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Notre Dame Journal of Formal Logic
Volume 31, Number 2, Spring 1990

Models for Inconsistent and Incomplete Differential Calculus

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Abstract In Section 2, a nilpotent ring is defined. In Section 3, nonclassical model theory is sketched and an incomplete model is defined. In Section 4, it is shown that the elements of equational differential calculus hold in this model, and a comparison with synthetic differential geometry is made. In Section 5, an inconsistent theory is defined with many, though not all, of the same properties.

1 Introduction This paper extends the nonclassical model theory for inconsistent first-order equational theories developed in [4], [6], and [7], to the case of inconsistent equational theories strong enough for a reasonable notion of differentiation. The aim is to show that inconsistency does not cripple such an equational differential calculus. There have been a number of calls recently for inconsistent calculus, some appealing to the history of the calculus in which inconsistent claims abound (see, e.g., [9]). However, inconsistent calculus has resisted development, for at least two reasons. First, the functional structure of fields interacts with inconsistency to produce triviality in even the purely equational part of first-order theories with terms of finite length (as pointed out in [6], [7], and [9]), in a way which standard contradiction-containment devices, such as weakening *ex contradictione quodlibet*, do not prevent. Stronger theories, those including set membership, terms of infinite length, order, limits, and integration are then infected with the same triviality. Second, the functional structure of inconsistent set theory remains difficult to control, and seems to require sacrifice of logical principles in addition to, and more natural than, *ex contradictione quodlibet* (see, e.g., [2], [5], or [8], pp. 178–180). But unless there

Received August 14, 1987; revised May 26, 1988

are distinctive inconsistent theories of the order of strength of classical analysis, the claim that the history of the calculus supports paraconsistency is seriously undermined; and furthermore, claims that inconsistent mathematics can generate substantial new insights are correspondingly weakened.

This paper addresses the former of these two points, by adapting the methods of [6] and [7], in presenting an inconsistent but nontrivial equational theory of polynomial differentiation. So far, inconsistent methods seem to be useful mostly in a fairly restricted application, namely treating congruences as literal identities (inconsistently), in accordance with informal mathematical terminological practice. In Section 2, a congruence on a subset of the classical hyperreals gives the functional structure of the domain of the models and is independently interesting in connection with the classical theory of nilpotent rings. Section 3 sketches nonclassical model theory. It turns out that the structure developed in Section 2 is usefully studied in the first instance by modifying inconsistent model theory using a finite-valued intuitionist logic, since the resulting first-order "intuitionist" equational differential calculus has some similarities with a corresponding fragment of synthetic differential geometry (see [1] or [3]), particularly in respect of incompleteness, nilpotence, Taylor formulas, differentiation, and continuity. This is done in Section 4, and the advantages (in particular, that of simplicity) and limitations of the comparison are outlined. In Section 5, the same results are then obtained for inconsistent polynomial theory. The limitations of the present approach and some further developments are outlined in the final section. It is argued on the basis of these results that the fact that the same functional structure underlies all of the incomplete, inconsistent, and classical consistent theories suggests that the functional aspects of mathematics are more important than squabbles at the sentential level over *ex contradictione quodlibet*, inconsistency, incompleteness, etc.

2 A nilpotent ring We begin with the usual arithmetic of the field of hyperreal numbers R^*, with operations $+$, $-$, $\times$, $/$. The subfield of real numbers is denoted by R. For each nonzero x in R^*, the binary relation $\underset{x}{\sim}$ is defined by: $x_1 \underset{x}{\sim} x_2 =_{\text{def}} (x_1/x)$ is at most infinitesimally different from (x_2/x), (i.e., $x_1/x \approx x_2/x$, i.e., $(x_1 - x_2)/x$ is infinitesimal, i.e., $(x_1 - x_2)/x \approx 0$; see the relation $\approx \delta$ in [10]). For fixed x this is an equivalence relation on R^*, as is easy to verify. It is not however a congruence. For example, if $x_1 \underset{x}{\sim} x_2$, then if $(x_1 - x_2)/x$ is infinite with respect to x_3, then $(x_1/x_3) \underset{x}{\sim} (x_2/x_3)$ does not in general hold. However, if x is set equal to an arbitrarily chosen infinitesimal δ, then a congruence with respect to $+$, $-$, $\times$, and an associated ring of equivalence classes is obtained. Let S be the set of noninfinite hyperreals, i.e., of the form $x + d$ where x is any real number and d is any infinitesimal, with the additional property that for some positive integer k, $d^k \underset{\delta}{\sim} 0$. Then

Proposition 1 *The relation $\underset{\delta}{\sim}$ on S is a congruence with respect to the operations $+$, $-$, $\times$.*

Proof: If $(x_1 + d_1) \underset{\delta}{\sim} (x_2 + d_2)$, i.e. $((x_1 - x_2) + (d_1 - d_2))/\delta \approx 0$, then $(((x_1 + x_3) - (x_2 + x_3)) + ((d_1 + d_3) - (d_2 + d_3)))/\delta \approx 0$, i.e. $((x_1 + d_1) + (x_3 + d_3)) \underset{\delta}{\sim} ((x_2 + d_2) + (x_3 + d_3))$. Also, let $d_1^{k_1}/\delta \approx 0$ and $d_3^{k_3}/\delta \approx 0$. Now

