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On Type Definable Subgroups of a Stable Group

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Abstract We investigate the way in which the minimal type-definable subgroup of a stable group G containing a set A originates. We give a series of applications on type-definable subgroups of a stable group G.

1 Introduction It is not known how to construct a stable group "ab ovo". The stability of a given group structure is deduced usually from some stronger properties, for example the group's being abelian-by-finite, or definable in some stable structure. So at least one could wonder what type-definable subgroups of a stable group G are possible to obtain. We address this problem here. In a way, our results generalize Zilber's ideas (cf. Zilber [12]) on generating subgroups by indecomposable subsets of an ω-stable group G.

Throughout, we work with a stable group $G = (G, \cdot, e)$, which is sufficiently saturated (i.e., G is a monster model). L is the language of G. Given a type-definable subset A of G we know that there is $\tilde{A}$, the minimal type-definable subgroup of G containing A (cf. Poizat [9]). We investigate here the relationship between A and $\tilde{A}$. For simplicity, usually we consider A which is type-definable almost over $\emptyset$. A finite set Δ of formulas of L is invariant under translation if it consists of formulas of the form $\varphi(u \cdot x \cdot v; \bar{y})$ ($u, v, \bar{y}$ are parameter variables here). Except in Section 2, Δ with possible subscripts will denote a finite set of formulas invariant under translation. One of the basic concepts of stable group theory is that of generic type, due to Poizat ([9]; see also Hrushovski [4]). Recall that if H is a type-definable subgroup of G then a strong 1-type r of elements of H is generic (for H) iff for every Δ, $R_\Delta(r) = R_\Delta(H)$, where R_Δ is the Morley Δ-rank (see Wagon [11]). Notice that as Δ is invariant under translation, R_Δ also is invariant under translation, meaning that for each definable subset X of G and $a \in G$, $R_\Delta(X) = R_\Delta(a \cdot X) = R_\Delta(X \cdot a)$. (This is the idea of "stratified order" from [9]; cf. also [4].) Let Mlt_Δ denote the Morley Δ-multiplicity. $R_\Delta(a/A)$ abbreviates $R_\Delta(\mathrm{tp}(a/A))$. Let $\vec{R}(p)$ denote $\langle R_\Delta(p) : \Delta \subseteq L$ is finite and invariant

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under translation⟩. $\vec{R}(p) \le \vec{R}(q)$ means that for every Δ, $R_\Delta(p) \le R_\Delta(q)$. Let $\mathrm{gen}(H)$ denote the set of generic types of H. H^0 is the connected component of H. We give a description of $\mathrm{gen}(\tilde{A})$ in topological terms, and prove some corollaries. We formulate also some open problems. Recall the following remark from [4], which can be taken as a definition of generic type.

1.1 Remark Assume H is a type-definable subgroup of G. Then r, a strong 1-type of elements of H, is generic for H iff for every $b \in H$ and a satisfying $r|b$, $a \cdot b \downarrow b$.

In our notation we usually follow Baldwin [1] and Wagon [11]. For background on stable groups see [9], [4], and Hrushovski [5]. By [11] we have

1.2 Remark $a \downarrow X$ iff for every Δ, $R_\Delta(a/X) = R_\Delta(a)$.

1.2 gives a rank equivalent for the forking relation. However this equivalent has one drawback. Condition $R_\Delta(a/X) = R_\Delta(a)$ may involve formulas not in Δ, as it may happen that $R_\Delta(\mathrm{tp}_\Delta(a)) > R_\Delta(a)$. In 1.3 we give another characterization of forking. Let $R^r_\Delta(p) = R_\Delta(p|\Delta)$ and $\vec{R}^r(p) = \langle R^r_\Delta(p) : \Delta \subseteq L\rangle$. r in R^r_Δ stands for "restricted".

1.3 Lemma *Assume $A \subseteq B$. If $\vec{R}^r(a/B) = \vec{R}^r(a/A)$ then $a \downarrow B(A)$. Moreover, if for some model $M \subseteq A$, $a \downarrow A(M)$, then $a \downarrow B(A)$ implies $\vec{R}^r(a/B) = \vec{R}^r(a/A)$.*

Proof: The first part follows by [11], Section III. By Lachlan [7], if $p \in S(M)$ then $\mathrm{Mlt}_\Delta(p|\Delta) = 1$. This implies the "moreover" part.

2 A theorem For simplicity we work here with sets type-definable almost over the empty set of parameters, however all the proofs generalize immediately to the case of arbitrary set of parameters. "Type-definable" will always mean in this section "type-definable almost over $\varnothing$". Let S be the set of strong 1-types over $\varnothing$, with the standard topology τ. Notice that there is an obvious correspondence between closed subsets of S and type-definable subsets of G. By the open mapping theorem, the mapping $p \to \hat{p} = p|G$ is a homeomorphic embedding of S into $S(G)$. We equip S with the following strong topology τ'. Let $(I,\le)$ be a directed set (i.e., $\le$ is a partial order on I and for all $a, b \in I$ there is $c \in I$ with $c \ge a, b$) and $\bar{p} = \langle p_i, i \in I\rangle$ be a net of types from S. We say that $\bar{p}$ is strongly convergent to $q \in S$ (or: q is a strong limit of $\bar{p}$, $q = \mathrm{slim}\,\bar{p}$) if for every Δ there is $i \in I$ such that for every $j \in I$, $j \ge i$ implies $\hat{p}_j|\Delta = \hat{q}|\Delta$. In particular, a strong limit of $\bar{p}$ is a limit of $\bar{p}$ in the usual sense. To distinguish between τ and τ', all topological notions regarding τ' will be called strong. Notice that if q is a strong limit of $\bar{p}$ then $\vec{R}^r(\hat{q})$ is a pointwise limit of $\vec{R}^r(\hat{p}_i)$, $i \in I$. For $p \in S$ let $R'_\Delta(p) = R^r_\Delta(\hat{p})$ and let $\vec{R}'(p) = \langle R'_\Delta(p) : \Delta \subseteq L\rangle$.

We define binary operation $*$ and unary operation $^{-1}$ on S as follows. For $p, q \in S$, $p * q = \mathrm{stp}(x \cdot y)$ and $p^{-1} = \mathrm{stp}(x^{-1})$, where x, y are independent realizations of p and q, respectively. Clearly this definition does not depend on a particular choice of x and y. Similarly we define $*$ on $S(G)$. Notice that $q = p * r$ iff $\hat{q} = \hat{p} * \hat{r}$. Differing somewhat from the common notation, we let p^n denote $p * \ldots * p$ (n times), and $p^{-n} = p^{-1} * \ldots * p^{-1}$ (n times). If P is a set of types then let $P(A)$ denote the set of elements of A realizing some type from P. For $P \subseteq S$ let $\langle P \rangle$ be the minimal type-definable subgroup of G containing $P(G)$. Clearly $\langle P \rangle$ is type-definable almost over $\emptyset$ anyway. If $P = \{p_1, \ldots, p_n\}$, then we write $\langle p_1, \ldots, p_n \rangle$ instead of $\langle P \rangle$. Theorem 2.3 below explains how $\langle P \rangle$ is formed. Let $\mathrm{cl}(P)$ denote the topological closure of P, and let $*$P denote the closure of P under $*$. Let $\mathrm{gen}(P)$ be the set of $r \in \mathrm{cl}(*P)$ such that there is no $q \in \mathrm{cl}(*P)$ with $R_\Delta(r) \leq R_\Delta(q)$, with some of the inequalities strict. As in [4] we have

2.1 Fact If $P \subseteq S$ is nonempty then $\mathrm{gen}(P)$ is nonempty, too. Moreover, $\mathrm{gen}(P)$ is a closed subset of S.

Following [4], for $p \in S$ and $x \in G$ let ${}^x p = r * p$, where $r = \mathrm{stp}(x)$. For $P \subseteq S$ let ${}^x P = \{{}^x p : p \in P\}$.

2.2 Lemma

(a) $*$ *is associative and continuous coordinate-wise.*
(b) *If $P \subseteq S$ is closed, then for every $x \in G$, ${}^x P$ is closed, too.*
(c) $R_\Delta(p * q) \geq R_\Delta(p), R_\Delta(q)$.
(d) $R'_\Delta(p * q) \geq R'_\Delta(p), R'_\Delta(q)$.

Proof: (a) That $*$ is continuous coordinate-wise follows by the open mapping theorem from Lascar and Poizat [8]. (b) follows from (a) and the fact that S is compact. (c) and (d) are easy.

