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A Modal Analog for Glivenko's Theorem and its Applications

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Abstract This paper gives a modal analog for Glivenko's Theorem. It is proved that $(\Box\Diamond A \to \Box\Diamond B) \in K4$ iff $(\Diamond A \to \Diamond B) \in S5$. Some applications of this analog are obtained. A formula ϕ is called an NP-formula if ϕ is built up on its own subformulas of the form $\Box\Diamond B$. It is shown that if ϕ is an NP-formula then the logic $\Lambda + \phi$ is decidable or has the finite model property if $\Lambda \supseteq K4$ and Λ has this property.

Introduction Glivenko [2] proved long ago his remarkable result for the intuitionistic propositional calculus H. Glivenko's Theorem may be formulated in this way: a formula $A \equiv B$ is derived in the classical propositional calculus $C1$ if and only if the formula $\neg A \equiv \neg B$ is proved in H. In this paper an analog of Glivenko's Theorem is found for modal logic. This analog has prompted us to investigate a special class of NP-formulas (which are built up on subformulas of the form $\Box\Diamond A$). It is shown that adding a finite number of such formulas to an arbitrary modal logic containing $K4$ preserves decidability and the finite model property. Some applications of these results are given.

1 We recall that K denotes the minimal normal propositional logic. Let Λ be a modal logic and A be a modal formula. Then $\Lambda + A$ denotes the smallest normal modal logic containing Λ and A. So in these denotations,

$$K4 \leftrightharpoons K + (\Box p \to \Box\Box p)$$
$$S4 \leftrightharpoons K4 + (\Box p \to p)$$
$$S5 \leftrightharpoons S4 + (\Diamond p \to \Box\Diamond p)$$
$$Grz \leftrightharpoons S4 + (\Box(\Box(p \to \Box p) \to p)).$$

Throughout this paper we assume some familiarity with algebraic and Kripke relational semantics for modal logics (see, for example, Rasiowa and Sikorski [4] or Segerberg [8]).

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Theorem 1 (Modal analog of Glivenko's Theorem) *Let A and B be arbitrary modal formulas. Then* $(\Box\Diamond A \to \Box\Diamond B) \in K4$ *iff* $(\Diamond A \to \Diamond B) \in S5$.

Proof: If $(\Box\Diamond A \to \Box\Diamond B) \in K4$ holds, then $(\Diamond A \to \Diamond B) \in K4$ holds as well because $(\Diamond C \leftrightarrow \Box\Diamond C) \in S5$, and $K4 \subseteq S5$.

As is well known, $K4$ has the finite model property (FMP) (see [7]). Therefore it is sufficient for completing the proof of Theorem 1 that if $(\Diamond A \to \Diamond B) \in S5$ then $\Box\Diamond A \to \Box\Diamond B$ is valid in all finite transitive Kripke models. Let $(\Diamond A \to \Diamond B) \in S5$, and let $\boldsymbol{W} = \langle W, R, \phi\rangle$ be a finite transitive Kripke model. Let $\alpha \in W$, and assume that $\alpha \Vdash_\phi \Box\Diamond A$. We proceed to check that $\alpha \Vdash_\phi \Box\Diamond B$.

Assume that $\beta \in W$ and $\alpha R\beta$. First observe that there exists $z \in W$ such that βRz. Indeed otherwise $\beta \nVdash_\phi \Box\Diamond A$ and hence $\alpha \nVdash_\phi \Box\Diamond A$—a contradiction. Since W is finite, there exists a $c \in W$ such that βRc, and if cRd then dRc.

We introduce the set V and the relation R_1 on V by $V \leftrightharpoons \{x \mid x \in W \wedge cRx\}$, and $R_1 \leftrightharpoons R \cap V^2$. The valuation ψ and the model $\boldsymbol{V}$ are defined by $\forall p\ \psi(p) \leftrightharpoons \phi(p) \cap V$, $\boldsymbol{V} \leftrightharpoons \langle V, R_1, \psi\rangle$.

