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Type-Free Property Theory, Exemplification and Russell's Paradox

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Abstract This paper presents a type-free property-theoretic system in the spirit of a framework proposed by Menzel and then supplements it with a theory of truth and exemplification. The notions of a truth-relevantly complex (simple) sentence and of a truth-relevant subsentence are introduced and then used in order to motivate the proposed theory. Finally, it is shown how the theory avoids Russell's paradox and similar problems. Some potential applications to the foundations of mathematics and to natural language semantics are sketched in the introduction.

1 Introduction In the last few years, several type-free frameworks that can be thought of as conveying a theory of properties—more generally properties, relations, and propositions[1] (PRPs)—have been proposed (Bealer [1], Castañeda [4], Cocchiarella [8], Feferman [11], Jubien [14], Menzel [17], Scholck [19], Turner [20], etc.).

A type-free property-theoretic framework appears to be a better framework than one based on type theory for certain problems in the foundations of mathematics and natural language semantics. For example, Feferman [11] argues for the relevance of a type-free framework in order to account for important aspects of mathematics, and Cocchiarella [8] and Bealer [1] have both used type-free systems in order to reconstruct logicism. Regarding natural language semantics, Chierchia [6] and Chierchia and Turner [7] have used type-free systems to develop a Montague-style semantics for fragments of English and have argued for the superiority of such an approach, as opposed to one based on type theory (e.g., Montague [18]), in order to account for a number of important linguistic phenomena.

A major constraint in developing a type-free property theory is constituted by the need to circumvent Russell's paradox. As a matter of fact, Russell's paradox is at least two paradoxes, one regarding sets and the membership relation, the other regarding properties and predication. There is a widespread agreement

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that the iterative conception of set has provided a satisfactory solution to the first paradox, but nothing comparable has happened for the latter (cf. [2]).

Ideally, one seeks a solution that can be motivated independently of Russell's and similar paradoxes. The system proposed by Menzel [17] is particularly interesting in this respect. It is based on the idea that complex PRPs must be thought of as "built up" from simpler ones on the basis of logical operations. This framework avoids Russell's paradox by ruling out the a priori postulation of a Russellian property of non-self-predication. On the other hand, the same constraints that rule out non-self-predication also rule out, *inter alia*, those PRPs that can be used to model in a natural way the natural numbers and natural language quantifiers, thereby undermining the application of Menzel's framework to the foundation of mathematics and to natural language semantics.[2] For example, according to standard treatments, the number zero could be modelled as a property with no instances, i.e., $[\lambda f\ -(\exists x)(f(x))]$, and some and every by $[\lambda fg\ \exists x(f(x)\ \&\ g(x))]$ and $[\lambda fg\ \forall x(f(x) \rightarrow g(x))]$, respectively. But none of these PRPs is available if Menzel's constraints are accepted (cf. Section 3 below).

Nevertheless, a Menzelian framework does not rule out that there might be primitive simple exemplification relationships (of different n-adicity), in particular a dyadic one, which we might designate by "I^2" (where the letter "I" is meant to be remindful of "instantiation"). If this relation is admitted, a version of "useful" PRPs such as the ones mentioned above can be provided. For example, some and every could be interpreted as $[\lambda fg\ \exists x(I^2(f,x)\ \&\ I^2(g,x))]$ and $[\lambda fg\ \forall x(I^2(f,x) \rightarrow I^2(g,x))]$, respectively. These and analogous definitions of crucial natural language notions open the road to the use of a Menzelian framework in natural language semantics, e.g., along the lines in which a system such as that of Turner [20] has been used in [7].

Furthermore, zero could be interpreted as $[\lambda f\ -(\exists x)(I^2(f,x))]$, and, in a similar vein, Fregean-style definitions of the other primitive notions involved in Peano's arithmetics could be provided. By supplying fine-grained identity conditions for PRPs, Peano's axioms could then be proven, along the lines of [1].

On the other hand, once dyadic exemplification is available, a complex property of non-self-exemplification can be built up from it. Hence, the hypothesis that there are exemplification relations must be developed with care. Obviously, they will have to be governed by axioms or rules which somehow justify the use of the term "exemplification," and, if one is not careful enough, these axioms (rules) might generate a new version of Russell's paradox, once non-self-exemplification is predicated of itself (if not in other ways).

The main purpose of this paper is to present a theory of exemplification that can be mounted on a Menzel-style framework and which escapes Russell's paradox and similar "logical nightmares".

Although this theory appears resistant to Russellian arguments and the like, so far I have not been able to provide a consistency result for it. Nevertheless, since the characteristic axioms of this theory can be intuitively motivated independently of the paradoxes, and since it differs substantially (as far as I know) from any other type-free property theory that can be found in literature, it is well worth presenting it in its own right. The field of type-free property theory is still relatively young and it is difficult to tell at this stage which lines of research will prove more fruitful. Hence, all the available approaches which show some promise should be seriously considered and compared in the dia-philosophical spirit of Castañeda [5]. Furthermore, it is hoped that interest in the theory presented in this paper will more easily lead to a (relative) consistency proof for it.