$(d_1 + d_3)^{k_1+k_3}/\delta = \left(\sum_{i=0}^{k_1+k_3} \binom{k_1 + k_3}{i} d_1^{k_1+k_3-i} d_3^i\right)\Big/\delta$. But each term ≈ 0, so the whole sum is. Hence $(x_1 + d_1) + (x_3 + d_3)$ is in S, and so $(x_2 + d_2) + (x_3 + d_3)$ also obviously is. The subtraction case is the same. For multiplication, if $((x_1 - x_2) + (d_1 - d_2))/\delta \approx 0$, then if x_3 is real, then $(((x_1 - x_2) + (d_1 - d_2)) \times (x_3 + d_3))/\delta \approx 0$ also. Hence $(x_1 + d_1) \times (x_3 + d_3) \underset{\delta}{\sim} (x_2 + d_2) \times (x_3 + d_3)$. Now also, $(x_1 + d_1) \times (x_3 + d_3) = (x_1 \times x_3) + (x_1 \times d_3) + (x_3 \times d_1) + (d_1 \times d_3)$. Clearly, however, $0 \approx (x_1 \times d_3)^{k_3}/\delta \approx (x_3 \times d_1)^{k_1}/\delta \approx (d_1 \times d_3)^{\min(k_1,k_3)}/\delta$. Hence $x_1 \times d_3$, $x_3 \times d_1$, $d_1 \times d_3$, and $x_1 \times x_3$ are all in S, so their sum is also in S, as in the proof of the addition case. That is, $(x_1 + d_1) \times (x_3 + d_3)$ is in S; and also obviously $(x_2 + d_2) \times (x_3 + d_3)$ is.

Note that the proof of congruence breaks down for the case of division because $(((x_1 - x_2) + (d_1 - d_2))/(x_3 + d_3))/\delta$ might not be infinitesimal, e.g., if $x_3 = 0$ and d_3 is infinitesimal with respect to $((x_1 - x_2) + (d_1 - d_2))/\delta$. It follows from Proposition 1 that the set of equivalence classes of members of S is a ring (call it $\mathbb{R}$) with respect to the induced operations $+$, $-$, $\times$. Denote the equivalence class of any element $x + d$ by $|x + d|$. $\mathbb{R}$ has the following properties:

Proposition 2
(1) *For any real numbers* x_1,x_2, $|x_1| = |x_2|$ *iff* $x_1 = x_2$
(2) *For any infinitesimals* d_1,d_2, *if* $|d_1^2| = |d_2^2| = |0|$, *then* $|d_1| \times |d_2| = 0$
(3) *For any nonnegative integer k, there is some infinitesimal d with* $|d^{k+1}| = |0|$ *while* $|d^k| \neq |0|$.

Proof: (1) If x_1,x_2 are real, then not $(x_1 - x_2)/\delta \approx 0$ unless $x_1 = x_2$. (2) Let $d_1' = d_1^2/\delta$ and $d_2' = d_2^2/\delta$. By hypothesis, $d_1' \approx 0 \approx d_2'$. But $d_1 d_2/\delta = (d_1^2 d_2^2/\delta^2)^{1/2} = (d_1')^{1/2}(d_2')^{1/2}$, which is infinitesimal if d_1' and d_2' are. (3) Consider $\delta^2, \delta, \delta^{1/2}, \delta^{1/3}, \ldots$, etc.

The following lemma is useful for Propositions 3, 4, and 6.

Lemma *For any infinitesimal* δ *and any positive integer k, there is an infinitesimal d such that* d^{k+1}/δ *is infinitesimal while* d^k/δ *is infinite.*

Proof: Let $d = \delta^{(k+1)/k(k+2)}$. Now $d^{k+1}/\delta = \delta^{(k+1)^2/k(k+2)}/\delta = \delta^{(k^2+2k+1)/k(k+2)}/\delta^{(k^2+2k)/(k^2+2k)} = \delta^{1/(k^2+2k)} \approx 0$. But $d^k/\delta = \delta^{k(k+1)/k(k+2)}/\delta = \delta^{(k+1)/(k+2)}/\delta^{(k+2)/(k+2)} = \delta^{-1/(k+2)} = 1/\delta^{1/(k+2)}$, which is infinite.

Define D_0 to be $|0|$, and for all positive integers k, let $D_k = \{|d| : |d^{k+1}| = |0|$ while $|d^k| \neq 0\}$. Let D be $\bigcup_{\forall k>0} D_k$. Then

Proposition 3 *For all positive integers k*
(1) *There is a* $|d|$ *in* D_k *such that for all* $|d_1|$ *in D,* $|d_1| \times |d^k| = |0|$
(2) *There is a* $|d|$ *in* D_k *and a* $|d_1|$ *in* D_{k+2} *such that* $|d_1| \times |d^k| \neq |0|$.