Assume that $V = \varnothing$ then $c \nVdash_\phi \Diamond A$ holds, and then $\alpha \nVdash_\phi \Box\Diamond A$—a contradiction. So $V \neq \varnothing$. It can be easily shown by induction on the length of a formula D that

$$\forall x \in V\ (x \Vdash_\phi D \Leftrightarrow x \Vdash_\psi D). \tag{1}$$

Now if $x \in V$ then xR_1x, and if $x, y \in V$ then xR_1y and yR_1x. This follows directly from the choice of c and from the fact that R is transitive. Thus R_1 is an equivalence relation. Therefore for an arbitrary formula M, if $M \in S5$ then $\langle V, R_1, \psi\rangle \Vdash M$. Taking αRc into account, we conclude that $c \Vdash_\phi \Diamond A$. This, by (1), gives us $c \Vdash_\psi \Diamond A$. By $(\Diamond A \to \Diamond B) \in S5$ and $\langle V, R_1\rangle \Vdash S5$, we obtain $c \Vdash_\psi \Diamond B$. Then $c \Vdash_\phi \Diamond B$ by (1). Therefore $\beta \Vdash_\phi \Diamond B$ holds. Thus it is true that $\alpha \Vdash_\phi \Box\Diamond B$. Hence we have proved that $\langle W, R, \phi\rangle \Vdash (\Box\Diamond A \to \Box\Diamond B)$, which proves the theorem.

In an obvious way, we obtain from Theorem 1 the following:

Corollary 2 *Let A be a formula, then* $\Diamond A \in S5$ *iff* $\Box\Diamond A \vee \Box\Diamond\bot \in K4$.

Indeed $\Diamond A \in S5$ iff $\Diamond T \to \Diamond A \in S5$. The latter is equivalent to $(\Box\Diamond T \to \Box\Diamond A) \in K4$ by Theorem 1, but $(\Box\Diamond T \to \Box\Diamond A) \in K4$ iff $\Box\Diamond A \vee \Box\Diamond\bot \in K4$.

Glivenko's Theorem for H is among the corollaries of Theorem 1. Indeed, let $A \equiv B \in \mathfrak{a}$. Let T be a Gödel translation of a propositional formula into a modal proposition. As is well known (see Dummet and Lemmon [1]) by a translation theorem, $C \equiv D \in \mathfrak{a}$ iff $(\mathrm{T}(C) \leftrightarrow \mathrm{T}(D)) \in S5$ for arbitrary formulas C, D. Since $(\mathrm{T}(A) \leftrightarrow \mathrm{T}(B)) \in S5$ and $(\Diamond\neg\mathrm{T}(A) \leftrightarrow \Diamond\neg\mathrm{T}(B)) \in S5$ holds, by Theorem 1 it is true that $(\Box\Diamond\neg\mathrm{T}(A) \leftrightarrow \Box\Diamond\neg\mathrm{T}(B)) \in K4$. Then $(\Box\neg\Box\mathrm{T}(A) \leftrightarrow \Box\neg\Box\mathrm{T}(B)) \in S4$ by $K4 \subseteq S4$. As is well known (see [1]), $\mathrm{T}(\neg C) \leftrightharpoons \Box\neg\mathrm{T}(C)$ and $(\Box\mathrm{T}(C) \leftrightarrow \mathrm{T}(C)) \in S4$. Therefore $(\mathrm{T}(\neg A) \leftrightarrow \mathrm{T}(\neg B)) \in S4$, and by the Gödel translation theorem (see [1]) $(\neg A \equiv \neg B) \in H$.

2 The derived modal analog of Glivenko's Theorem has a number of applications. We say that a modal propositional formula A is an *NP-formula* if A is obtained from a formula G by substituting formulas of the form $\Box\Diamond D$ for all the propositional variables in G.

Theorem 3 *If Λ is a decidable modal logic containing system $K4$ and A is an NP-formula, then $\Lambda + A$ is also decidable.*

Proof: Let $A \leftrightharpoons A_0(\Box\Diamond D_j)$ and let $B(p_1, \dots, p_n)$ be a formula having no propositional variables not in $\{p_1, \dots, p_n\}$. It can easily be seen that

$$(2) \qquad (B \in \Lambda + A) \Leftrightarrow (((\Box Q_0 \wedge Q_0) \to B) \in \Lambda)$$

for some Q_0 which is a conjunction of instances of A having no propositional variables not in B.