2.3 Theorem *Assume P is a nonempty subset of S. Then $\langle P \rangle = \{x \in G :$ ${}^x\mathrm{gen}(P) = \mathrm{gen}(P)\}$. Also, $\mathrm{gen}(P)$ is the set of generic types of $\langle P \rangle$.*

The rest of this section is devoted to the proof of this theorem. So we fix a $P \subseteq S$. If $p, q \in S$ satisfy $p(G), q(G) \subseteq \langle P \rangle$, then also $p * q(G) \subseteq \langle P \rangle$. Also, if $Q \subseteq S$ and $Q(G) \subseteq \langle P \rangle$ then $\mathrm{cl}(Q)(G) \subseteq \langle P \rangle$. Hence the set $\mathrm{cl}(*P)$ is our first approximation of $\langle P \rangle$: we know that $\mathrm{cl}(*P)(G) \subseteq \langle P \rangle$. It is surprising to find out that this is quite a good approximation: by 2.3 all generics of $\langle P \rangle$ belong to $\mathrm{cl}(*P)$, hence 2.3 implies in fact $\langle P \rangle = \mathrm{cl}(*P)(G) \cdot \mathrm{cl}(*P)(G)$ ($X \cdot Y$ is the complex product of $X, Y \subseteq G$). First notice that iteration of cl and $*$ does not increase $\mathrm{cl}(*P)$ anymore.

2.4 Fact $*\mathrm{cl}(*P) = \mathrm{cl}(*P)$.

Proof: Let $p, q \in \mathrm{cl}(*P)$. It suffices to prove that within any open U containing $p * q$, there is r from $*P$. By 2.2, if q' is close enough to q then $p * q'$ belongs to U, and for fixed q', if p' is close enough to p then $p' * q'$ belongs to U. We can choose p' and q' from $*P$, so we are done.

Let $\mu = |L|$, and let $\Delta_\alpha, \alpha < \mu$, be an enumeration of finite sets of formulas in L invariant under translation. We define by induction on $\alpha \leq \mu$ closed subsets P_α of $\mathrm{cl}(*P)$ as follows. $P_0 = \mathrm{cl}(*P)$, $P_\delta = \bigcap_{\alpha<\delta} P_\alpha$ for limit δ. $P_{\alpha+1}$ is the set of $p \in P_\alpha$ such that $R_{\Delta_\alpha}(p) = R_{\Delta_\alpha}(P_\alpha(G))$. Notice that if we start with $P = S$, then this procedure leads to $P_\mu = \mathrm{gen}(G)$ (cf. the introduction to [4]), whence P_μ does not depend on the particular choice of Δ_α's in this case. We will see that this is always true, i.e. that $P_\mu = \mathrm{gen}(\langle P \rangle)$, and so does not depend on the choice of Δ_α's.

Let $n_\alpha = R_{\Delta_\alpha}(P_\alpha(G))$ and $k_\alpha = \mathrm{Mlt}_{\Delta_\alpha}(P_\mu(G))$. Let $\varphi_{\alpha,i}(x)$, $i < k_\alpha$, be disjoint formulas almost over $\emptyset$ of Δ_α-rank n_α and Δ_α-multiplicity 1 with $P_{\mu(G)} \subseteq \bigcup_i \varphi_{\alpha,i}(G)$. Define $\varphi_{\alpha,i,a}(x)$ as $\varphi_{\alpha,i}(a\cdot x)$. Let $X = \{a \in G: {}^aP_\mu = P_\mu\}$.

2.5 Claim *$X = \bigcap_{\alpha<\mu}\{a \in G$: for each $i < k_\alpha$, $R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap P_\mu(G)) = n_\alpha\}$. In particular, $X = \{a \in G: {}^aP_\mu \subseteq P_\mu\}$, i.e. ${}^aP_\mu \subseteq P_\mu$ implies ${}^aP_\mu = P_\mu$.*

Proof: Notice that if ${}^aP_\mu \subseteq P_\mu$ then for each α and i, $R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap P_\mu(G)) = n_\alpha$, hence $a \in X$, and we are done.

Notice that "$R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap P_\mu(G)) = n_\alpha$" is a definable almost over $\emptyset$ property of a. Indeed, $R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap P_\mu(G)) = n_\alpha$ iff for some (unique) j, $R_{\Delta_\alpha}(\varphi_{\alpha,i,a}(G) \cap \varphi_{\alpha,j}(G)) = n_\alpha$, the latter property of a being definable over the parameters of $\varphi_{\alpha,j}, j < k_\alpha$. Also, X is closed under taking inverses. In particular we get that X is a type-definable almost over $\emptyset$ subgroup of G. The next lemma concludes the proof of 2.3.

2.6 Lemma *$P(G) \subseteq X$, also P_μ is the set of generic types of X. In particular, $X = \langle P\rangle$, P_μ does not depend on the choice of Δ_α's, $n_\alpha = R_{\Delta_\alpha}$ $(\mathrm{cl}(*P))(G)$ and $P_\mu = \mathrm{gen}(P)$.*

Proof: If $p \in P$ and $q \in P_\mu$ then we have $p * q \in \mathrm{cl}(*P) = P_0$. By induction on $\alpha < \mu$, by 2.2(c) we see that $R_{\Delta_\alpha}(p * q) = n_\alpha$, i.e. $p * q \in P_\mu$. This shows that $P(G) \subseteq X$. X is type-definable, hence also $\langle P\rangle \subseteq X$, and in particular $P_\mu(G) \subseteq X$. If r is a generic type of X then we have $r * P_\mu = P_\mu$, hence by 2.2(c) and our definition of generic type, $n_\alpha = R_{\Delta_\alpha}$ $(X) = R_{\Delta_\alpha}(r)$, and each type from P_μ is generic for X. We need to show yet that every generic of X belongs to P_μ (this will imply $X \subseteq \langle P\rangle$, and finish the proof). Let $r \in \mathrm{gen}(X)$ and $p \in P_\mu$. Let $q = r * p$. So $q \in P_\mu$. Let a, b be independent realizations of r, p respectively and $c = a\cdot b$. By 1.2, looking at the Δ_α-ranks of $\mathrm{tp}(c/b)$, we get $b \downarrow c$, hence $a = c\cdot b^{-1}$ satisfies $q * p^{-1}$, i.e. $r = q * p^{-1}$. We have $P_\mu * P_\mu = P_\mu$, hence $P_\mu * p \subseteq P_\mu$. Similarly as in 2.5 we get $P_\mu * p = P_\mu$, i.e. there is $r' \in P_\mu$ with $r' * p = q$. Again we get $r' = q * p^{-1}$, hence $r = r'$ and $r \in P_\mu$. This proves the lemma.

3 Applications and corollaries Let T be a stable theory. Hrushovski proved in [5] that if p is a strong type and $\cdot$ is a definable partial binary operation with some natural properties, defined for independent pairs of elements realizing p, then (in $\mathfrak{C}^{\mathrm{eq}}$) there is a type-definable connected group $(G, \cdot)$ and a definable embedding $f: p\ (\mathfrak{C}) \to G$ preserving $\cdot$, such that $f(p)$ is the generic type of G. In other words: a definite place plus less definite binary operation on it yields a definable group. Here we prove an analogous result: a definite group operation on a less definite place also yields a definable group, namely,

3.2 Theorem *Assume T is stable, $A \subseteq \mathfrak{C}$ and $\cdot$ is a definable binary operation such that $(A, \cdot)$ is a group. Then (in $\mathfrak{C}^{\mathrm{eq}}$) there is a definable group $H = (H, \circ)$ and a definable group monomorphism $h: A \to H$.*

Proof: The proof is an adaptation of the proof of Hrushovski's result from [5], modulo Section 2. Hence we give a sketch only. Wlog A is contained in the set of constants of the language of T. As in Section 2, S denotes the set of strong 1-types over $\emptyset$. For $a \in \mathfrak{C}$ let $p_a = \mathrm{stp}(a)$, and let $P = \{p_a : a \in A\}$. First we proceed as if we were acting within a group structure in Section 2. So for $p,q \in S$ we define $p * q$ as $\mathrm{stp}(x\cdot y)$, where x,y are independent realizations of p,q respectively, provided $x\cdot y$ is defined. Notice that $p_a * p_b$ is always defined for $a,b \in A$, and equals $p_{a\cdot b}$. It follows that $*P = P$, hence we can skip one step from the construction in Section 2, and consider just $\mathrm{cl}(P)$ (which equals $\mathrm{cl}(*P)$ here). By the open mapping theorem, if $a,b \in \mathrm{cl}(P)(\mathfrak{C})$ are independent, then $a\cdot b$ is defined, and also belongs to $\mathrm{cl}(P)(\mathfrak{C})$ (see the proof of 2.4). In particular, $*$ is defined on $\mathrm{cl}(P)$ and $*\mathrm{cl}(P) = \mathrm{cl}(P)$. Within $\mathrm{cl}(P)$ we look for "generic types" of the group we are going to define. We proceed as in the proof of 2.3; however, as in [4], we have to modify the meaning of Δ from Section 2. Wlog $e\cdot x$ and $x\cdot e$ are defined for every $x \in \mathfrak{C}$, and equal x, where e is the identity element of A. Now Δ ranges over sets of the form $\{\varphi(a\cdot x\cdot b;\bar{y}): \varphi(u\cdot x\cdot v;\bar{y}) \in \Delta'$ and $a,b \in \mathrm{cl}(P)(\mathfrak{C})\}$ for some finite set Δ' of formulas of $L = L(T)$. Most importantly, for this new meaning of Δ, 1.2 continues to hold and 2.2(c) remains true for $p,q \in \mathrm{cl}(P)$; hence we are able to carry on reasonings typical for generic types in a stable group. Let $\mu = |T|$, and let $\Delta_\alpha, \alpha < \mu$ be an enumeration of the finite subsets of $L(T)$ invariant under $\cdot$-translation. We define P_μ as in the proof of 2.3, and similarly as in Section 2 we prove the following claim.