Proof: (1) Let d be $\delta^{1/k}$. Now $d^k/\delta = 1$, not ≈ 0. But $d^{k+1}/\delta = 1.\delta^{1/k} \approx 0$. Hence $|d|$ is in D_k. Moreover, for any infinitesimal d_1, $d_1 d^k/\delta = d_1 \approx 0$, so $|d_1| \times |d^k| = |0|$. (2) Let d be $\delta^{(k+1)/k(k+2)}$ as in the lemma, and let d_1 be

δ/d^k. Now by the argument of the lemma, $|d|$ is in D_k. Furthermore, $d_1 = \delta/d^k = \delta/\delta^{(k+1)/(k+2)} = \delta^{1/(k+2)}$. So $d_1^{k+2}/\delta = \delta^{(k+2)/(k+2)}/\delta^{(k+2)/(k+2)} = 1$, not ≈ 0; and $d_1^{k+3}/\delta = \delta^{(k+3)/(k+2)}/\delta^{(k+2)/(k+2)} = \delta^{1/(k+2)} \approx 0$; hence d_1 is in D_{k+2}. Finally, $(d_1 d^k)/\delta = 1$, not ≈ 0, so $|d_1| \times |d^k| \neq |0|$.

Proposition 2(1) shows that $\mathbb{R}$ has a subfield isomorphic to the real numbers R; this field of equivalence classes will also be referred to as R where no confusion results. Now as usual we can write $|x|^k$ for $|x^k|$ and drop the $|\ |$ and multiplication signs in $\mathbb{R}$ where no confusion results. An element d of an algebra is *strictly nilpotent of degree k* if $d^{k+1} = 0$ while $d^k \neq 0$, and an algebra is *strictly nilpotent of degree k* if it has strictly nilpotent elements of degree k. Proposition 2(3) shows that $\mathbb{R}$ is strictly nilpotent of all positive integral degrees. Proposition 2(2) is relevant to the comparison with synthetic differential geometry in Sections 4 and 5. While all elements of D_k go to zero on being raised to the $k+1$st power and not for any lesser integral power, Proposition 3 shows that these elements fall into two nonnull classes: those whose kth power when multiplied by any nilpotent element goes to zero, and those whose kth power has a nonzero product with some nilpotent element. This is also relevant to the results of Section 4.

3 Summary of nonclassical model theory and the construction of an incomplete model This section sketches basic nonclassical model theory as developed in [4], [6], and [7], and applies it to the construction of an incomplete theory using a three-valued intuitionist logic.

A *logic* will be said to be a complete lattice L together with a filter $\nabla \subset L$, called the *designated elements* of L. The *language* $\mathcal{L}$ considered here consists of a set of *simple terms* (names) in 1–1 correspondence with the noninfinite hyperreal numbers (think of them as naming themselves). *Complex terms* are produced by closure with respect to the operations $+$, $-$, $\times$. A *term* is either a simple term or a complex term. The metalinguistic variables $t, t_0, t_1, \ldots$ range over terms. If t_1, t_2 are terms, then an *atomic sentence* is of the form $t_1 = t_2$. The language has two sorts of object language *variables*, each with several sorts of associated *quantifiers*: variables $x, x_0, x_1, \ldots$ with the four associated quantifiers $(\forall \in \mathbb{R}), (\exists \in \mathbb{R}), (\forall \in R), (\exists \in R)$; and variables $d, d_0, d_1, \ldots$ with the associated quantifiers $(\forall \in D), (\exists \in D)$, and, for each positive integer k, $(\forall \in D_k)$ and $(\exists \in D_k)$. The metalinguistic variables $v, v_0, v_1, \ldots$ range over variables of any sort.

The language also has the sentential operators $\neg$, &, $\vee$, $\rightarrow$, and $A \supset B$ is defined as $\neg A \vee B$ and $A \equiv B$ as $(A \supset B)$ & $(B \supset A)$. $(\mathrm{E}!x \in R)(Fx)$ is defined as $(\exists x \in R)(Fx \;\&\; (\forall x_0 \in R)(Fx_0 \rightarrow x = x_0))$. Wffs and sentences are defined in the usual way.

In this section the logic L is the three element chain (Hasse Diagram):

*	$\circ$	T	True
	$\circ$	N	Neither
	$\circ$	F	False

with set of designated elements $\nabla = \{T\}$. (Designated elements are starred on the Hasse Diagram.) An *assignment* is a function I: (closed sentences of $\mathfrak{L}$) $\to$ $\{T,N,F\}$ satisfying:

(1) For any atomic sentence $t_1 = t_2$, $I(t_1 = t_2) \in \{T,N,F\}$

(2) $I(A \;\&\; B) = \text{glb}(I(A),I(B))$, $I(A \vee B) = \text{lub}(I(A),I(B))$, while $I(A \to B)$ and $I(\neg A)$ are given by the table:

	$\to$	T	N	F	$\neg$
*	T	T	F	F	F
	N	T	T	F	F
	F	T	T	T	T

(note that L is intuitionist since its $(\&,\vee,\neg)$-structure is exactly that of the well-known three-element interior algebra)

(3) For every quantified sentence of the form $(\forall v \in X)Fv$, $I((\forall v \in X)Fv) =$ glb $\{y$: for some term t, $I(t)$ is in X and $I(F(t/v)) = y\}$; and for every sentence of the form $(\exists v \in X)Fv$, $I((\exists v \in X)Fv) =$ lub $\{y$: for some term t, etc.$\}$, where v is any variable and X is $\mathbb{R}, R, D$ or D_k (subject to proper matching of types).