There are only finitely many formulas having only the propositional variables of B up to equivalence in $S5$. Moreover, all of these formulas may be effectively constructed (see Maksimova [3]). Let $\{c_i \mid 1 \le i \le k\}$ denote the set of all these formulas.

We take Q to be the conjunction of all formulas which are obtained from A by all possible replacing of the propositional variables by formulas from $\{c_i \mid 1 \le i \le k\}$. Let us prove that each conjunct R_0 of Q_0 is equivalent in Λ to some conjunct R of Q. Indeed, if

$$R_0 \leftrightharpoons A_0(\Box\Diamond D_j(\alpha_V(p_1, \dots, p_n)))$$

then, by Theorem 1,

$$((\Box\Diamond D_j(\alpha_V(p_1, \dots, p_n))) \leftrightarrow \Box\Diamond D_j(c_{i_V})) \in \Lambda$$

for some $c_{i_V} \in \{c_i \mid 1 \le i \le k\}$. Therefore (2) implies

$$(3) \qquad (B \in \Lambda + A) \Leftrightarrow (((\Box Q \wedge Q) \to B) \in \Lambda).$$

The logic Λ is decidable. Therefore (3) gives the algorithm for recognizing if $B \in \Lambda + A$ holds. This proves the Theorem.

The following theorem regarding FMP is similar to Theorem 3.

Theorem 4 *Let Λ be a modal logic which has FMP and $\Lambda \subseteq K4$. Let A be an NP-formula. Then $\Lambda + A$ has FMP as well.*

Proof: Let $B(p_1, \dots, p_n)$ be a modal formula having only the propositional variables in $\{p_1, \dots, p_n\}$, and let $A \leftrightharpoons A_0(\Box\Diamond D_j)$ where all propositional variables of A are in $\{q_1, \dots, q_m\}$. Assume that $B \notin \Lambda + A$. As we have shown in the proof of Theorem 3 (cf. (3)), $((\Box Q \wedge Q) \to B(p_1, \dots, p_n)) \notin \Lambda$. By the theorem condition, Λ has FMP. Therefore there exists a finite modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \models \Lambda$, but $\mathfrak{A} \not\models (\Box Q \wedge Q) \to B$. Then there exists an n-tuple, $(a_1, \dots, a_n)$, $a_i \in \mathfrak{A}$, such that

$$\mathfrak{A} \not\models \Box Q \wedge Q(a_1, \dots, a_n) \le B(a_1, \dots, a_n).$$

We take $\mathfrak{L}$ to be the subalgebra of the algebra $\mathfrak{A}$ which is generated by $\{a_1, \dots, a_n\}$. Then $\forall c \in \Lambda\ \mathfrak{L} \models c$ holds, and $\mathfrak{L} \not\models \Box Q \wedge Q(a_i) \le B(a_i)$. The filter Δ, where $\Delta \leftrightharpoons \{x \mid x \in \mathfrak{L},\ \Box Q \wedge Q(a_i) \le x\}$, is obviously an I-filter. This means that $\forall \beta \in \mathfrak{L}\ \beta \in \Delta \Rightarrow \Box\beta \in \Delta$. The quotient algebra $\mathfrak{L}/\Delta$ with respect to this I-filter Δ is a homomorphic image of $\mathfrak{L}$. This implies $\forall c \in \Lambda\ \mathfrak{L}/\Delta \vDash c$. By choice of Δ, $B(a_i/\Delta) \neq 1/\Delta$.