*2 **Property theory and Russell's paradox*** Prima facie, an appropriate formal framework for property theory is constituted by a multi-sorted second-order language with individual variables and predicate variables of different n-adicity, and which allows for: (i) predicate terms occurring in both subject and predicate position, and (ii) the formation of complex predicates, by means of a rule such as the following[3]:

(GR) If A is a wff and $a_1, \ldots, a_n$ are n distinct individual variables, $[\lambda a_1 \ldots a_n A]$ is an n-adic predicate term (which is also called, for convenience, a *lambda-abstract*).[4]

As regards the logistic associated with such a language, it is natural to assume standard axioms and rules for classical logic with identity supplemented with a lambda-conversion axiom, which has the function of giving complex predicates their intended meaning. It can be stated as follows:

(λ-conv) $[\lambda a_1 \ldots a_n A] b_1 \ldots b_n \leftrightarrow A[b_1/a_1, \ldots, b_n/a_n]$ (provided b_i is free for a_i in A, $1 \leq i \leq n$).

Unfortunately, (GR) grants the formation of the "Russellian" expression "$[\lambda x \exists f(f = x \ \& \ -f(x))]$", and since predicate terms are allowed to occur in both subject and predicate position, the full lambda-conversion scheme (λ-conv) and obvious transformations immediately lead to

(r1) $[\lambda x \exists f(f = x \ \& \ -f(x))][\lambda x \exists f(f = x \ \& \ -f(x))] \leftrightarrow -[\lambda x \exists f(f = x \ \& \ -f(x))][\lambda x \exists f(f = x \ \& \ -f(x))]$.

*3 **The system M*** One way to block the paradox is by restricting in a systematic way the formation of lambda-abstracts. This can be done, e.g., by replacing (GR) with the weaker (GR′) below:

(GR′) If A is a wff in which (i) no bound variable occurs in predicate position and (ii) $a_1, \ldots, a_n$ are variables which do not occur free in predicate position in A, or in any term that occurs in predicate position in A, then $[\lambda a_1 \ldots a_n A]$ is a predicate term.

The system resulting from the one outlined in Section 2 by means of such a replacement will be called "M". In turn, the language of M will be called "$L(M)$".

It is easy to see that (GR′) makes the Russellian expression "$[\lambda x \exists f(f = x \ \& \ -f(x))]$" ill-formed. But this is not its only merit, since (GR′) can be motivated on philosophical grounds, independently of Russell's paradox. However, since the weakening of (GR) in favor of (GR′) constitutes the central feature of the system LPRP of [17], I refer the reader to this work for a philosophical motivation of (GR′) and hence, indirectly, of the system M. Here I shall only provide a quick sketch that will do for our present purposes.

Drawing from [1], [16], and [21], Menzel individuates a number of basic logical operations that can be performed on given PRPs. Examples are negation, conjunction, universalization and existentialization (of a given argument position), reflexivization (of two given argument positions), etc. Now, each lambda abstract that (GR′) generates can be thought of as expressing the result of performing one such operation on given PRPs. For example, $[\lambda x\, -Px]$ can be regarded as the negation of P; $[\lambda xyzw\,(Px\ \&\ Qyzw)]$ as the conjunction of P and Q; $[\lambda xy\, Qxyx]$ as the reflexivization of the first and third argument position of Q; $[\lambda xw\, \forall y(Qyxw)]$ as the universalization of the first argument position of Q. On the contrary, each predicate delivered by (GR), but not by (GR′), cannot be regarded in such a way. Examples are the Russellian predicate "$[\lambda x\, \exists f(f = x\ \&\ -f(x))]$", as well as the useful predicates "$[\lambda f - (\exists x)(f(x))]$", "$[\lambda\, \exists x(f(x)\ \&\ g(x))]$" and "$[\lambda fg\, \forall x(f(x) \rightarrow g(x))]$", mentioned in the introduction.[5]

In order to illustrate this last point, let us focus on the Russellian predicate "$[\lambda x\, \exists f(f = x\ \&\ -f(x))]$". Prima facie, this predicate could be regarded, roughly speaking, as the result of applying reflexivization and then existentialization to a complex PRP resulting from the conjunction of two simpler PRPs. One of them is identity, assuming of course that this is the relation denoted by the predicate constant "$=$", and the other should be a complex PRP resulting from the application of negation to a simpler PRP. But there is no candidate for this simpler PRP, since in the Russellian expression the formula dominated by negation is "$f(x)$" and "f" is not a predicate constant, but a predicate variable bound by an existential quantifier.

Menzel gives rigorous definitions of the logical operations that I have mentioned, as well as of additional ones. Furthermore, Menzel's system LPRP includes a number of axioms that jointly constitute a sort of fine-grained identity theory for PRPs, and is given an algebraic semantics in which the denotation of complex lambda-abstracts depends systematically on that of simpler ones. LPRP is proven to be consistent as well as valid and complete with respect to this semantics.

An important difference between M and Menzel's LPRP is that whereas M supports full lambda-conversion, in LPRP lambda-conversion fails in a few special cases. Full lambda-conversion, in any case, is not against the spirit of Menzel's approach. Furthermore, since in M all the Russellian predicates are ruled out at the outset by (GR′), it can plausibly be conjectured that M is no less consistent than LPRP, notwithstanding the difference in question.