A sentence A *holds in an assignment* I, written $\vdash_I A$, iff $I(A) \in \nabla$. A set of sentences holding in an assignment is a *theory*. If $A \in Th$ where Th is a theory, we write $\vdash_{Th} A$, dropping the subscript when Th is clear. I and Th are *consistent* if for no A is it the case that both $\vdash A$ and $\vdash \neg A$, and *complete* if for all A either $\vdash A$ or $\vdash \neg A$. I is an *assignment with functionality* iff for all terms t_1, t_2, if $t_1 = t_2$ holds, then for all atomic sentences Ft_1 containing t_1, Ft_1 holds iff Ft_2 holds, where Ft_2 is like Ft_1 except that t_2 replaces t_1 in one or more places. I is an *assignment with identity* iff for all terms t_1, t_2, if $t_1 = t_2$ holds, then for all closed sentences Ft_1 containing t_1, Ft_1 holds iff Ft_2 holds.

A *model* is a pair $\langle \mathfrak{D}, I \rangle$ where $\mathfrak{D}$ is a domain and I an assignment having the additional properties that: (1) I assigns to every simple term a member of $\mathfrak{D}$; (2) I assigns to every n-ary functional expression an n-ary partial function on $\mathfrak{D}$; (3) The assignment to complex terms is given by $I(f(t_1, \ldots, t_n)) = I(f)(I(t_1), \ldots, I(t_n))$, provided that this is defined; (4) I is required to be onto $\mathfrak{D}$, so that every element of the domain is assigned to some term; (5) I satisfies: $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$. If $\langle \mathfrak{D}, I \rangle$ is a model and I an assignment with functionality (identity), then $\langle \mathfrak{D}, I \rangle$ is a *model with functionality (identity).*

In this section, we construct a theory by specifying further features of the assignment function I. The domain $\mathfrak{D}$ is taken to be the nilpotent ring $\mathbb{R}$ of equivalence classes of the previous section. The specifications are as follows: (1) For every name t, $I(t) = |t|$; (2) $I(+)$, $I(-)$, $I(\times)$ are the ring operations on $\mathbb{R}$ induced by the congruence $|\ |$; this determines the interpretation of all complex terms; (3) Set $I(t_1 = t_2) = \text{T}$ iff $I(t_1) = I(t_2)$, set $I(t_1 = t_2) = \text{N}$ if $I(t_1) \neq I(t_2)$ but the hyperreal number $(t_1 - t_2)/\delta$ is noninfinite, and set $I(t_1 = t_2) =$ F otherwise. The values of all nonatomic sentences are then determined as above.

We observe that the model just described is a model with identity. (This follows from the facts: (1) That $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$, and hence (2) That if $t_1 = t_2$ holds, then $I(F(t_1)) = I(F(t_2))$ for any atomic F, if these are defined. The latter is then the basis clause of an obvious induction on the lengths of sentences.) Being a model with identity suffices for being a model with functionality, and the latter permits calculations by substitution of identicals, whether or not the background logic is classical. In Section 5, substitutivity of identity is weakened, but in a controlled way. The model and its associated theory are intuitionist in another sense, namely that they are incomplete: Since $I(\delta = 0) =$ N, $I(\neg\delta = 0) =$ F, so that neither $\vdash\delta = 0$ nor $\vdash\neg\delta = 0$, although $\vdash\neg\neg\delta = 0$, $\vdash\delta^2 = 0$, and $\vdash\neg\delta^{1/2} = 0$.

Note finally that a wholly classical (two-valued) theory of $\mathbb{R}$ can be obtained by a different I: by setting $I(t_1 = t_2) =$ T iff $I(t_1) = I(t_2)$, and F otherwise. This also shows that classical two-valued model theory can be recovered as a special case.

4 Incomplete differential calculus, and comparison with synthetic differential geometry In this section it is shown that Taylor's formula and the polynomial differentiation laws hold in the model. A definition of limits can be given, and it is proved that every function is continuous. It is shown that the theory has significant similarities with a corresponding part of synthetic differential geometry, and the dissimilarities are outlined.

A *functional expression* (abbreviated to *function*) is the result of replacing any term or terms in an atomic wff by variables. A function with no remaining terms denoting infinitesimals is called a *real* function. If f is a function with a single free variable v of any sort, then we indicate this by $f(v)$. The result of replacing v throughout by a term t is denoted by $f(t)$. *If* v_1 and v_2 are variables of any sort, then $f(v_1 + v_2)$ is the result of replacing v by $v_1 + v_2$ throughout. Similarly for $-$ and $\times$. (E!$x_1, \ldots, x_k \in R$) is defined as (E!$x_1 \in R$)$\ldots$ ($E\rangle x_k \in R$). Then we have:

Proposition 4 *If $f(x)$ is any real function, then for every positive integer k,*

$$\vdash(\forall x \in R)(\mathrm{E}!x_1, \ldots, x_k \in R)(\forall d \in D_k)(f(x + d) = f(x) + x_1 d + \ldots + x_k d^k).$$