3.2 Claim

(a) *If* $p \in \mathrm{cl}(P)$, *then* $p * P_\mu = P_\mu * p = P_\mu$.

(b) P_μ *does not depend on the choice of* Δ_α*'s, and* $R_{\Delta_\alpha}(P_\mu(\mathfrak{C})) = R_{\Delta_\alpha}(\mathrm{cl}(P)(\mathfrak{C}))$.

Let $P' = P_\mu$. Notice that P' is a closed subset of $\mathrm{cl}(P)$. If P' consisted of a single type, the further proof would be nearly the same as in [5]. However, even if P' may have more elements than one, notice that:

(1) for each Δ, $P'|\Delta$ is finite.

On the set of functions f from $\mathfrak{C}^{\mathrm{eq}}$ uniformly definable by instances of some fixed formula, with $\{y \in P'(\mathfrak{C}) : y \downarrow f\} \subseteq \mathrm{Dom}(f)$, we define an equivalence relation $\sim$ by: $f \sim f'$ iff for $y \in P'(\mathfrak{C})$ with $y \downarrow f,f'$, $f(y) = f'(y)$.

By (1), $\sim$ is a definable equivalence relation, hence $f/\!\sim$ is an element of $\mathfrak{C}^{\mathrm{eq}}$. If $g = f/\!\sim$ and $y \in P'(\mathfrak{C})$ is independent from g, then $g(y)$ is defined in an obvious way. In particular, every $a \in \mathrm{cl}(P)(\mathfrak{C})$ determines a P'-germ g_a defined for $c \downarrow a$ by $g_a(c) = a\cdot c$. Let $F_0 = \{g_a : a \in \mathrm{cl}(P)(\mathfrak{C})\}$ and let F be the set of P'-germs of all definable functions $f \in \mathfrak{C}^{\mathrm{eq}}$ with $\{y \in P'(\mathfrak{C}) : y \downarrow f\} \subseteq \mathrm{Dom}(f)$ such that for $y \in P'(\mathfrak{C})$ with $y \downarrow f$, $f(y) \downarrow f$. Hence for $g \in F$ and $y \in P'(\mathfrak{C})$ with $y \downarrow g$ we have $g(y) \downarrow g$. Notice that F_0 is type-definable almost over $\emptyset$. By the choice of P', 3.2 and 1.2, F_0 is contained in F.

For $g_1, g_2 \in F$ let $g_1 \circ g_2$ be the P'-germ of the composition of g_2 and g_1. By the choice of F, $g_1 \circ g_2$ is properly defined and belongs to F. Now we define h. For $a \in \mathrm{cl}(P)(\mathfrak{C})$ let $h(a) = g_a \in F_0$. We check that $h|A$ is an embedding and maps $\cdot$ to $\circ$.

Indeed, if $a \neq a' \in A$ then for any $b \in P'(\mathfrak{C})$ with $b \downarrow a,a'$, $a\cdot b \neq a'\cdot b$ (this follows by the open mapping theorem and the fact that A is a group, i.e. satisfies the right cancellation law). Hence $h|A$ is an embedding.

Now let $a,b \in \mathrm{cl}(P)(\mathfrak{C})$. We have trivially

(2) if $a \downarrow b$ and $c = a\cdot b$ then $g_a \circ g_b = g_c$.

Of course $c \in \mathrm{cl}(P)(\mathfrak{C})$. (2) amounts to saying that for $d \in P'(\mathfrak{C})$ with $d \downarrow a,b,c$, $(a \cdot b) \cdot d = c \cdot d$, which is trivial.

We need yet to find the type-definable group H containing F_0. Let F_1 be the closure of F_0 under $\circ$. As in [5] we see that F_1 satisfies the right cancellation law (in the proof we use the fact that for each $g \in F_1$ and $r \in P'$ there is $y \in P'(\mathfrak{C})$ with $y \downarrow g$ such that $g(y)$ satisfies r, this follows as in 3.2). Let F_2 be the closure of $\{g_a : a \in P'(\mathfrak{C})\}$ under $\circ$. F_2 is a subset of F_1. We will show that F_2 is type-definable. As in [5] it suffices to prove that if $a,b,c \in P'(\mathfrak{C})$ then for some $u,v \in P'(\mathfrak{C})$, $g_a \circ g_b \circ g_c = g_u \circ g_v$. By 3.2, for each $u \in P'(\mathfrak{C})$ and $x \in P'(\mathfrak{C})$ with $x \downarrow u$ there is $y \in P'(\mathfrak{C})$ with $y \downarrow x$ and $y \downarrow u$ such that $u \cdot y = x$. Applying this to $x = b$, we can choose $u,v \in P'(\mathfrak{C})$ such that $u \cdot v = b$, u and v are independent from b and $u,v \downarrow a,b,c(b)$. It follows that $u \downarrow a,b,c$ and $v \downarrow a,b,c$. By (2), $g_a \circ g_b \circ g_c = g_a \circ g_u \circ g_v \circ g_c = g_{a \cdot u} \circ g_{v \cdot c}$. $a \downarrow v$ and $u \downarrow c$ imply $a \cdot u$, $v \cdot c \in P'(\mathfrak{C})$. Now, F_2 is a type-definable semigroup with the right cancellation law, hence by [5], F_2 is a group. If $a \in \mathrm{cl}(P)(\mathfrak{C})$ and $b \in P'(\mathfrak{C})$ are independent, then $a \cdot b = c \in P'(\mathfrak{C})$, and by (2), $g_a \circ g_b = g_c$. As F_2 is a group, for some $u,v \in P'(\mathfrak{C})$, $(g_u \circ g_v) \circ g_b = g_c = g_a \circ g_b$. By the right cancellation law in F_1 we get $g_a = g_u \circ g_v$. This shows that $F_1 = F_2$, and $H = F_2$ satisfies our demands.

As in [5] we can prove that h is 1-1 on $P'(\mathfrak{C})$, and the proof above shows that h maps P' onto $\mathrm{gen}(H)$.

Another application of 2.3 consists in showing that existence of a subgroup of G with some properties yields existence of type-definable subgroup of G with these properties. Suppose $W(x_1, \ldots, x_n)$ is a formula of L. We say that a subset A of G satisfies W if all $\bar{a} \subseteq A$ satisfy W. If H is a type-definable subgroup of G then we say that H satisfies W generically iff all independent tuples $\bar{a} \subseteq H$ of elements realizing generic types of H satisfy W.

3.3 Corollary *If a subgroup A of G satisfies W then the minimal type-definable subgroup of G containing A satisfies W generically.*

Proof: Wlog A is a set of constants. Let $P = \{\mathrm{stp}(a) : a \in A\}$. Then obviously each independent tuple $\bar{a} \subseteq \mathrm{cl}(P)(G)$ of suitable length satisfies W. By 2.3, the generic types of the minimal type-definable subgroups of G containing A belong to $\mathrm{cl}(P)$, hence we are done.

Notice that if H is generically abelian then H is abelian. In particular, we get another proof of an old result (cf. Baldwin and Pillay [2]).

3.4 Corollary *If A is an abelian subgroup of G then $\tilde{A}$ is also abelian.*

Another application concerns the existence of free subgroups of G. Even if it is not known if a free group with ≥ 2 generators is stable, at least we will see that there are "generically free" stable groups. Let $\mathcal{F}(I)$ denote the free group generated by the set I. We say that a type-definable subgroup H of G is generically free if for every $n < \omega$, for each nontrivial word $v(x_1, \ldots, x_n)$ in $\mathcal{F}(x_1, \ldots, x_n)$, H satisfies generically $v(x_1, \ldots, x_n) \neq e$.

