptions; in Russell's example there is no important difference between the approaches. For to assume that $(s) \cdot s$ is false is just to assume that there is some term which is not a true proposition. In behalf of Russell's emphasis on negation, it should be noted that negation is still not satisfactorily expressible if existential quantification is taken as a primitive, along with implication and universal quantification.

Still, I think my emphasis on existence is beneficial in that it brings out the connection of this difficulty with a persistent difficulty faced by logicism. As noted earlier, Russell came to hold that certain existential statements, notably the Axiom of Infinity and the Axiom of Choice, were essential for the proof of certain mathematical truths but were not logical truths. So Russell, and indeed many other philosophers, hold that the strong logicist thesis that all mathematical truths are logical truths fails because there are existence assumptions necessary in mathematics which are not logical truths. Here the concern has been with the weaker thesis that all concepts necessary for mathematics are definable using only concepts of logic. But problems for the weaker thesis arise around the same concept: existence. On a well-motivated, Russellian account of the matter, universal quantification and implication emerge as the fundamental *logical* constants. However, these concepts do not suffice to define all concepts needed for the *expression* of mathematical claims unless certain existence assumptions are made. Since existence, in the form of the existential quantifier, is one of the concepts needed, the proposed basis of logical concepts is inadequate.

NOTES

1. Tarski also has noted the centrality of universal quantification and implication as logical constants (see [19], p. 418).

2. This is strikingly illustrated in Russell's contrasting attitude toward independence proofs in logic and Peano arithmetic with arithmetical primitives (see [15], pp. 15, 125).

3. This shows itself in Russell's claim that rules of inference point "to a certain failure of formalism in general" ([15], p. 34).

4. Manuscript evidence makes it clear that the notions of propositional function and formal implication emerge late in the development of *Principles*, perhaps as late as 1902. For instance, in Part V, which was written in November 1900, Russell speaks of propositions containing variables implying other propositions containing variables. Russell changed all this terminology at the proofreading stage so as to talk of propositional functions and formal implications. (See [15], pp. 263–264, and Russell Archives document number 230-030350 F-11, pp. 13–14.)

5. This is suggested at several places in *Principles*, for example, at Section 8. In Griffin [7] it is maintained that in *Principles* Russell was attracted by the idea that the *necessity* of logic could be used to explain the *necessity* of mathematics. But Griffin's view in fact comes quite close to the view I hold, since he argues that Russell's view of necessity applies this term primarily to propositional functions true for absolutely all values of the variable (see [7], pp. 118–121).

6. A similar motivation for treating material equivalence as the sole propositional primitive is found in Tarski's paper "On the primitive term of logistic", p. 2 in [19].
7. In *Principia Mathematica*, of course, this proposal is adopted. Its adoption is urged on the grounds of the "simplicity of primitive ideas and symmetry of treatment" (see [17], p. 6.)
8. I here treat the verb "implies" as a statement connective.
9. In defining singular terms, identity occurs: "$s =$ the α". No parallel argument for the indefinability of identity can be given because identity is a relational predicate. A definition of identity will involve, as auxiliary apparatus, the universal quantifier and implication, not identity itself.
10. I have ignored the fact that, strictly speaking, the definitional argument seems to argue for the indefinability of the biconditional, not the one-sided conditional.
11. The definition goes as follows:

> And logical constants are classes or relations whose extension either includes everything, or at least has as many terms as if it included everything. And a collection has as many terms as if it included everything, when there is a relation which every possible term, without exception, has to one and only one term of the given collection, provided that to every term of the given collection some term has the given relation.
>
> Russell Archives document 230.030350-F2, p. 1.

12. In addition to [15], pp. 45–46, Russell also presents the argument to Frege in his letters of June 24 and July 10, 1902 (see [5], pp. 134, 138; Frege's replies are found on pp. 135–136 and 141–142). A formal system which embodies Russell's ideas is found in [3].
13. For a discussion of Russell's 1903 conception of the propositions of logic, see [7], pp. 123–135.
14. On p. 523 of *Principles* Russell maintains that the objects of daily life are classes, not propositions. As an example, a person is identified with a class of psychical existents. This is distinctly less plausible than the view that a person is a whole, or unity, of related psychical existents.
15. As I suggest later, we might let "pq" denote $\{p, q\}$ if either p or q is a nonproposition.
16. It seems impossible to set up a definition of conjunction that will render "pq" always false when one component is nonpropositional, though I cannot yet construct an argument to show this.
17. This must be borne in mind when reading Russell's early post-Peano writings. To read '$\exists$' as an existential quantifier often leads to nonsense; for such a mistake, see the editor's footnote in [5], p. 151.
18. Commentators on *Principles*, including Russell himself later on, have cited this and related passages to foster the view that the semantic theory of *Principles* is a rather simplistic one in which each meaningful word in a sentence must be correlated with some entity. See Russell's "Introduction to the Second Edition", p. x. Two more recent examples where this criticism may be found are [18], p. 16, and [1], p.190. I think that the semantic theory found in Chapters 3 to 9 of *Principles* is, of necessity, subtler than Russell's general remarks allow.

19. The necessity of there being a clear difference in the treatment of predicates and relations, on the one hand, and propositional functions on the other, is brought out in [4], pp. 78–79.

20. In their correspondence, Frege responds to Russell's worries about the inexpressibility of type distinctions in precisely this way. He first says that "the words 'function' and 'concept' should properly speaking be rejected" and secondly that "if we want to express ourselves precisely, our only option is to talk about words or signs" ([5], p. 141).

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