4 Menzel's approach and exemplification As mentioned in the introduction, Menzel's machinery to generate complex PRPs does not grant the existence of any PRP that can be taken to correspond to the notion of dyadic exemplification. At the same time, it does not rule out its existence either. Assuming that there is such a PRP and that we designate it by "I^2," the most obvious way to capture the meaning of "I^2" is, prima facie, by means of the following axiom:

(I) $I^2R^1a \leftrightarrow R^1a.$

Nevertheless, once the language of M is suitably augmented and (I) is added to M's axiom schemes, a version of Russell's paradox arises, for if I^2 is a relation, then one can generate the complex property $[\lambda x\, -I^2xx]$, via application of reflexivization and negation. By lambda-conversion

(r2) $[\lambda x\, -I^2xx][\lambda x\, -I^2xx] \leftrightarrow -I^2[\lambda x\, -I^2xx][\lambda x\, -I^2xx]$

and, by (I)

(r3) $[\lambda x\, -I^2xx][\lambda x\, -I^2xx] \leftrightarrow -[\lambda x\, -I^2xx][\lambda x\, -I^2xx]$. (Cf. [17], p. 20.)

Menzel concludes that

> there cannot be such a thing as r ["I^2" in my terminology] [cf. [17], p. 20] . . . or else (much more doubtfully, I think) it must have a very different logical behavior than what we intuitively think it should have; cf. again Bealer [1] 94 ff. [cf. [17] note 15, p. 56]

Let us generalize a bit on this issue. The first step is to supplement $L(M)$ by introducing among the primitive predicate constants of M, for each $n \geq 1$, the predicate constant I^n, meant to represent the n-adic instantiation or exemplification relation. Let us call the resulting language $L(MI)$.

In general, a formula of the form "$I^{n+1}R^na_1 \ldots a_n$" could be read as "(the n-adic relation) R^n is instantiated (exemplified) by $a_1, \ldots, a_n$ (taken in that order)." Note that, as we are in a type-free setting, I have not ruled out iterated occurrences of exemplification predicates. For example, "$I^4I^3R^2a_1a_2$" could be read as "I^3 is instantiated by R^2, a_1, a_2 (taken in that order).

Since we can naturally take propositions to be zero-adic properties, we can, by the same token, take truth to be monadic exemplification (taking zero-adic properties, i.e. propositions, as its "intended" arguments). Accordingly, I shall informally use the more suggestive "T" instead of "I^1" (cf. [20], p. 457).

Now that the language we are working with encompasses a family of n-adic exemplification relations, let us consider a generalized version of (I):

($\mathbf{I^n}$) $I^{n+1}R^na_1 \ldots a_n \leftrightarrow R^na_1 \ldots a_n$ (where R^n is any n-adic predicate term).

Note that, for $n = 1$,

($\mathbf{I^1}$) $T[\lambda A] \leftrightarrow [\lambda A]$

and, by (λ-conv),

(T) $T[\lambda A] \leftrightarrow A$.

Certainly, (T) is prima facie the most obvious way to capture the notion of truth, yet we should be suspicious about it, because of paradoxes such as the liar. As far as the case of $n > 2$ goes, it is well-known that, for any n, there is an n-adic version of Russell's paradox of predication (see, e.g., [3]) which of course, given ($\mathrm{I^n}$), would be reproducible as a paradox of exemplification. In conclusion, ($\mathrm{I^n}$) as a whole must be abandoned and replaced.

5 A theory of truth and exemplification I shall address Menzel's skepticism about exemplification's having a logical behavior different from the one embodied in (I), and more generally in ($\mathrm{I^n}$), by providing an alternative theory of exemplification that can be mounted on M and that is suggested by rather intuitive distinctions and principles.

Essentially, (I^n) tells us how to treat $I^{n+1}R^n a_1 \ldots a_n$, for any sentence of the form $R^n a_1 \ldots a_n$. I take it that an alternative theory of exemplification should do the same, but, being alternative, not for any sentence of the form $R^n a_1 \ldots a_n$ should this theory treat $I^{n+1}R^n a_1 \ldots a_n$ in the way demanded by (I^n).

We need therefore: (i) a relevant classification of the sentences of $L(MI)$, and (ii) for each kind of sentence resulting from this classification, principles that suggest to us how to treat the members of the kind in question. In other words, these principles should suggest to us whether the members of a given kind should be treated according to (I^n), and, if not, according to which principle.

Ideally, this classification and these principles should have an intuitive value and a *raison d'être* independent of Russell's paradox. The classification I shall provide is based on the notions of a *truth-relevantly complex (simple) sentence* and of a *truth-relevant subsentence.* These notions can be characterized by means of the following examples.

Example 1
(1) Snow is white.

At least prima facie (1) does not appear to contain any subsentence, and thus, a fortiori, it does not contain any subsentence whose truth is relevant to its truth. Hence, intuitively, it should be classified as truth-relevantly simple.

Example 2
(2) That snow is white is a proposition.

Certainly, (1) can be considered a subsentence of (2), but not a truth-relevant one, for the truth of (2) does not in any sense depend on the truth of (1).

Example 3
(3) Snow is not white.

By any reasonable standard, (1) is a subsentence of (3) and, moreover, a truth relevant subsentence, for certainly the truth of (3) depends on whether or not (1) is the case. Accordingly, (1) should be classified as a truth-relevant subsentence of (3), and (3) as truth-relevantly complex.

Example 4
(4) Snow is nonwhite.

Although superficially (4) has a simple subject–predicate structure, it hides a connective in the complex predicate "nonwhite." Intuitively, this makes it truth-relevantly complex in a way analogous to (3), for the truth of (4) depends, in the same way as that of (3), on the truth of (1). This latter sentence should thus be considered a truth-relevant subsentence of (4).