Proof: If $f(x)$ is any real function, then by the polynomial laws of R^*, for any term t, $I(f(t))$ is identical with $I(t_0 + t_1 t + \ldots + t_n t^n)$, where the t_i are simple terms denoting real numbers, since identities are not destroyed in passing from R^* to $\mathbb{R}$. So we may restrict attention to functions of the form $t_0 + t_1 x + \ldots + t_n x^n$, where the t_i denote real numbers, i.e., where the $I(t_i)$ are in R. We abbreviate these functions by $\sum_{i=0}^{n} t_i x^i$. Then for any such $f(x)$ and any term t from R and any term d with $I(d)$ in some D_k, $f(t + d)$ is $t_0 + t_1(t + d) + \ldots + t_n(t + d)^n$. So $I(f(t + d)) = I(t_0) + (I(t_1) \times (I(t) + I(d))) + \ldots$ etc., where $+$ and $\times$ are the induced operations on $\mathbb{R}$. These operations obey the R^* polynomial laws, so we can compute this sum using the binomial expansion. If $n \leq k$, the nilpotence of the element d does not affect this expansion, and (α) below follows by normal arithmetic. If $k < n$, those terms in the binomial expansion of $I(f(t + d))$ which contain $|d^{k+1}|$ as a factor are identical with $|0|$. So $I(f(t + d))$ computes to

$$I(t_0 + t_1 t + t_2 t^2 + \ldots + t_n t^n) + I\left(\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right) d\right) \ldots$$
$$+ I\left(\left(\sum_{i=1}^{n} \binom{i}{k} t_i t^{i-k}\right) d^k\right). \qquad (\alpha)$$

Hence, by the assignment conditions for quantifiers

$$\vdash (\forall x \in R)(\exists x_1, \ldots, x_k \in R)(\forall d \in D_k)\left(f(x + d) = f(x) + \sum_{i=1}^{k} x_i d^i\right).$$

The next part of the argument (proving uniqueness) uses the postulate that the $I(t_i)$ are real. We need to conjoin to (α) the following: $(\forall x_{k+1}, \ldots, x_{2k} \in R)\left((\forall d \in D_k)\left(f(t + d) = f(t) + \sum_{i=1}^{k} x_{k+i} d^i\right) \to ((t_1 = x_{k+1}) \,\&\ldots\&\, (t_k = x_{2k}))\right)$, where the t_i are a relabelling of the coefficients of (α). Eliminating quantifiers to appropriately assigned terms, we need to prove that:

$$\vdash (\forall d \in D_k)\left(f(t + d) = f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right) \to \&_{i=1}^{k} (t_i = t_{k+i}). \qquad (\beta)$$

If the consequent takes the value T, then (β) holds by the tables for $\to$. If the consequent does not take the value T, then there are two cases: either (i) $t_k = t_{2k}$ does not hold or (ii) some other $t_i = t_{k+i}$ does not hold. If $t_k = t_{2k}$ does not hold, then $I(t_k) \neq I(t_{2k})$. Now, since t_k and t_{2k} are real, $(t_k - t_{2k})/\delta$ is infinite, so $I(t_k = t_{2k})$ is F. But by the lemma of Section 2, there is some infinitesimal hyperreal number d such that d^k/δ is infinite, hence $(t_k - t_{2k})d^k/\delta$ is infinite. If every other $t_i = t_{k+i}$ holds, then $|t_i| = |t_{k+i}|$ and $t_i = t_{k+1}$ in R^*. So in R^*, $f(t + d) - \left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right) = f(t) + \sum_{i=1}^{k} t_i d^i - \left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right) = (t_k - t_{2k})d^k$. But the latter is infinite with respect to δ. So in $\mathbb{R}$, $I(f(t + d)) \neq I\left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right)$. But also in R^*, $\left(f(t + d) - \left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right)\right)\Big/\delta$ is infinite. Hence the antecedent of (β) is F, and (β) holds by the table for $\to$.

Otherwise, if some other $t_i = t_{k+1}$ does not hold, let i be the least integer for which $t_i = t_{k+i}$ does not hold. Then choosing the same d, in R^* $f(t + d) - \left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right) = (t_i - t_{k+i})d^i$ + higher powers of d. But the first term is infinite with respect to δ if d^k is. So, as for (i), in $\mathbb{R}$ $I(f(t + d)) \neq I\left(f(t) + \sum_{i=1}^{k} t_{k+i} d^i\right)$. But in R^* $\left(f(t + d) - \left(f(t) + \left(\sum_{i=1}^{k} t_{k+i} d^i\right)\right)\right)\Big/\delta$ is infinite. Hence again the antecedent of (β) is F and so (β) holds.