3.5 Lemma *If A is a free subgroup of G with ≥ 2 generators then $\tilde{A}$ is generically free.*

Proof: Suppose I is the set of free generators of A, and wlog I is a set of constants of L. We say that a word w in letters from I is positive if a^{-1} does not occur in w for any $a \in I$. Choose $a \neq b \in I$. Let $v_n = a^{-n}ba^{-n}b$, $n > 0$. We say that a word $w(x_1, \ldots, x_n)$ in letters $x_1, \ldots, x_n$ is nontrivial if it is nonempty and no $x_i x_i^{-1}$ or $x_i^{-1} x_i$ occurs in w. The following claim can be proved by induction on the length of w.

3.6 Claim *Assume $w(x_1, \ldots, x_m)$ is a nontrivial word in letters $x_1, \ldots, x_m, n_i, k_i$, $i \leq m$, are natural numbers. If $n_1, k_1, n_2, k_2, \ldots, n_m, k_m$ grows fast enough then for any positive words w_i, $i \leq m$, of length k_i, $w(v_{n_1} w_1, \ldots, v_{n_m} w_m) \neq e$ holds in A.*

Let A_0 be the semi-group generated by I. If $c \in A_0$ then $c = c_1 \ldots c_n$ for some $c_1, \ldots, c_n \in I$. We define $\ell(c) = n$. Applying 2.3 in the language expanded by adding constants for elements of A_0 we see that each generic type r of $\tilde{A}$ is in the closure of $\{\mathrm{stp}(c) : c \in A_0\}$. Also, as in 2.5, for every $v, w \in A_0$, the mappings $r \to \mathrm{stp}(v) * r$ and $r \to r * \mathrm{stp}(w)$ are permutations of $\mathrm{gen}(\tilde{A})$. In particular, by 2.2(a), for every $v, w \in A_0$, $\mathrm{gen}(\tilde{A}) \subseteq \mathrm{cl}(\{\mathrm{stp}(vcw) : c \in A_0\})$. Hence for every n, k we have

(1) $\mathrm{gen}(\tilde{A}) \subseteq \mathrm{cl}(\{\mathrm{stp}(v_n c) : c \in A_0 \text{ and } \ell(c) \geq k\})$.

Now suppose the lemma is false. This means that for some nontrivial word $w(x_1, \ldots, x_m)$, $w(x_1, \ldots, x_m) = e$ belongs to $r_1(x_1) \otimes \ldots \otimes r_m(x_m)$ for some $r_1, \ldots, r_m \in \mathrm{gen}(\tilde{A})$. By the open mapping theorem this means that $\exists U_1 \forall p_1 \in U_1 \exists U_2 \forall p_2 \in U_2 \ldots \exists U_m \forall p_m \in U_m$, $w(x_1, \ldots, x_m) = e \in p_1(x_1) \otimes \ldots \otimes p_m(x_m)$, where U_i ranges over open neighborhoods of r_i. By (1) and 3.6 we get an easy contradiction.

It is well-known (cf. Shelah [10]) that there are two rotations of $\mathbf{R}^3$ which generate a free group. By 3.5 we see that there is a type-definable subgroup H of the group of linear automorphisms of $\mathbf{C}^3$, which is generically free. But the field of complex numbers is ω-stable, hence H is definable, and stable in itself.

4 On connected type-definable subgroups of G From now on, "a subgroup of G" will always mean "a type-definable almost over $\emptyset$ subgroup of G". So if H is a subgroup of G then $\mathrm{gen}(H)$ is a subset of S. Suppose H is a connected subgroup of G and $r \in \mathrm{gen}(H)$. Then $r * r = r$ and $\langle r \rangle = H$. In fact, by 2.3 we have

4.1 Proposition *Let $r \in S$. Then the following are equivalent.*
(a) $r * r = r$
(b) *$\langle r \rangle$ is connected and r is the generic type of $\langle r \rangle$. In particular, $r * r = r$ implies $r = r^{-1}$.*

Proposition 4.1 suggests the following problem. Is it possible to characterize, using only $*$ and topological notions, the class of $r \in S$ such that $\langle r \rangle$ is connected?

We can think of $*$ and topology as our syntactical means, while $\langle r \rangle$ being connected is a kind of semantical notion. Another way to state this problem is as follows: What are the possible syntactical reasons that make $\langle r \rangle$ connected?

In this section we find an ample subset Con of S such that $\langle r\rangle$ is connected for $r \in$ Con.

4.2 Remark *Let H be a subgroup of G and $p \in S$. Then*
(a) $p(G) \subseteq H$ *iff for some (every)* $r \in \mathrm{gen}(H)$, $p * r \in \mathrm{gen}(H)$
(b) $p(G) \subseteq H^0$ *iff for some (every)* $r \in \mathrm{gen}(H)$, $p * r = r$.

Proof: (a) $\rightarrow$ is obvious by 1.2. $\leftarrow$. Let a,b be independent realizations of p,r respectively. Then $c = a\cdot b \in H$, hence $a = c\cdot b^{-1} \in H$.

(b) Let r_0 be the generic type of H^0. Then by (a), $p(G) \subseteq H^0$ iff $p * r_0 = r_0$.

$\rightarrow$. Let $r \in \mathrm{gen}(H)$. Then $r_0 * r = r$, and $p * r = p * (r_0 * r) = (p * r_0) * r = r_0 * r = r$.

$\leftarrow$. Suppose $p * r = r$ for some $r \in \mathrm{gen}(H)$. Let a,b be independent realizations of p,r respectively. Then $a\cdot b$ realizes r, b and $a\cdot b$ are in the same H^0-coset of H. It follows that $a = (a\cdot b)\cdot b^{-1} \in H^0$.

Notice that by 2.2(d) and 4.2(a), if $p(G) \subseteq H$ and $r \in \mathrm{gen}(H)$ then $\vec{R}'(p) \leq \vec{R}'(r)$, and $p \in \mathrm{gen}(H)$ iff $\vec{R}'(p) = \vec{R}'(r)$. This again shows that any reasonable rank of a generic type is maximal possible. The next fact will be often used.

4.3 Fact *Let H be a subgroup of G and $p \in S$. Assume that for some $r \in \mathrm{gen}(H)$, $\vec{R}(r) = \vec{R}(p * r)$. Then $p^{-1} * p(G) \subseteq H^0$ and for every $r \in \mathrm{gen}(H)$, $\vec{R}(r) = \vec{R}(p * r)$.*

Proof: Choose a realizing p and b realizing r with $a \downarrow b$, where $r \in \mathrm{gen}(H)$ and $\vec{R}(r) = \vec{R}(p * r)$. By 1.2, $a\cdot b \downarrow a$, hence $a\cdot b \downarrow a^{-1}$, i.e. $a \cdot b$ and a^{-1} are independent realizations of $p * r$ and p^{-1} respectively. It follows that $b = a^{-1}\cdot(a\cdot b)$ realizes $p^{-1} * (p * r) = (p^{-1} \in p) * r$, i.e. $(p^{-1} * p) * r = r$ ($*$ is associative). By 4.2(b), $p^{-1} * p(G) \subseteq H^0$. Hence, by 4.2(b) and 2.2(c), for every $r' \in \mathrm{gen}(H)$, $\vec{R}(r') \leq \vec{R}(p * r) \leq \vec{R}(p^{-1} * p * r') \leq \vec{R}(r')$, which gives $\vec{R}(r') = \vec{R}(p * r')$.

Notice that $\vec{R}(r) = \vec{R}(p * r)$ is equivalent by 1.2 and 1.3 to $\vec{R}'(r) = \vec{R}'(p * r)$.