Example 5
(5) That snow is white is true.

Now, (5) is similar to (2) in that both can naturally be taken to have (1) as their only subsentence. However, they differ in that the truth of (1) certainly appears to be relevant to the truth of (5). Accordingly, (1) should be classified as a truth-relevant subsentence of (5) and (5) as truth-relevantly complex.

Example 6
(6) Snow exemplifies (being) white.

As noted, truth can be regarded as monadic exemplification. Hence, for reasons analogous to those of Example 5, and despite the fact that the surface grammatical structure of (6) may not suggest this, (1) can naturally be taken to be a truth-relevant subsentence of (6). The latter should therefore be regarded as truth-relevantly complex.

In the context of $L(MI)$, these intuitive ideas can be made rigorous by means of the following inductive definition:

(a) If A is a sentence of the form $(B \& C)$ then A is a truth-relevantly complex sentence and B and C are its truth-relevant subsentences;
(b) If A is a sentence of the form $-B$, A is a truth-relevantly complex sentence and B is its truth-relevant subsentence.
(c) If A is a sentence of the form $\forall bB$, then A is a truth-relevantly complex sentence and, for any term c such that $B[c/b]$ is a sentence, $B[c/b]$ is a truth-relevant subsentence of A.
(d1) If A is atomic of the form $R^n a_1 \ldots a_n$, R^n is I^n, and $a_1 \ldots a_n$ is a sentence, then A is truth-relevantly complex and $a_1 \ldots a_n$ is its truth-relevant subsentence.
(d2) If A is atomic of the form $[\lambda\, a_1 \ldots a_n B]\, c_1 \ldots c_n$, where B is atomic, then A is truth-relevantly complex and $B[c_1/a_1 \ldots c_n/a_n]$ is its truth-relevant subsentence.
(d3) If A is atomic of the form $[\lambda\, a_1 \ldots a_n -B]\, c_1 \ldots c_n$, then A is truth-relevantly complex and $B[c_1/a_1 \ldots c_n/a_n]$ is its truth-relevant subsentence.
(d4) If A is atomic of the form $[\lambda\, a_1 \ldots a_n (B \& C)]\, c_1 \ldots c_n$, then A is truth-relevantly complex and $B[c_1/a_1 \ldots c_n/a_n]$ and $C[c_1/a_1 \ldots c_n/a_n]$ are its truth-relevant subsentences.
(d5) If A is atomic of the form $[\lambda\, a_1 \ldots a_n \forall bB]\, c_1 \ldots c_n$, then A is truth-relevantly complex and, for any term d such that (i) d is free for b in $A[c_1/a_1 \ldots c_n/a_n]$ and (ii) $A[c_1/a_1 \ldots c_n/a_n][d/b]$ is a sentence, $A[c_1/a_1 \ldots c_n/a_n][d/b]$ is a truth-relevant subsentence of A.
(d6) If A is atomic and does not fall in any of the above cases, then A is truth-relevantly simple.

(I shall say that in case (a), conjunction *makes* A truth-relevantly complex. Mutatis mutandis, I shall use a similar terminology for the other cases, except, of course, for the last one.)

I have thus distinguished between truth-relevantly simple and truth-relevantly complex subsentences of $L(MI)$, and in turn I have distinguished eight kinds of truth-relevantly complex sentences, on the basis of the logical notions of $L(MI)$ generating truth-relevant complexity. Among the logical notions of $L(MI)$ I have included not only connectives and quantifiers, but also, essentially following [1], for any n, n-adic exemplification.

Intuitively, the truth-relevantly simple sentences are such that the only thing we can and need do in order to determine whether or not they are true is, roughly speaking, to "look at" the world as it contingently happens to be. The truth-value of truth-relevantly complex sentences, on the other hand, cannot be determined in a similar way. The laws governing the logical notions become relevant. In fact, these laws tell us how we can try to compute the truth-value of truth-relevantly complex sentences, on the assumption that the contingent world informs us about the truth-value of the appropriate truth-relevantly simple sentences.

These ideas can be captured by means of the following principles:

(T1) The truth of a truth-relevantly complex (nominalized)[6] sentence depends on the truth of its truth relevant (nominalized) subsentences, on the basis of the laws governing the logical notion that makes A truth-relevantly complex.

(T2) The truth of a truth-relevantly simple (nominalized) sentence A depends on whether or not A.[7]

As the above remarks introducing (T1) and (T2) indicate, the nine-folded classification of sentences that I have provided, and the principles (T1) and (T2), are supported by epistemological considerations which are quite independent of Russell's paradox. It is therefore interesting to see which set of axioms they suggest.

Now, (T2) immediately suggests

(T-collapse′) $T[\lambda Ra_1 \ldots a_n] \leftrightarrow Ra_1 \ldots a_n$ (provided R is a primitive predicate constant other than I^n).

Since the laws governing negation, conjunction, and the universal quantifiers are assumed from classical logic, and since lambda-conversion extends the applicability of these laws so as to cover cases (d2)–(d5), (T1) suggests the following principles:

(T-neg′) $T[\lambda -A] \leftrightarrow -T[\lambda A]$.

(T-conj′) $T[\lambda A \;\&\; B] \leftrightarrow T[\lambda A] \;\&\; T[\lambda B]$.

(T-univ′) $T[\lambda \forall aA] \leftrightarrow \forall aT[\lambda A]$.