Consider the case where $k = 1$. Then for any $|d|$ in D_1 and any real t, $\vdash f(t + d) = f(t) + t_1 d$ for some term t_1 with $I(t_1)$ in R. A functional expression $g(x)$ is called *a derivative of* $f(x)$, if for any d in D_1 and any t with $I(t)$

in R, $\vdash f(t + d) = f(t) + g(t)d$. We know independently from real number calculus that there is always at least one derivative for any real function $f(x)$. If $g(x)$ is a derivative of $f(x)$, we can also denote it by $f'(x)$. Thus for any derivative $f'(x)$, we have the *Taylor formula* $\vdash f(t + d) = f(t) + d \cdot f'(t)$, or $\vdash (\forall x \in R)(\forall d \in D_1)(f(x + d) = f(x) + d \cdot f'(x))$. Define an *n-th degree polynomial in the indeterminate* x to be any functional expression of the form $t_0 + t_1 x + \ldots + t_n x^n$, where the t_i are simple terms, that is $\sum_{i=0}^{n} t_i x^i$.

Proposition 5 (Polynomial Differentiation) *If f is any polynomial of the form $\sum_{i=0}^{n} t_i x^i$ with real coefficients t_i, and $f'(x)$ is a derivative of f, then* $\vdash (\forall x \in R)\left(f'(x) = \sum_{i=1}^{n} i t_i x^{i-1}\right)$.

Proof: From the Taylor formula, $\vdash (\forall x \in R)(f(x + d) = f(x) + d \cdot f'(x))$, where $I(d)$ is in D_1. Whence $I(f(t + d)) = I(f(t)) + (I(d) \times I(f'(t)))$ for any term t with $I(t)$ in R. But $I(f(t + d)) = I\left(\sum_{i=0}^{n} t_i (t + d)^i\right)$. As in Proposition 4, this computes to $I\left(\sum_{i=0}^{n} t_i t^i\right) + \left(I\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right) \cdot I(d)\right) + \left(I\left(\sum_{i=2}^{n} \binom{i}{2} t_i t^{i-2}\right) \cdot I(d^2)\right) + \ldots$ higher powers of d. Since $I(d^2) = I(d^3) = \ldots = 0$, all products of $d^2, d^3, \ldots$ may be dropped. Thus we have $I(f(t + d)) = I(f(t)) + (I(d) I(f'(t)))$ and also $= I(f(t)) + I(d) I\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right)$. So, since minus is a congruence, $I(d) I(f'(t)) = I(d) I\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right)$. But since $I(d)$ is in D_1, and $I(t), I(t_i)$ are in R, this can happen only if $I(f'(t)) = I\left(\sum_{i=1}^{n} \binom{i}{1} t_i t^{i-1}\right)$. But t was arbitrarily chosen. Hence $\vdash (\forall x \in R)\left(f'(x) = \left(\sum_{i=1}^{n} i t_i x^{i-1}\right)\right)$, as required.

A definition of two-sided limits can be given. Define '$\lim_{x \to t} f(x) = t_1$' to mean '$(\forall d \in D)(f(t + d) = t_1 \vee (\exists d_1 \in D)(f(t + d) - t_1 = d_1))$'. One-sided limits can also be defined by introducing the notions of positivity and negativity for members of D, but that is not done here because of the following proposition. It is also noted that in the above definition of limit the case where not $\vdash f(t) = t_1$ does not arise, as the following proposition shows. For any real function $f(x)$, define in the usual way 'f is continuous at t' to mean '$\lim_{x \to t} f(x) = f(t)$', and '$f$ is continuous' to mean '$(\forall x \in R)$(f is continuous at x)'. Then

Proposition 6 *For every real function $f(x)$, $\vdash f$ is continuous.*

Proof: It has to be proved, for every real term t, that $\vdash (\forall d \in D)(f(t + d) = f(t) \vee (\exists d_1 \in D)(f(t + d) - f(t) = d_1))$. But it follows from Proposition 4 that $\vdash (\forall d \in D_k)(f(t + d) = f(t) + t_1 d + \ldots + t_k d^k)$. If not all the real $t_i = 0$, then $\vdash f(t + d) - f(t) = t_1 d + \ldots + t_k d^k$. It is obvious that raising the right-hand side to power k is not (considered as a hyperreal number) infinitesimal with respect to δ (since its first term is not), while raising it to power $k + 1$ is infinitesimal with respect to δ (since each term is), so that the right-hand side is in D_k. Thus $\vdash (\exists d_1 \in D)(f(t + d) - f(t) = d_1)$. The result follows by disjoining the alternatives and universal generalization.

Synthetic differential geometry (SDG), as expounded in [3] (see also [1]), is likewise an incomplete theory (with neither $\delta = 0$ nor $\neg\delta = 0$ holding). The theory of [3] has elements which are strictly nilpotent of all degrees, while that of [1] restricts consideration to D_1. Neither proceeds from a construction on the classical hyperreals, however, nor utilizes a three-valued model theory. In these theories, also, every function is continuous. The method of obtaining derivatives from the Taylor formula as in Proposition 5 is similar to that of [3], and is a variant of the usual classical treatment. Like SDG, Propositions 4 and 5 utilize the calculatory advantages of nilpotent elements, since these ensure that higher-order differentials can ultimately be ignored.