4.4 Corollary *$p \in S$ and $p(G) \subseteq \langle P\rangle$ then $p * p^{-1}$ $(G) \subseteq \langle P\rangle^0$.*

4.5 Definition We define an increasing sequence of sets $\mathrm{Con}_0 \subseteq \mathrm{Con}_1 \subseteq \mathrm{Con}_2 \subseteq S$. The definitions of Con_0,Con_1,Con_2 reflect more and more sophisticated reasons for $\langle r\rangle$ to be connected. Let $*$ denote the group operation in $\mathcal{F} = \mathcal{F}(\{x_n : n < \omega\})$. The expression $w(x_1,\ldots,x_n)$ of the form $a_1 * \ldots * a_k$, where each a_i is either x_j or x_j^{-1} for some $j \leq n$, is called a $*$-tuple. If $r_1,\ldots,r_n \in S$ and $w(x_1,\ldots,x_n)$ is a $*$-tuple, then $w(r_1,\ldots,r_n)$ is the type from S obtained by substituting in $w(x_1,\ldots,x_n)$ r_i for x_i. We call w a 0-$*$-tuple if $w(\bar{x}) = e$ holds in $\mathcal{F}$. Let

$\mathrm{Con}_0 = \{w(r_1,\ldots,r_n) : w(x_1,\ldots,x_n)$ is a 0-$*$-tuple, $n < \omega$ and $r_1,\ldots,r_n \in S\}$,
$\mathrm{Con}_1 = \{p \in S : p = \mathrm{stp}(a_1) * \ldots * \mathrm{stp}(a_n)$ for some n, $a_i \in G$ and $a_1\cdot\ldots\cdot a_n = e\}$ and
$\mathrm{Con}_2 = \{p \in S$: there is an infinite indiscernible set $I = \{\bar{a}^1,\bar{a}^2,\ldots\}$ with $\bar{a}^i = \{a_1^i,\ldots,a_n^i\}$, $a_1^1\cdot\ldots\cdot a_n^1 = e$ and $p = \mathrm{stp}(a_1^1\cdot a_2^2\cdot\ldots\cdot a_n^n)\}$.

Finally, let Con $= \mathrm{cl}(\mathrm{Con}_2)$.

It is easy to see that indeed $\mathrm{Con}_0 \subseteq \mathrm{Con}_1 \subseteq \mathrm{Con}_2$. Also, Con_0,Con_1,Con_2 are all closed under $*$, hence by 2.4 Con is closed under $*$. If $\langle r\rangle$ is connected and r is the generic of $\langle r\rangle$ then $r \in \mathrm{Con}_0$, hence $r \in \mathrm{Con}$. The following was the motivation to define Con_2. Suppose we define $\mathrm{Con}_1(G)$ in $S(G)$ like Con_1 in S. Assume some possibly forking extension $r \in S(G)$ of $p \in S$ belongs to $\mathrm{Con}_1(G)$. Then $\langle r\rangle$ is connected (to be shown below), hence also $\langle p\rangle$ is connected. The definition of Con_2 grasps the syntactical meaning of the fact that there exists an $r \in S(G)$ extending p, which belongs to $\mathrm{Con}_1(G)$.

In the next lemma we use local forking. However due to the remark after 1.2 we have to use R^r_Δ instead of R_Δ. Recall that for $q \in S$, $\hat{q} = q\,|\,G$.

4.6 Lemma *If $r \in$ Con and $R^r_\Delta(q * r) = R^r_\Delta(q)$ then $(\hat{q} * \hat{r})|\Delta = \hat{q}\,|\,\Delta$.*

Proof: First assume $r \in \mathrm{Con}_1$. Let $r = \mathrm{stp}(a_1) * \ldots * \mathrm{stp}(a_k)$ with $a_1 \cdot \ldots \cdot a_k = e$. Let $p_i = \mathrm{stp}(a_i)$. Choose b realizing q, independent from $a_1, \ldots, a_k$. Wlog $a_1, \ldots, a_k, b \downarrow G$. By 2.2(d) we have

(1) $R^r_\Delta(q) = R^r_\Delta(q * p_1) = \ldots = R^r_\Delta(q * p_1 * \ldots * p_k) = R^r_\Delta(q * r)$.

By induction on $i \leq k$ we show

(2) $b\cdot a_1 \cdot \ldots \cdot a_i$ realizes $(\hat{q} * \hat{p}_1 * \ldots * \hat{p}_i)|\Delta$ and $R^r_\Delta(b\cdot a_1\cdot \ldots \cdot a_i/G \cup \{a_1, \ldots, a_k\}) = R^r_\Delta(b\cdot a_1 \cdot \ldots \cdot a_i/G)$.

For $i = 0$, (2) holds vacuously. Suppose (2) holds for $i = t$, we will prove it for $i = t + 1$. We have $\mathrm{Mlt}_\Delta((\hat{q} * \hat{p}_1 * \ldots * \hat{p}_t)|\Delta) = 1$, hence if c realizes $\hat{q} * \hat{p}_1 * \ldots * \hat{p}_t$ and $c \downarrow a_1, \ldots, a_k(G)$, then $r = \mathrm{tp}_\Delta(c/G \cup \{a_1, \ldots, a_k\}) = \mathrm{tp}_\Delta(b\cdot a_1 \cdot \ldots \cdot a_t/G \cup \{a_1, \ldots, a_k\})$. We have $c\cdot a_{t+1}$ satisfies $\hat{q} * \hat{p}_1 * \ldots * \hat{p}_{t+1}$. Clearly, r determines $\mathrm{tp}_\Delta(c\cdot a_{t+1}/G \cup \{a_1, \ldots, a_k\})$ (as Δ is invariant under translation).

Also, by (1) we have $R^r_\Delta(c\cdot a_{t+1}/G \cup \{a_1, \ldots, a_k\}) = R^r_\Delta(c\cdot a_{t+1}/G)$. Hence we get $\mathrm{tp}_\Delta(c\cdot a_{t+1}/G \cup \{a_1, \ldots, a_k\}) = \mathrm{tp}_\Delta(b\cdot a_1 \cdot \ldots \cdot a_{t+1}/G \cup \{a_1, \ldots, a_k\})$ and (2) holds for $i = t + 1$.

Applying (2) for $i = k$, using $a_1 \cdot \ldots \cdot a_k = e$, we get that b realizes $(\hat{q} * \hat{r})|\Delta$, i.e. $\hat{q}\,|\,\Delta = (\hat{q} * \hat{r})|\Delta$.

Now suppose $r \in \mathrm{Con}_2$. Let G' be a large saturated extension of G. Wlog we can choose $I = \{\bar{a}^1, \bar{a}^2, \ldots\}$, an indiscernible set witnessing $r \in \mathrm{Con}_2$, such that $r = \mathrm{stp}(a^1_1 \cdot \ldots \cdot a^n_n)$, $I \downarrow G$ and I is based on G', so that $\{\bar{a}^1, \bar{a}^2, \ldots\}$ is independent over G'. Thus, $a^1_1 \cdot \ldots \cdot a^n_n$ realizes over G the type $\hat{r}$. Choose b realizing $q\,|\,G \cup I$. It suffices to prove that $\mathrm{tp}_\Delta(b/G) = \mathrm{tp}_\Delta(b\cdot a^1_1 \cdot \ldots \cdot a^n_n/G)$. We shall prove more, namely

(3) $\mathrm{tp}_\Delta(b/G') = \mathrm{tp}_\Delta(b\cdot a^1_1 \cdot \ldots \cdot a^n_n/G')$.

Let $q' = \mathrm{tp}(b/G')$, $r' = \mathrm{tp}(a^1_1 \cdot \ldots \cdot a^n_n/G')$ and $p_i = \mathrm{tp}(a^1_i/G')$. We see that $r' = p_1 * \ldots * p_n$ (in $S(G')$), and $a^1_1 \cdot \ldots \cdot a^n_n = e$, hence $r' \in \mathrm{Con}_1(G')$ defined in $S(G')$ like Con_1 in S. Also $\mathrm{tp}(b\cdot a^1_1 \cdot \ldots \cdot a^n_n/G') = q' * r'$. But $b\cdot a^1_1 \cdot \ldots \cdot a^n_n$ realizes over $G\hat{q} * \hat{r}$, hence $\hat{q} * \hat{r} = q' * r'\,|\,G$. By the assumptions of Lemmas 2.2(d) and 1.3 we get

(4) $R^r_\Delta(\hat{q}) = R^r_\Delta(q') \leq R^r_\Delta(q' * r') \leq R^r_\Delta(\hat{q} * \hat{r}) = R^r_\Delta(\hat{q})$.

Thus $R^r_\Delta(q') = R^r_\Delta(q' * r')$. Now we can repeat the first part of the proof with $r := r'$, $q := q'$ and $G := G'$ to get $q'|\Delta = q' * r'|\Delta$, i.e. (3).

Finally suppose that $r \in \mathrm{Con} \setminus \mathrm{Con}_2$ and $R'_\Delta(q * r) = R'_\Delta(q)$. For $n < \omega$, the set of $p \in S$ with $R'_\Delta(p) \geq n$ is closed. By 2.2(a), for $p \in \mathrm{Con}_2$ close enough to r we have $R'_\Delta(q * p) = R'_\Delta(q)$, hence $\hat{q} * \hat{p}|\Delta = \hat{q}|\Delta$. Again by 2.2(a), $\hat{q} * \hat{r}|\Delta = \hat{q}|\Delta$.