(T-lambda′) $T[\lambda[\lambda a_1 \ldots a_n A]b_1 \ldots b_n] \leftrightarrow [\lambda a_1 \ldots a_n T[\lambda A]]b_1 \ldots b_n$.

An attempt to use (T1) to understand what axiom(s) should cover Case (d1) seems to take us into a loop, for the laws governing, for any n, n-adic exemplification, are precisely what we are trying to uncover. On the other hand, we have already granted (T-collapse′)–(T-lambda′). This means that we know the answer to our problem at least for some special cases. For example, if the $Ra_1 \ldots a_n$ of (d1) is $T[\lambda -A]$, then given (T-neg′), the truth of $[\lambda -A]$ depends on that of $-T[\lambda A]$, which suggests $T[\lambda\, T[\lambda -A]] \leftrightarrow T[\lambda -T[\lambda A]]$. Furthermore, given (T-neg′), the righthand side of this biconditional is equivalent to $-T[\lambda\, T[\lambda A]]$.

In sum, as regards (d1), (T1) suggests that we generalize on (T-collapse)–(T-univ) as follows:

(T-collapse) $\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} Ra_1 \ldots a_n \underbrace{] \ldots]}_{m \text{ times}} \leftrightarrow Ra_1 \ldots a_n$ (provided R is a primitive predicate constant other than I^n).

(T-neg) $\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} -A \underbrace{] \ldots]}_{m \text{ times}} \leftrightarrow -\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} A \underbrace{] \ldots]}_{m \text{ times}}.$

(T-conj) $\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} A \,\&\, B \underbrace{] \ldots]}_{m \text{ times}} \leftrightarrow (\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} A \underbrace{] \ldots]}_{m \text{ times}}$
$\& \underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} B \underbrace{] \ldots]}_{m \text{ times}}).$

(T-univ) $\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} \forall aA \underbrace{] \ldots]}_{m \text{ times}} \leftrightarrow \forall a \underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} A \underbrace{] \ldots]}_{m \text{ times}}.$

(T-lambda) $\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} [\lambda a_1 \ldots a_n A] b_1 \ldots b_n \underbrace{] \ldots]}_{m \text{ times}} \leftrightarrow$
$[\lambda a_1 \ldots a_n \underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} A \underbrace{] \ldots]}_{m \text{ times}}] b_1 \ldots b_n.$

((T-collapse)–(T-lambda) should appear no less intuitive than their counterparts without iterated truth predicates, for essentially they just add the further information that the logical behavior of iterated occurrences of the truth predicate is the same as that of just a single occurrence.)

At this point, in order to capture all the cases covered by (d1), all we have to do is to establish an intuitively obvious "bridge" between truth and exemplification:

(T/I) $\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} I^{n+1} R^n a_1 \ldots a_n \underbrace{] \ldots]}_{m \text{ times}} \leftrightarrow$
$\underbrace{T[\lambda \ldots T[\lambda}_{m \text{ times}} T[\lambda R^n a_1 \ldots a_n] \underbrace{] \ldots]}_{m \text{ times}}$[8], where R^n is any predicate.

(Compare (T/I) to Turner's (S) in [20], p. 458.)

As Appendix I shows, in the presence of (T/I), (I-collapse) and (I-transfer) below are just a more compact version of (T-collapse)–(T-lambda). Accordingly, I elect to take them as axiom schemes. I shall refer by "*MI*" to the system which results from *M* by adding (T/I), (I-transfer) and (I-collapse) to the axioms of *M*, modulo extension of the language of *M* to $L(MI)$.

(I-collapse) $I^{n+m} \ldots I^{n+1} R a_1 \ldots a_n \leftrightarrow R a_1 \ldots a_n$ (provided: (i) R is a primitive predicate constant, and (ii) R is not I^n).

(I-transfer) $I^{n+m} \ldots I^{n+1} [\lambda b_1 \ldots b_n A] a_1 \ldots a_n \leftrightarrow [\lambda b_1 \ldots b_n A\{I_m\}] a_1 \ldots a_n.$

In general, $A\{I_m\}$ is defined to be the wff which results from A by replacing every atomic subwff $Rc_1 \ldots c_k$ of A which does not occur within a lambda-abstract occurring in A with $I^{k+m} \ldots I^{k+1} Rc_1 \ldots c_k$.

Example 1 If A is $-P^2xy$, $A\{I_1\}$ is $-I^3P^2xy$.

Example 2 If A is $(-I^2fz \,\&\, (\forall g)(Q^3fgx \rightarrow [\lambda x -Px]g))$, $A\{I_2\}$ is $(-I^4I^3I^2fz \,\&\, (\forall g)(I^5I^4Q^3fgx \rightarrow I^3I^2[\lambda x -Px]g))$.

*6 **Exemplification-normal vs. exemplification-abnormal predicates*** We have seen that (T1)-(T3) suggest the principles (T-collapse)-(T/I). It is worth adding that, as far as I can see, in no way do (T1)-(T3) directly suggest (I^n), that is, the axiom which generates a version of Russell's paradox, by granting $I^{n+1}Ra_1 \dots a_n \leftrightarrow Ra_1 \dots a_n$, for any predicate R.

In the absence of a consistency proof, we cannot exclude, of course, that (I^n) is implied by classical logic plus (λ-conv) and (T-collapse)-(T/I). Nevertheless, the Russellian argument, as reconstructed in *MI*, strongly suggests that this is not the case.