The case where $x = 0$, $k = 1$ of Proposition 4 is Axiom′ 1 of [3], with the proviso that R in Proposition 4 be replaced by the whole domain there; the case where $x = 0$ is Axiom 1′ of [3] with the same proviso. If, however, R were replaced by $\mathbb{R}$ here, then Proposition 4 would fail, as follows. Choose any d_1 in D_1 and let f be the function $f(x) =_{df} d_1 x$. Then certainly $\vdash (\exists x \in R)(\forall d \in D_1)(f(d) = f(0 + xd))$, the x in question being d_1. However, this x is not unique: for any other d_2 in D we have $\vdash (\forall d \in D_1)(dd_2 = dd_1 = 0)$ while not $\vdash d_1 = d_2$, so that the antecedent of $(\forall d \in D_1)(f(d) = f(0) + d_2 d) \rightarrow d_1 = d_2$ holds while the consequent does not hold. Indeed, f could even have a noninfinitesimal coefficient, $f(x) = (5 + \delta)x$, say; for then the coefficient fails to be unique, since $\vdash (\forall d \in D_1)((5 + \delta)d = 0 = (5 + 2\delta)d)$ while not $\vdash 5 + \delta = 5 + 2\delta$. Thus the present theory is a theory of functions with real slopes as in nonstandard analysis, and so is less general than SDG.

The essential difference with the nilpotent elements in SDG is that the D_1 part of the domain is postulated in SDG to contain elements d_1, d_2 such that not $\vdash d_1 d_2 = 0$, while in the present model this is not so (Proposition 2(2)). Correspondingly the SDG cancellation principle, $(\forall d \in D_1)(dt_1 = dt_2) \rightarrow t_1 = t_2$, fails: for example, when $I(t_1) = |\delta|$ and $I(t_2) = |2\delta|$ the antecedent is T and the consequent N. However, the cancellation principle holds for cases where the difference between $I(t_1)$ and $I(t_2)$ is infinite with respect to δ if they are different at all, as for example $\vdash (\forall x_1 x_2 \in R)((\forall d \in D_1)(dx_1 = dx_2) \rightarrow x_1 = x_2)$.

The failure of the law of excluded middle (LEM) is of interest. The account of [3] links it to the holding of the cancellation principle and the continuity of every function. However we can see that the failure of LEM in the present paper is rather independent of the functional part of the construction, since the latter can also produce a wholly classical model (end of Section 3). The same point pertains to the inconsistent theory of the next section. This does not show that the 'correct' description is that of classical two-valued logic, however; to the contrary it suggests that functionality is mathematically prior to sentential logic.

SDG in [3] employs the mathematical machinery of Cartesian closed categories, which is much stronger than that of the present paper, which aims rather at studying equational theories. On the other hand, there is here some simplicity in the presentation of the ideas of incompleteness, nilpotence, differentiability, limits, continuity, etc., within the model-theoretic framework, albeit a nonclassical one. Furthermore, the present approach permits the investigation of similar theories with different background logics (see Sections 5 and 6). Another point is that while [3] maintains that SDG is an *essentially geometric* treatment of analysis, one might argue that it is interesting how much one can get of SDG with resources merely from model theory and algebraic number theory.

5 Inconsistent differential calculus The background logic is now altered to the well-known logic RM3.

		$\to$	T	B	F	$\neg$
*	True	*T	T	F	F	F
*	Both	*B	T	B	F	B
	False	F	T	T	T	T

The set of designated elements $\nabla = \{\mathrm{T},\mathrm{B}\}$. There are a number of options for the assignment function for the values of atomic sentences. The one used here illustrates the possibility of controlling substitutivity of identity even though a full model with identity is absent (another option is mentioned in the final section): (1) Set $I(t_1 = t_2) = \mathrm{T}$ if $t_1 = t_2$; that is, if t_1, t_2 are considered hyperreal numbers; (2) $I(t_1 = t_2) = \mathrm{B}$ if $t_1 \neq t_2$ but $I(t_1) = I(t_2)$ in $\mathbb{R}$; and (3) $I(t_1 = t_2) = \mathrm{F}$ if $I(t_1) \neq I(t_2)$.

Note that in consequence of (1) and (2), $I(\delta^2 = 0) \neq \mathrm{T}$, but rather $I(\delta^2 = 0) = \mathrm{B}$ and so $I(\neg\delta^2 = 0) = \mathrm{B}$. The theory is thus inconsistent. Again, $\vdash (5 + \delta^2 = 5)\ \&\ \neg(5 + \delta^2 = 5)$. In consequence of (3), $I(\delta = 0) = \mathrm{F}$ and $I(\neg\delta = 0) = \mathrm{T}$, unlike SDG. Indeed, all theories which are constructed by assigning to the set of atomic equations values from the above logic in the above fashion are complete.

The present model is a model with functionality but not with identity. (Proof of functionality: By inspection $t_1 = t_2$ holds iff $I(t_1) = I(t_2)$. But I is a congruence; so for any term $t(t_1)$ containing t_1, $I(t(t_1)) = I(t(t_2))$. Hence if $t_3(t_1) = t_4(t_1)$ holds, then $I(t_3(t_1)) = I(t_4(t_1))$; so $I(t_3(t_2)) = I(t_4(t_2))$, hence $t_3(t_2) = t_4(t_2)$ holds. Proof of nonidentity: $\vdash\delta^2 = 0$, but while $\vdash\neg\delta^2 = 0$, not $\vdash\neg 0 = 0$ nor $\vdash\neg\delta^2 = \delta^2$.) This means that, on the one hand, calculations may be carried out utilizing the advantages of $\vdash\delta^2 = 0$, as in earlier sections; while on the other hand, one does not have to submit to $\vdash\neg t = t$ for any term t, an improvement on [4], [6], and [7].