When G is categorical, Zilber proved in [12] that if $\{A_i : i < \omega\}$ is a family of indecomposable definable subsets of G, then $\bigcup\{A_i : i < \omega\}$ generates a definable subgroup of G. This result was generalized to the superstable context in Berline and Lascar [3]. Unfortunately, in the stable case we do not have such a measure of types as Morley rank in the ω-stable case or U-rank in the superstable case. Here we consider the following problem. Suppose H_i, $i \in I$, are connected subgroups of G. We know that H, the minimal type-definable subgroup containing all the H_i's, is connected. How is H related to the H_i's? As a surrogate for Zilber's result, given $p_i \in \mathrm{Con}$ such that $H_i = \langle p_i \rangle$, we describe topologically how to find $p \in \mathrm{Con}$ with $\langle p \rangle = H$. Theorem 4.7(c) is the first step in this direction. For $P \subseteq S$ and $r \in S$ let $P * r = \{p * r : p \in P\}$. Similarly we define $r * P$.

4.7 Theorem

(a) *If* $r \in \mathrm{Con}$ *then* $\langle r \rangle$ *is connected, moreover* $\langle r^n, n < \omega \rangle$ *strongly converges to the generic type of* $\langle r \rangle$. *So if* q *is the generic type of* $\langle r \rangle$ *then* $\vec{R}'(q)$ *is the pointwise limit of* $\vec{R}'(r^n)$, $n < \omega$.

(b) *If* $P \subseteq S$ *and* $r \in \mathrm{Con}$ *then* $\langle P \cup \{r\} \rangle = \langle r * P \rangle$. *Also,* $\langle P * r \rangle = \langle r * P \rangle$.

(c) *If* $p_1, \ldots, p_n \in \mathrm{Con}$ *then* $\langle p_1, \ldots, p_n \rangle = \langle q \rangle$, where $q = p_1 * \ldots * p_n \in \mathrm{Con}$.

Proof: (a) By 2.2(d), for each Δ, $\langle R'_\Delta(r^n), n < \omega \rangle$ is nondecreasing, and bounded by $R_\Delta(x = x)$, which is finite. Hence there is $n(\Delta)$ such that for $n > n(\Delta)$, $R'_\Delta(r^n) = R'_\Delta(r^{n(\Delta)})$ and by 4.6, $\hat{r}^n|\Delta = \hat{r}^{n(\Delta)}|\Delta$. Thus $\langle r^n, n < \omega \rangle$ strongly converges to some $q \in S$. Also, $r * q = q$. By Theorem 2.3, q is a generic of $\langle r \rangle$. By 4.2(b), $r(G) \subseteq \langle r \rangle^0$, hence $\langle r \rangle = \langle r \rangle^0$ is connected.

(b) Let $p \in P$. It suffices to prove that $r(G), p(G) \subseteq \langle r * P \rangle$. Let q be a generic of $\langle r * P \rangle$. By 2.2 we have

$$(*) \quad \vec{R}'(q) \leq \vec{R}'(q * r) \leq \vec{R}'(q * r * p)$$

$r * p \in r * P$, hence by 4.2(a), $q * (r * p) \in \mathrm{gen}(\langle r * P \rangle)$. It follows that $\vec{R}'(q * r * p) = \vec{R}'(q)$, and in $(*)$ equalities hold. By 4.6, $q = q * r$, hence by 4.2(a), $r(G) \subseteq \langle r * P \rangle$. Also, $q * p = q * (r * p)$ is a generic of $\langle r * P \rangle$, hence by 4.2(a) again, $p(G) \subseteq \langle r * P \rangle$. Similarly, we show $\langle P \cup \{r\} \rangle = \langle P * r \rangle$.

(c) follows from (b).

4.8 Corollary *Assume* $P = \{p_i : i \in I\} \subseteq \mathrm{Con}$. *If* $j = \{i_1, \ldots, i_n\} \subseteq I$ *then we define* $q_j = p_{i_1} * \ldots * p_{i_n}$. *Assume* $q \in R = \bigcap_{i \in I} \mathrm{cl}(\{q_j : j \in j \subseteq I$ *and* j *is finite*$\})$. *Then* $q \in \mathrm{Con}$ *and* $\langle q \rangle = \langle P \rangle$.

Proof: Clearly, $\langle q \rangle \subseteq \langle P \rangle$. Suppose H is an almost-$\emptyset$-definable subgroup of G containing $\langle q \rangle$. By 4.7(c), for every i, $\langle p_i \rangle \subseteq H$, hence $\langle P \rangle \subseteq H$. It follows that $\langle q \rangle = \langle P \rangle$.

Notice that if q_j in 4.8 were defined as generic of $\langle p_{i_1} * \ldots * p_{i_n} \rangle$, then any $q \in R$ would be the generic of $\langle P \rangle$, hence in fact R would be a singleton in such a case. We can say more. By 4.7 and 4.6, if r is the generic of $\langle P\rangle$ then $\vec{R}'(r)$ is the pointwise supremum of $\{\vec{R}'(p) : p \in *P\}$. Also, r is the strong limit of some net of types from $*P$.

In case when the U-rank of G is finite, we get a more exact counterpart of Zilber's result.

4.9 Corollary *Assume G is a superstable group with finite U-rank and $p \in$* Con. *Then for some n, p^n is the generic type of $\langle p\rangle$. In particular, $\langle p\rangle = p^n(G)\cdot p^n(G)$.*

Proof: From 2.2(c) and 1.2 it follows that for $q \in S$, $U(q * r) \geq U(q), U(r)$. Hence we can choose n such that for $m > n$, $U(p^n) = U(p^m)$. It follows that also $\vec{R}'(p^n) = \vec{R}'(p^m)$, and by 4.6, $p^m = p^n$. By Theorem 2.3 we are done.

*5 **A special case*** In this section we focus our attention on the special case of $\langle p\rangle$ for a single type $p \in S$. For $P \subseteq S$ in Theorem 2.3 we explain where the generic types of $\langle P\rangle$ lie. However, in some respect, the results of Section 3 improved greatly Theorem 2.3: if $p \in$ Con and q is the generic of $\langle p\rangle$ then $q = \mathrm{slim}_n p^n$. This formula uses only the topological notion of limit and independent multiplication of types $*$, and does not mention any ranks at all! The following question arises.

5.1 Question Assume $P \subseteq S$. Is it possible to find a generic type of $\langle P\rangle$ (say, the generic type of $\langle P\rangle^0$) using only topological terms and $*$?

The first natural conjecture regarding this question was the statement (C) below. For $p \in S$ let $\mathcal{L}(p)$ be $\liminf \{p^n : n < \omega\} = \{q \in S$: every open U containing q contains p^n for cofinally many $n < \omega\}$.

(C) For $p \in S$, $\mathrm{gen}(\langle p\rangle) = \mathcal{L}(p)$.

By Theorem 2.3 we have of course $\mathrm{gen}(\langle p\rangle) \subseteq \mathcal{L}(p)$. Unfortunately Hrushovski found an easy counterexample to (C). Namely, let $G = (\mathbf{Q},+,1,P)$, where $P = \{2^{n^2} : n < \omega\} \subseteq \mathbf{Q}$. $\mathrm{Th}(G)$ is ω-stable with Morley rank ω, $P(x)$ is strongly minimal, $\langle \mathrm{stp}(1)\rangle$ = all of G, but the strongly minimal type in P is in $\mathcal{L}(\mathrm{stp}(1))$ and is not a generic of G.

We show however that (C) is true for several cases, for example for all stable groups of bounded exponent. In a way we shall answer positively question 5.1 in case when $P \subseteq S$ is a singleton, in the double step Theorem 5.12 below. We start with comparing $\langle p\rangle$ and $\langle q\rangle$ for various $p, q \in \mathrm{Con}_0$. We need some additional notation. Let $w(x_1,\ldots,x_n) = a_1 * \ldots * a_k$ be a $*$-tuple. For $i \leq k$ let w_i be the shortest $*$-tuple such that in $\mathcal{F}(\{x_n : n < \omega\})$, $a_1 * \ldots * a_i = w_i$ holds. Let $\mathrm{In}_0(w) = \{w_i : i \leq k\}$ and $\mathrm{In}(w) = \{v \in \mathrm{In}_0(w) : v$ is not a proper initial segment of any $v' \in \mathrm{In}_0(w)\}$. As an example notice that if $w = w(x_1)$, then $\mathrm{In}(w)$ has at most two elements which are of the form $x_1 * \ldots * x_1$ or $x_1^{-1} * \ldots * x_1^{-1}$.

5.2 Theorem *Assume $w(x_1,\ldots,x_n), v(x_1,\ldots,x_n)$ are 0-$*$-tuples and $r_1,\ldots,r_n \in S$.*

(a) $\langle w(r_1,\ldots,r_n)\rangle = \langle\{w'(r_1,\ldots,r_n) * w'(r_1,\ldots,r_n)^{-1} : w' \in \mathrm{In}(w)\}\rangle$.