Consider (r2) again. It is a theorem of *MI*, yet (I^n) is no longer available to derive (r3) from it. Since "$[\lambda x -I^2xx]$" is not primitive, (I-collapse) is not applicable. Repeated applications of (I-transfer) and lambda-conversion however lead to this chain of equivalences:

(r4) $-I^2[\lambda x -I^2xx][\lambda x -I^2xx] \leftrightarrow I^3I^2[\lambda x -I^2xx][\lambda x -I^2xx]$
(r5) $I^3I^2[\lambda x -I^2xx][\lambda x -I^2xx] \leftrightarrow -I^4I^3I^2[\lambda x -I^2xx][\lambda x -I^2xx]$ etc.[9]

The derivability of (r2) might prima facie appear to be an unfortunate result. Even though it is not a source of inconsistency for *MI*, it is, one might think, a bizarre, if not paradoxical, statement. (r2) might in fact be read as "non-self-exemplification *is non-self-exemplifying* iff non-self-exemplification *does not exemplify non-self-exemplification*".

Far from being unfortunate, this result was quite to be desired, for of course it must be true of any given property *P*, that *P* is non-self-exemplifying iff *P* does not exemplify *P*. Non-self-exemplification itself ($[\lambda x -I^2xx]$) cannot be an exception!

In other words, (r2) is a (logical) truth as trivial as

(Tr) $[\lambda x -I^2xx]$ (red) $\leftrightarrow -I^2$(red,red).

Many philosophers and logicians have pointed out that predicating self-exemplification of itself does not lead to contradiction, but only to trivialities. Hence, they have wondered why predicating, in a parallel way, *non*-self-exemplification of itself leads instead to contradiction. *MI* solves the puzzle by taking both predications to result, democratically, in trivialities.

On the other hand, the "innocuous" character of self-exemplification, representable with "$[\lambda xI^2xx]$," shows up with the fact that "$[\lambda xI^2xx][\lambda xI^2xx] \leftrightarrow I^{n+2} \dots I^2[\lambda x I^2xx][\lambda x I^2xx]$" (for any $n \geq 2$), is a theorem of *MI*. This can be shown by repeated applications of (λ-conv) and (I-transfer) to the left-hand side of the biconditional.

In general, call an atomic sentence $Ra_1 \dots a_n$ of $L(MI)$ *m-exemplification normal* (*m-exemplification abnormal*) iff $\vdash_{MI} I^{n+m} \dots I^{n+1}Ra_1 \dots a_n \leftrightarrow Ra_1 \dots a_n (\vdash_{MI} -(I^{n+m} \dots I^{n+1}Ra_1 \dots a_n \leftrightarrow Ra_1 \dots a_n))$.

Analogously, call a predicate R^n of $L(MI)$ *m-exemplification normal* (*m-exemplification abnormal*) iff $\vdash_{MI} \forall a_1 \dots \forall a_n(I^{n+m} \dots I^{n+1}Ra_1 \dots a_n \leftrightarrow Ra_1 \dots a_n)(\vdash_{MI} -\forall a_1 \dots \forall a_n(I^{n+m} \dots I^{n+1}Ra_1 \dots a_n \leftrightarrow Ra_1 \dots a_n))$.

Given these definitions, the above results show that: (i) for any odd m, "$[\lambda x -I^2xx][\lambda x -I^2xx]$" is m-exemplification abnormal, (ii) for any even m, it is m-exemplification normal, and (iii) for any m, "$[\lambda xI^2xx][\lambda xI^2xx]$" is m-exemplification normal.

Curry's paradox and the liar offer us two more cases of exemplification abnormal predicates and sentences. The reader can verify it by considering the following "Curry" and "liar" sentences:

(C) $[\lambda x I^2 xx \rightarrow A][\lambda x I^2 xx \rightarrow A]$.
(L) $[\lambda \exists y (\forall z$ (this-proposition(z) $\leftrightarrow z = y)$ & $-Ty)]$.[10]

Classes of exemplification-normal predicates and sentences can also be characterized. For example, any predicate R^n of $L(M)$ with no variable in predicate position, or any predicate of L(MI) obtained therefrom by prefixing sequences $I^{p+k} \dots I^{p+1}$ to atomic subwffs $R^p a_1 \dots a_p$ of R^n can be proven to be, for any m, m-exemplification normal. The reader can convince herself of this, by noting that, given

(N) $I^{n+m} \dots I^{n+1}[\lambda a_1 \dots a_n A] b_1 \dots b_n$,

where $[\lambda a_1 \dots a_n A]$ is any such predicate, repeated applications of (I-transfer) and (λ-conv) finally succeed in "transferring" the leftmost occurrence of "$I^{n+m} \dots I^{n+1}$" in (N) exactly to the relevant subwffs $I^{p+k} \dots I^{p+1} R^p c_1 \dots c_p$ of $A[b_1/a_1 \dots b_n/a_n]$, where R^p is a primitive predicate constant other than I^n. At that juncture, (I-collapse) can be applied, thereby getting rid, so to speak, of the sequence $I^{n+m} \dots I^{n+1}$. The precise proof is by induction on the rank of (N), where the rank of a wff is defined essentially along the lines of [17], p. 27. For brevity's and simplicity's sake, I shall omit these technical details.