Let us prove that $\mathfrak{L}/\Delta \vDash A$. By choice of Δ, we have

$$(\Box Q \wedge Q)(a_i/\Delta) = 1/\Delta. \tag{4}$$

Assume there exist y_ρ $(1 \leq \rho \leq m)$ such that $A(y_\rho/\Delta) \neq 1/\Delta$. Because the $\mathfrak{L}$ is generated by $\{a_1, \ldots, a_n\}$ there exist the terms $t_1, \ldots, t_m$ which are built up on $a_1, \ldots, a_n$ and such that $y_\rho = t_\rho(a_1, \ldots, a_n)$, $1 \leq \rho \leq m$. Then $A(t_\rho(a_1, \ldots, a_n)/\Delta) \neq 1/\Delta$. Hence $A(t_\rho(a_1, \ldots, a_n)) \notin \Delta$ holds. Moreover

$$\forall \rho (t_\rho(p_1, \ldots, p_n) \leftrightarrow c_{i_\rho}(p_1, \ldots, p_n) \in S5$$

where $\{c_i \mid 1 \leq i \leq k\}$ is the set of all formulas up to equivalence in $S5$ with variables in $\{p_1, \ldots, p_n\}$ which was constructed in the proof of Theorem 3.

We replace in $A(t_\rho)$ all t_ρ by the equivalent $S5$ formulas c_{i_ρ} from $\{c_i \mid 1 \leq i \leq k\}$. By Theorem 1, as A is an NP-formula, we obtain

$$A(t_\rho(p_1, \ldots, p_n) \leftrightarrow A(c_{i_\rho}(p_1, \ldots, p_n)) \in K4.$$

Hence $A(c_{i_\rho}(a_1, \ldots, a_n)) \notin \Delta$ holds. Then it is true that $Q(a_1, \ldots, a_n) \notin \Delta$ such as $A(c_{i_\rho}(a_1, \ldots, a_n))$ is a conjunct member of $Q(a_1, \ldots, a_n)$, since $Q(a_1/\Delta, \ldots, a_n/\Delta) \neq 1/\Delta$ holds—a contradiction of (4). Thus $\mathfrak{L} \vDash A$ holds. Moreover $\mathfrak{L}/\Delta \nvDash B$ and $\forall c \in \Lambda\ \mathfrak{L}/\Delta \vDash c$. Thus $\Lambda + A$ has FMP and Theorem 4 is proved.

A number of modal systems have been obtained by adding NP-formulas. For example,

$S4.1 \leftrightharpoons S4 + (\Box\Diamond p \to \Diamond\Box p)$ (see Segerberg [7]),
$S4.2 \leftrightharpoons S4 + (\Diamond\Box p \to \Box\Diamond p)$,
$K4.1 \leftrightharpoons K4 + (\Box\Diamond p \to \Diamond\Box p)$,
$K4.2 \leftrightharpoons K4 + (\Diamond\Box p \to \Box\Diamond\Box p)$ (see [8]).

The theorems which ascribe decidability and FMP to these systems (see [7] and [8]) follow directly from Theorems 3 and 4.

Introduce the formulas $\forall n \in \boldsymbol{N}$:

$$\alpha_n \leftrightarrow \bigwedge_{0 \leq i \leq n} \Diamond\Box p_i \to \bigvee_{i \neq j} \Diamond\Box(p_1 \wedge p_j);$$

$$\beta_n \leftrightarrow \Box\Diamond \bigvee_{0 \leq i \leq n} \Box\left(p_i \to \bigvee_{i \neq j} p_j\right).$$

By Theorems 3 and 4, the logics $Grz + \alpha_n$ and $S4 + \beta_k$ are decidable and have FMP because $S4$ and Grz have these properties (see, for example, [8]).

It has been shown in Theorems 12 and 13 of Rybakov [5] that Theorems 3 and 4 cannot be generalized to the case of adding an infinite but recursive set of NP-formulas. The logic $Grz + \{\theta_i \mid i \geq 1\}$ was constructed in [5], where θ_i are NP-formulas, which is incomplete in Tomason's sense as shown in [9]. This means, in particular, that this logic is incomplete by Kripke (and of course does not possess FMP) and has no finite number of axioms. In passing we note that a modal propositional logic containing $S4$ that is decidable but noncompact by Tomason (i.e., incomplete by Kripke and such that it has no finite number of axioms) has been found by Rybakov [6].

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