(b) *If every $w' \in \mathrm{In}(w)$ is an initial segment of some $v' \in \mathrm{In}(v)$, then $\langle w(r_1,\ldots,r_n)\rangle \subseteq \langle v(r_1,\ldots,r_n)\rangle$.*

Proof: (a) $\supseteq$. First we prove that for each $w' \in \mathrm{In}_0(w)$, $\langle w'(r_1,\ldots,r_n) * w'(r_1,\ldots,r_n)^{-1}\rangle \subseteq \langle w(r_1,\ldots,r_n)\rangle$.

The proof is similar to that of 4.6 and 4.7(b). Let q be the generic of $\langle w(r_1,\ldots,r_n)\rangle$, $r = w'(r_1,\ldots,r_n)$, and it suffices to prove that $q * (r * r^{-1}) = q$. As $w' \in \mathrm{In}_0(w)$, there is a $p \in S$ such that $q * r * p$ is the generic of $\langle w(r_1,\ldots,r_n)\rangle$. Hence, $\vec{R}(q) = \vec{R}(q * r)$. By 4.3, $\langle r * r^{-1}\rangle \subseteq \langle w(r_1,\ldots,r_n)\rangle$.

$\subseteq$. Let $H = \langle\{w'(r_1,\ldots,r_n) * w'(r_1,\ldots,r_n)^{1-} : w' \in \mathrm{In}(w)\}\rangle$, and let q be the generic of H. Choose $b_1,\ldots,b_n \in G$ realizing $r_1,\ldots,r_n$ respectively, and if $w(r_1,\ldots,r_n) = p_1 * \ldots * p_k$, where $p_i = r_j^\epsilon$, $\epsilon = \pm 1$, then put $a_i = b_j^\epsilon$. Thus, $a_1 \cdot \ldots \cdot a_k = e$. Choose c realizing q, independent from $b_1,\ldots,b_n$. As in 4.6 (the case $r \in \mathrm{Con}_1$) we prove that for every $i \leq k$, $c \cdot a_1 \cdot \ldots \cdot a_i$ realizes $q * p_1 * \ldots * p_i$ (the proof relies on the definition of H). This implies $\langle w(r_1,\ldots,r_n)\rangle \subseteq H$.
(b) follows from (a).

By 5.2 and 4.4 we get the following corollary.

5.3 Corollary *Let $p \in S$. Then $\langle p^n * p^{-n}\rangle \subseteq \langle p^{n+1} * p^{-(n+1)}\rangle \subseteq \langle p\rangle^0$.*

One could wonder whether $\langle p^n * p^{-n}\rangle = \langle p^{-n} * p^n\rangle$. This seems unlikely, although by 5.2 and 4.8 it is not hard to prove that $\langle\{p^n * p^{-n} : n < \omega\}\rangle = \langle\{p^{-n} * p^n : n < \omega\}\rangle$. In the next lemma we shall see that the relationship between $\{p^n * p^{-n} : n < \omega\}$ and $(p^{-n} * p^n : n < \omega\}$ is even closer.

5.4 Lemma *Let q be the generic type of $\langle\{p^n * p^{-n} : n < \omega\}\rangle = \langle\{p^{-n} * p^n : n < \omega\}\rangle$.*
(a) $q = \mathrm{slim}_n\, p^n * p^{-n} = \mathrm{slim}_n\, p^{-n} * p^n$.
(b) $\vec{R}'(q) = \lim_n \vec{R}'(p^n)$ (the limit is pointwise here).

Proof: First notice that $\vec{R}'(q) \geq \lim_n \vec{R}'(p^n)$, as $\vec{R}'(p^n * p^{-n}) \geq \vec{R}'(p^n)$. On the other hand we know that $q \in \mathrm{cl}(*P)$, where $P = \{p^n * p^{-n} : n < \omega\}$. For a finite Δ choose m such that for $n \geq m$, $R'_\Delta(p^n) = R'_\Delta(p^m)$. As in the proof of 4.6, for every $r \in *P$, $\hat{p}^m * \hat{r}|\Delta = \hat{p}^m|\Delta$. By 2.2(a), $\hat{p}^m * \hat{q}|\Delta = \hat{p}^m|\Delta$. By 2.2(d), $R'_\Delta(p^m) = R'_\Delta(p^m * q) \geq R'_\Delta(q)$. This shows (b).

Now let $r \in \bigcap_m \mathrm{cl}(\{p^n * p^{-n} : n > m\})$. Then $\vec{R}'(r) \geq \lim_n \vec{R}'(p^n * p^{-n}) \geq \lim_n \vec{R}'(p^n) = \vec{R}'(q)$. So by 4.2(a), $\vec{R}'(r) = \vec{R}'(q)$, and r is the generic of $\langle\{p^n * p^{-n} : n < \omega\}\rangle$. It follows that $q = r$, i.e. $q = \lim_n p^n * p^{-n}$. But $\vec{R}'(q) = \lim_n \vec{R}'(p^n * p^{-n})$, hence we see that q is the strong limit of $\{p^n * p^{-n} : n < \omega\}$.

5.5 Corollary *Let $p \in S$. There is a connected type-definable almost over $\varnothing$ subgroup H of $\langle p\rangle^0$ such that $\vec{R}'(H) = \lim_n \vec{R}'(p^n)$.*

The q from Lemma 5.4 might be called $p^\omega * p^{-\omega}$ or $p^{-\omega} * p^\omega$. It is not hard to prove that $p * q * p^{-1} = q$, hence such a notation would imply $p^{(1+\omega)} * p^{-(1+\omega)} = p^\omega * p^{-\omega}$, which agrees well with $\omega = 1 + \omega$.

Now let us see what the connection is between $\langle P\rangle$ and $\langle P\rangle^0$ for $P \subseteq S$. First we deal with $P = \{p\}$.

5.6 Lemma *Let $p \in S$. Then $[\langle p\rangle : \langle p^n\rangle]$ is finite for each $n > 0$. Also, $\langle p\rangle^0 = \bigcap_n \langle p^n\rangle$. In particular, $[\langle p\rangle : \langle p\rangle^0] \leq 2^{\aleph_0}$.*

Proof: By 5.3, for $i \leq n$, $p^i * p^{-i}(G) \subseteq \langle p^n\rangle$, hence $p^i(G)$ is contained in one left (and one right) $\langle p^n\rangle$-coset of $\langle p\rangle$. Thus also for every $j < \omega$, $p^j(G)$ is contained in one left $\langle p^n\rangle$-coset and it follows that there are only finitely many left

$\langle p^n \rangle$-cosets containing some $p^i(G)$. In particular, for q_0, the generic type of $\langle p \rangle^0$, $q_0(G)$ is contained in one $\langle p^n \rangle$-coset of $\langle p \rangle$. As $q_0 = q_0 * q_0^{-1}$, we have $q_0(G) \subseteq \langle p^n \rangle$ and $\langle p \rangle^0 \subseteq \langle p^n \rangle$.

Thus if q is a generic of $\langle p \rangle$ then $q(G)$ is contained in one left $\langle p^n \rangle$-coset of $\langle p \rangle$. Also, $q \in \mathcal{L}(p)$ and there are only finitely many $\langle p^n \rangle$-cosets containing some $p^i(G)$. Thus there are only finitely many $\langle p^n \rangle$-cosets containing $q(G)$ for some $q \in \mathrm{gen}(\langle p \rangle)$. This implies $[\langle p \rangle : \langle p^n \rangle]$ is finite.

Now suppose that H is a relatively definable almost over $\emptyset$ subgroup of $\langle p \rangle$ with finite index in $\langle p \rangle$. Then $q_0(G) \subseteq H$, hence by 2.3 for some n, $p^n(G) \subseteq H$. It follows that $\langle p^n \rangle \subseteq H$, i.e. $\langle p \rangle^0 = \bigcap_{n<\omega} \langle p^n \rangle$.

Notice that if X is a free group with κ generators then there are $\leq (\kappa + \aleph_0)$-many normal subgroups of X with finite index in X. Hence by a similar proof we get

5.7 Corollary *If $P \subseteq S$ then $[\langle P \rangle : \langle P \rangle^0] \leq 2^{|P| + \aleph_0}$.*

Suppose for some k, $p(x) \vdash x^k = e$; that is, p is a type of elements of finite order. Then we have $p^k \in \mathrm{Con}_1$; hence by 5.6 we get the following corollary.