7 Conclusion The system *MI* embodies a Menzel-style property theory supplemented with a theory of exemplification. For reasons outlined in the introduction, a theory of this kind promises to be a useful framework for natural language semantics and the foundations of mathematics. In addition, *MI* is independently well-motivated and presents a number of intriguing aspects. Efforts must then be directed toward a development of a semantics and a proof of (relative) consistency for it.

A standard way to obtain a relative consistency result for a given formal system is by mapping it into a standard set theory such as Zermelo-Fraenkel's. This method has the further advantage that it may yield an intuitively well motivated set-theoretical semantics which provides further insight into the concepts that the system in question is supposed to model.

In the case of *MI*, an obvious way to proceed is by interpreting its exemplification relations by means of the predication relation(s) of extant property-theoretic systems which already have a formal semantics. However, this task may be more difficult than it appears to be at first glance. For example, systems such as those of Cocchiarella ([8],[9]) are very different in spirit from *MI*, since they restrict lambda-conversion by using the notion of stratification. Contrariwise, *MI* allows lambda-conversion for predicates (e.g., $[\lambda x (\exists f)(f = x \,\&\, Ixx)]$) which in an intuitive sense can be regarded as unstratified, provided that the exemplification predicates are seen as a counterpart of predication.

A system such as that of Jubien [14] does not seem a good candidate either, since, contrary to *MI*, it interprets predication essentially along the lines of a set theory without extensionality.[11]

At first glance, the system of Turner [20] might seem a better candidate, since its theorem "$T[-p(r,r)] \leftrightarrow p(r,r)$"[12] is certainly in the spirit of (r2) above. Nevertheless, Turner's system does not have a counterpart of (T-neg), a principle which essentially allows negation to be moved in and out of truth and exemplification contexts.

The model-theoretic version of Turner's system is based on an extension to predication of the methods used by Gupta [12] and Herzeberger [13] to model the notion of truth. These methods constitute an alternative to those expounded by Kripke [15], although they are in many respects in the same spirit. Hence, a model-theoretic account of property theory can similarly be developed by extending Kripke's account of truth.[13] It is not obvious, however, that this will yield models of *MI*, since Kripke's approach is based on truth-value gaps, whereas *MI* is firmly entrenched in classical logic.

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NOTES

1. Bealer's terminology in [1].
2. Unless of course one introduces specific meaning postulates for each concept involved in a given application, as suggested in footnote 19, p. 57, of [17].
3. Before we proceed, it will be convenient to fix a few conventions. I shall use A, B, C, etc. as metavariables ranging over wffs and R, a, b, c, d, etc. (with or without numerical superscripts or subscripts) as metavariables ranging over predicate and singular terms. Furthermore, I shall assume standard notions of freedom, bondage, and *free for* (for the proper substitution of variables). $A[b_1/a_1, \ldots, b_n/a_n]$ will be understood to be the expression which results from A by simultaneously replacing each free occurrence of a_i in A with b_i $(1 \leq i \leq n)$.

   I shall also assume standard notions of sub-wff, predicate (function), and subject (argument) position. However, since we shall be dealing with complex predicates of the form $[\lambda a_1 \ldots a_n A]$, this perhaps deserves a further comment. If A is an atomic wff of the form $Ra_1 \ldots a_n$, then R has an occurrence in predicate position and each a_i $(1 \leq i \leq n)$ has an occurrence in subject position in A or in any wff B such that A is a subwff of B. From this it does *not* follow that "x" occurs in predicate position in, e.g., "$[\lambda yw\, Qxyw]zz$," even though "x" is embedded in "$[\lambda yw\, Pxyw]$," which in turn occurs in predicate position in "$[\lambda yw\, Pxyw]zz$."

Moreover, it is assumed that in all the systems that I will consider, negation, conjunction, and the universal quantifier are taken as primitives, and the other connectives and the existential quantifier are defined in standard ways.

Finally, it will be convenient to assume that atomic sentences have the form $Ra_1 \ldots a_n$, although for readability I take sometime the liberty to use a notation of the form $R(a_1, \ldots, a_n)$.

4. The restriction to individual variables in (GR) captures the intuition that the range of application of PRPs is fully general (cf. [10], [17]). Given a more liberal version of (GR) wherein this restriction is dropped, Russell's property need not be represented as in (rl) below, since the simpler "$[\lambda f\, -f(f)]$" is available.

5. Similarly, "$[\lambda f\, -f(f)]$" (cf. Note 4 above) cannot be seen as expressing the result of applying negation to a given property.

6. Roughly speaking, nominalization is the grammatical operation which transforms a sentence into a name, thereby making it a possible subject of predication. Hence, in particular, nominalization makes predicating truth of a sentence possible. In English, "that," as prefixed to a sentence, constitutes the typical nominalization device. In a language such as $L(MI)$, "$[\lambda \ldots]$," i.e., a vacuous use of the lambda operator, can play the role of the English "that".

7. In (T1) and (T2) I talk of sentences rather than propositions, mainly for ease of exposition. As a matter of fact, I take propositions rather than sentences to be the primary bearers of truth and falsehood. Of course, there is no problem in recasting (T1) and (T2) so as to eliminate their nominalistic flavor, but it would take some additional machinery to do it rigorously. Note that as a consequence of the nominalistic flavor of (T1), (c) and (d5) above are somehow reminiscent of the substitutional account of quantifiers and of the problems connected with it. These associations would be avoided in a non-nominalistic reformulation of (T1).