It can be asked how much is lost from a theory if full substitutivity of identity in all contents is relaxed. This leads to a comparison first with the full $+$, $-$, $\times$-theory of the noninfinite hyperreals, and then with the theory of the previous section. It is shown (i) that every sentence holding in the $+$, $-$, $\times$, $\&$, $\vee$, $\neg$, $\forall$, $\exists$-theory of the noninfinite hyperreals holds in the present model, and (ii) that Propositions 4 to 6 may also be reproved utilizing the same calculatory advantages of nilpotent elements.

(i) is an immediate consequence of the extendability lemma (Proposition 1 of [7]), since the sets of sentences of the forms $t_1 = t_2$ and $\neg t_1 = t_2$ holding for the noninfinite hyperreals are respectively subsets of those holding in the present model.

As for (ii) we have:

Proposition 7 *If $f(x)$ is any real function, then for every positive integer k,*

$$\vdash (\forall x \in R)(\mathrm{E}!x_1, \ldots, x_k)(\forall d \in D_k)(f(x+d) = f(x) + x_1 d + \ldots + x_k d^k).$$

Proof: The proof that $I(f(t+d))$ computes to (α) as in Proposition 4 is identical. To prove uniqueness, we need to prove (β). If the consequent of (β) is T, then (β) holds. And, for real coefficients t_i, t_{k+i}, one never has $I(t_i = t_{k+i}) = \mathrm{B}$. Hence consider the case where $I(t_i = t_{k+i}) = \mathrm{F}$. Then $I(t_i) \neq I(t_{k+i})$. But also $(t_i - t_{k+i})/\delta$ is an infinite hyperreal number since the numerator is real and nonzero. Hence, as in Proposition 4, for some d with $|d|$ in D_k, $d^i(t_i - t_{k+i})/\delta$ is noninfinitesimal, so $I(d^i t_i) \neq I(d^i t_{k+i})$ and the antecedent is F as required.

Proposition 8 *If f is any polynomial of the form $\sum_{i=0}^{n} t_i x^i$ with real coefficients t_i, then $\vdash (\forall x \in R)\left(f'(x) = \sum_{i=1}^{n} i t_i x^{i-1}\right)$.*

Proof: Similar to the proof of Proposition 5.

Proposition 9 *For every real function f, $\vdash f$ is continuous.*

Proof: Similar to the proof of Proposition 6.

The $\rightarrow$-free part of this theory is a common inconsistent extension of the classical theories of (i) the ring of noninfinite infinitesimals of R^*, and (ii) the nilpotent ring $\mathbb{R}$, which cannot be achieved classically. To repeat an earlier point, inconsistent calculus is not being recommended as superior or truer, though its nilpotent elements have some of the calculatory advantages of synthetic differential geometry. The aim is only to demonstrate its existence, and to lend support to the claim that inconsistent theories are of mathematical interest.

6 Conclusion An inconsistent model with identity can be constructed with RM3: Set $I(t_1 = t_2) = \mathrm{B}$ iff $I(t_1) = I(t_2)$, and F otherwise, as with the models of [4], [6], and [7]. This produces $\vdash \delta^2 = 0 \;\&\; \neg\delta^2 = 0 \;\&\; \neg\delta = 0$ but not $\vdash \delta = 0$, all as in Section 5; but it also yields $\vdash t = t \;\&\; \neg t = t$ for every term t. There seems to be no reason not to adopt the more sensitive model of Section 5 which is functional but not with identity.

In following papers, it is proposed to report results on the following related topics: (1) corresponding inconsistent theories using Brazilian-style paraconsistent negation, and topological and Routley-* dualizations of these and the theories of the present paper; (2) order and set membership; (3) integration; (4) inconsistent superreals; (5) inconsistent polynomial rings in one or more indeterminates.

The congruence $\underset{\delta}{\sim}$ and particularly the associated inconsistent theories can be regarded as yet another approach to the idea of an "infinitesimal microscope" (see [10] or [11]). A microscope with "resolving power" δ can be said to be a theory which inconsistently identifies with zero and one another all sizes infinitesimal with respect to δ. One is unable to distinguish the behavior of quantities below this "order of infinitesimality" or "order of relative identity". These quantities have all of one another's properties if the theory has substitutivity of identity, or atomic properties if the theory/model has functionality.

Finally, inconsistent claims about infinitesimals have been around for as long as calculus. One must always try to see whether these stem from confusion, or from dim but genuine paraconsistent insights. The only possibility for giving the second kind of answer lies in the rigorous construction of inconsistent mathematical theories. Perhaps the present theories satisfy some of the intuitions of classical analysts; but even if they do not, inconsistent and incomplete mathematics needs investigation.

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