5.8 Corollary *If $p(x) \vdash x^k = e$ then $[\langle p \rangle : \langle p \rangle^0] \leq k$ and $\mathrm{gen}(\langle p \rangle) = \mathcal{L}(p)$ is finite. Let q be the generic of $\langle p \rangle^0$. Then $q = \mathrm{slim}_n\, p^{nk}$. Also, for $i < k$ $\mathrm{slim}_n\, p^{nk+i}$ exists and is a generic of $\langle p \rangle$, and every generic of $\langle p \rangle$ is obtained in this way.*

5.9 Corollary *If $\mathrm{Th}(G)$ is small and $P \subseteq S$ is finite then $\langle P \rangle$ is connected-by-finite.*

Proof: By adding a finite set of constants to L we can assume that $P \subseteq S(\emptyset)$. By Theorem 2.3, every generic of $\langle P \rangle$ is in $\mathrm{cl}(*P)$, hence $S(\emptyset)$ being countable implies that $\mathrm{gen}(\langle P \rangle)$ is countable, too, and $[\langle P \rangle : \langle P \rangle^0] < \omega$.

The next theorem shows that in many cases (C) is true. For the definition of weakly normal groups, see [6]. Notice that any pure group which is abelian-by-finite is weakly normal.

5.10 Theorem *Assume $p \in S$ and G has bounded exponent or is weakly normal. Then $\mathrm{gen}(\langle p \rangle) = \mathcal{L}(p)$.*

Proof: In case when G has bounded exponent the conclusion follows by 5.8. So suppose G is weakly normal. Choose any $q \in \mathcal{L}(p)$. We will prove that $q \in \mathrm{gen}(\langle p \rangle)$. Let r be the generic of $\langle p \rangle$ such that $q^{-1} * r(G) \subseteq \langle p \rangle^0$, that is $q(G)$ and $r(G)$ are in the same $\langle p \rangle^0$-coset of $\langle p \rangle$. We will prove that $q = r$. By Hrushovski and Pillay [6], every definable subset of G is a Boolean combination of cosets of almost over $\emptyset$ definable subgroups of G. Hence, fix an almost-$\emptyset$-definable $H < G$. It suffices to prove that for any $a \in G$, $r(G) \subseteq aH$ iff $q(G) \subseteq aH$.

Suppose $r(G) \subseteq aH$. Then $r^{-1} * r(G) \subseteq H$, hence $\langle p \rangle^0 \subseteq H$. As $q(G)$ and $r(G)$ are in the same $\langle p \rangle^0$-cosets, we get $q(G) \subseteq aH$.

Now suppose $q(G) \subseteq aH$. Then $\hat{q}(x) \vdash x \in aH$, and $q \in \mathcal{L}(p)$, so there are infinitely many n with $p^n(G) \subseteq aH$. Choose $n, k > 0$ with $p^n(G), p^{n+k}(G) \subseteq aH$. It follows that $p^k(G) \subseteq H$, hence again by 5.6 $\langle p \rangle^0 \subseteq H$. As above we get $r(G) \subseteq aH$.

It is easy to see that $*$ restricted to $\mathrm{gen}(\langle p\rangle)$ is continuous (as a binary function). Unfortunately, $*$ is not always continuous on $\mathcal{L}(p)$, because this implies (C) for p. Define $f_p : S \to S$ by $f_p(q) = p * q$, and similarly define $f_{p^{-1}}$.

5.11 Lemma *$f_p|\mathcal{L}(p)$ is a permutation of $\mathcal{L}(p)$. Also, $f_{p^{-1}} \circ f_p|\mathcal{L}(p) = \mathrm{id}_{\mathcal{L}(p)}$.*

Proof: Suppose $\langle p^{n_i} : i \in I\rangle$ is a net converging to $q \in \mathcal{L}(p)$, and wlog $\langle p^{n_i - 1} : i \in I\rangle$ converges to $q' \in \mathcal{L}(p)$. We see that $f_p(q') = q$, hence $\mathrm{Rng}(f_p|\mathcal{L}(p)) = \mathcal{L}(p)$. For a fixed Δ, as in the proof of 4.6 and 5.4, we see that if n is large enough then $\hat{p}^{-1} * \hat{p} * \hat{p}^n|\Delta = \hat{p}^n|\Delta$. It follows that $\langle p^{-1} * p * p^{n_i} : i \in I\rangle$ also converges to q. But this means that $f_{p^{-1}} \circ f_p|\mathcal{L}(p) = \mathrm{id}_{\mathcal{L}(p)}$, and we are done.

Let $p \in S$. Suppose we are given a task of getting a generic type of $\langle p\rangle$; we know topology, independent multiplication $*$, but cannot measure any ranks. The first guess would be to choose a $q_0 \in \mathcal{L}(p)$. We know that possibly $\mathrm{gen}(\langle p\rangle) \neq \mathcal{L}(p)$. So it may happen that $q_0 \notin \mathrm{gen}(\langle p\rangle)$. However q_0 in some respect is more similar to a generic of $\langle p\rangle$ than any p^n, for example any rank of q is $\geq$ that rank of p^n. Also, $\langle p\rangle^0 \subseteq \langle q_0\rangle \subseteq \langle p\rangle$, $\mathrm{gen}(\langle q_0\rangle) \subseteq \mathrm{gen}(\langle p\rangle)$ and $\mathcal{L}(q_0) \subseteq \mathcal{L}(p)$ (this is proved below). So maybe if we try again and choose $q_1 \in \mathcal{L}(q_0)$, then we are more lucky in getting a generic of $\langle p\rangle$. The next theorem confirms this guess.

5.12 Double step theorem *Assume $p \in S$, $q \in \mathcal{L}(p)$ and $r \in \mathcal{L}(q)$. Then r is a generic type of $\langle p\rangle$.*

Proof: First notice that

(1) $\langle p\rangle^0 \subseteq \langle q\rangle \subseteq \langle p\rangle$.

Indeed, any almost-$\varnothing$-definable subgroup H of G containing $\langle q\rangle$ contains $p^n(G)$ for some n, hence also $\langle p^n\rangle$. By 5.6, $\langle p\rangle^0 \subseteq \langle p^n\rangle \subseteq H$. Looking at ranks, (1) implies $\mathrm{gen}(\langle q\rangle) \subseteq \mathrm{gen}(\langle p\rangle)$. Also, $\mathcal{L}(p)$ is closed and closed under $*$, hence $\mathcal{L}(q) \subseteq \mathcal{L}(p)$. Now let $q_0 = q^{-1} * q$. We show that

(2) $q_0 \in \mathcal{L}(p)$.

Choose a net $\langle p^{n_i} : i \in I\rangle$ converging to q. Then $\langle p^{-n_i} : i \in I\rangle$ converges to q^{-1}. It suffices to find within an arbitrary open U containing q_0 a type from $\mathcal{L}(p)$. By 2.2(a) we can find an $i \in I$ such that $p^{-n_i} * q \in \mathsf{U}$. By 5.11, the mapping $s \to p^{-n_i} * s$ is a permutation of $\mathcal{L}(p)$, hence $p^{-n_i} * q \in \mathcal{L}(p)$.

By (1), $\langle p\rangle^0 \subseteq \langle q_0\rangle \subseteq \langle q\rangle \subseteq \langle p\rangle$, hence $\langle p\rangle^0 = \langle q_0\rangle^0 = \langle q\rangle^0$. But $q_0 \in \mathrm{Con}$, hence by 4.7, $\langle q_0^n, n < \omega\rangle$ is strongly convergent to q_1, the generic type of $\langle q_0\rangle = \langle p\rangle^0$. By 5.3, 5.4 and 2.2(d) it follows that $\vec{R}(q_1) = \lim_n \vec{R}'(q_0^n) = \lim_n \vec{R}'(q^n)$. We know that any $s \in \mathcal{L}(p)$ is a generic of $\langle p\rangle$ iff $\vec{R}'(s) = \vec{R}'(q_1)$, and for every $s \in \mathcal{L}(p)$, $\vec{R}'(s) \leq \vec{R}'(q_1)$. On the other hand, by 2.2(d), $\vec{R}'(r) \geq \lim_n \vec{R}'(q^n) = \vec{R}'(q_1)$, as $r \in \mathcal{L}(q)$. This implies $\vec{R}'(r) = \vec{R}'(q_1)$, hence r is a generic type of $\langle p\rangle$.

Take r from 5.12. By 5.6 we can define the generic type r' of $\langle p\rangle^0$ as $\mathrm{slim}_n r^{n!}$. A similar argument yields the following corollary.

5.13. Corollary *Let $p \in S$. The following conditions are equivalent.*
(a) *p is a generic type of $\langle p \rangle$.*
(b) $p = \lim_n p^{n!+1}$
(c) $p = \mathrm{slim}_n p^{n!+1}$.

A challenging problem is to generalize 5.12 for arbitrary $P \subseteq S$. This would tell us more about restrictions on the structure of G imposed by the stability assumption.

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