8. Contrary to (a)–(d6), (T-collapse)-(T/I) have to do with wffs in general and not just with sentences. This is not an incongruence. Intuitively, the notion of truth applies to sentences, hence in (a)–(d6) I talk of sentences. But typically in a calculus we deal with open wffs in order to carry out certain inferences, hence the restriction to sentences is lifted in (T-collapse)–(T/I).

9. Analogous results can be obtained in *MI* by "self-predicating" lambda-abstracts such as "$[\lambda x\, \exists f(f = x\ \&\ -I^2fx)]$". On the other hand, it is worth noting that, by eliminating the second proviso on (I-collapse), "$I^2[\lambda x\, -I^2xx][\lambda x\, -I^2xx]$," "$I^2[\lambda x\, \exists f(f = x\ \&\ -I^2fx)][\lambda x\, \exists f(f = x\ \&\ -I^2fx)]$," etc. lead to contradiction.

10. In (C), A is any sentence. As regards (L), note that in order to prove that it is exemplification abnormal, we need of course appropriate assumptions concerning the predicate "this-proposition."

11. Although one of the referees mentioned in the acknowledgements has suggested that it should be possible to find a model of MI inside a model of Jubien's impure property theory [14].

12. Where "p" stands for predication and "r" is an abbreviation for the Russellian property as represented in Turner's system.

13. This is implicit in Kripke [15]'s remarks on constructing fixed point models for satisfaction.

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APPENDIX

Theorem $\vdash_{MI}$ (I-transfer) & (I-collapse) $\leftrightarrow$ (T-lambda) & (T-neg) & (T-conj) & (T-univ) & (T-collapse).

Proof: For simplicity, I shall concentrate on the particular case of $m = 1$ (roughly speaking, m iterations of "T" and "I^n" in (I-transfer)-(I-collapse) and (T-lambda)- (T-collapse), respectively), though of course the proof can be generalized in obvious ways.

A. ($\Rightarrow$ direction)

(a) show (T-collapse):
($\Rightarrow$) Assume $T[\lambda\, Ra_1 \ldots a_n]$ (where R is primitive and is not I^n);
$I^{n+1}\, Ra_1 \ldots a_n$ (by (T/I));
$Ra_1 \ldots a_n$ (by (I-collapse)).
($\Leftarrow$) Proceed in reverse order.

(b) Show (T-neg):
($\Rightarrow$) assume $T[\lambda\, -A]$;
$-A\{I_1\}$ (by (I-transfer) and (λ-conv). Note that if $m > 1$, one should first apply (T/I) as many times as needed);
$-T[\lambda\, A]$ (by (λ-conv) and (I-transfer). Again (T/I) would be needed if $m > 1$).
($\Leftarrow$) Proceed in reverse order.

(c) Show (T-conj) and (d) show (T-univ): proceed as in (a).

(d) Show (T-lambda):
($\Rightarrow$) Assume $T[\lambda[\lambda\, a_1 \ldots a_n A]\, b_1 \ldots b_n]$ and proceed as in (a) above so as to get $T[\lambda\, A\,[b_1/a_1 \ldots b_n/a_n]]$. Then apply ($\lambda$-conv) in order to get $[\lambda\, a_1 \ldots a_n\, T[\lambda\, A]]\, b_1 \ldots b_n$.
($\Leftarrow$) Proceed in reverse order.

B. ($\Leftarrow$ direction)

(a′) Show (I-collapse): proceed as in (a) but in reverse order.

(b′) Show (I-transfer): the proof is by induction on the length of A.
1 (base case). A is $R^k c_1 \ldots c_k$.
($\Rightarrow$) Assume $I^{n+1}[\lambda\, b_1 \ldots b_n\, R^k c_1 \ldots c_k]\, a_1 \ldots a_n$;
$T[\lambda\, R^k c_1 \ldots c_k[a_1/b_1 \ldots a_n/b_n]]$ (by (T/I), (T-lambda) and (λ-conv))
$[\lambda\, b_1 \ldots b_n\, I^{k+1} Rc_1 \ldots c_k]\, a_1 \ldots a_n$ (by (T/I) and (λ-conv)).
($\Leftarrow$) Proceed in reverse order.

2 (negation). A is $-B$.
(⇒) Assume $I[\lambda\, b_1 \dots b_n\ -B]\, a_1 \dots a_n$;
$-T[\lambda\, B[a_1/b_1 \dots a_n/b_n]]$ (by (T/I), (T-lambda), (λ-conv) and (T-neg));
$-[\lambda\, b_1 \dots b_n\, T[\lambda\, B]]\, a_1 \dots a_n$ (by (λ-conv);
$-I[\lambda\, b_1 \dots b_n\, B]\, a_1 \dots a_n$ (by (T-lambda) and (T/I));
$-[\lambda\, b_1 \dots b_n\, B\{I_n\}]\, a_1 \dots a_n$ (by the induction hypothesis);
$-B\{I_n\}\, [a_1/b_1 \dots a_n/b]$ (by (λ-conv);
$[\lambda b_1 \dots b_n\ -B\{I_n\}]\, a_1 \dots a_n$ (by (λ-conv).
(⇐) Proceed as above but in reverse order.

3 (conjunction) and 4 (universal quantifier). Proceed as in 2, using (T-conj) and (T-univ) instead of (T-neg).