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Kripke-Type Semantics for Da Costa's Paraconsistent Logic C_ω

MATTHIAS BAAZ

This paper should be considered as an example of the formal derivation of semantics in connection with propositional logics extending positive logic. These constructions can be applied to many similar problems and should be compared to those proposed in Hacking [4]: The Kripke-type structure of the resulting semantics depends on the previous choice of Gentzen-style formulations. The details of the semantics are read off from the adopted introduction and elimination rules.

In Arruda [1], p. 27, problem 8, the problem is formulated to adapt world semantics to C_ω. The proposed construction of an adequate semantics for C_ω is based on Raggio's Gentzen-type formulation CG_ω of C_ω^* [5], and on the proof-theoretic analysis of intuitionistic logic in Takeuti [6], Section 8.

1 Definition of C_ω and CGP_ω *Define C_ω as usual*

(1) $A \supset (B \supset A)$
(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
(3) $\dfrac{A \quad A \supset B}{B}$
(4) $A \wedge B \supset C$
(5) $A \wedge B \supset B$
(6) $A \supset (B \supset A \wedge B)$
(7) $A \supset A \vee B$
(8) $B \supset A \vee B$
(9) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
(10) $A \vee \neg A$
(11) $\neg\neg A \supset A$.

(Cf. Da Costa [2] or Arruda [1].) Let CGP_ω be the restriction of CG_ω to propositional syntax: CGP_ω corresponds to the propositional part of Gentzen's system LK with

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$\supset$-right replaced by $\supset$-right$'$: $\dfrac{A, \Pi \to B}{\Pi \to A \supset B}$

$\neg$-left replaced by $\neg$-left$'$: $\dfrac{A, \Pi \to \Gamma}{\neg\neg A, \Pi \to \Gamma}$.

Proposition 1

(i) $\vdash_{CGP_\omega} \to A_1 \ldots A_n$ *iff* $\vdash_{C_\omega} A_1 \vee \ldots \vee A_n$

(ii) $\vdash_{CGP_\omega} A_1 \ldots A_n \to B_1 \ldots B_m$ *iff* $\vdash_{C_\omega} (A_1 \wedge \ldots \wedge A_n) \supset (B_1 \vee \ldots \vee B_m)$.

Proof: cf. Raggio [5], p. 360.

2 World semantics for C_ω and CGP_ω

Definition 1 Let a C_ω-Kripke model be defined as the quadruple

$$M = \langle W, \leq, T, \Vdash \rangle$$

such that

I. $(W, \leq)$ is a nonempty partially ordered set. (The objects $p, q, r, \ldots$ in W denote possible worlds.)

II. T is a function which takes as values sets of negated propositional forms such that $p \leq q$ implies $T(p) \subseteq T(q)$. ($T(p)$ denotes the set of negated forms, which are proposed to be true at p independently of the truth-values of its subformulas.)

III. $\Vdash$ is a relation such that

a. $\Vdash^{p} A$ is defined for all $p \in W$ and all propositional A

b. $\Vdash^{p} X$ iff for all q such that $p \leq q$, $\Vdash^{q} X$ (X atomic)

c. $\Vdash^{p} A \wedge B$ iff $\Vdash^{p} A$ and $\Vdash^{p} B$

d. $\Vdash^{p} A \vee B$ iff $\Vdash^{p} A$ or $\Vdash^{p} B$

e. $\Vdash^{p} A \supset B$ iff for all q such that $p \leq q$, $\Vdash^{q} A$ implies $\Vdash^{q} B$

f. Let $\neg^0 A \equiv A$, $\neg^{n+1} A \equiv \neg(\neg^n A)$ for $A \not\equiv \neg B$
$\Vdash^{p} \neg^1 A$ iff $\neg^1 A \in T(p)$ or there is a $q \leq p$ such that $\nVdash^{q} A$
$\Vdash^{p} \neg^{n+2} A$ iff $\neg^{n+2} A \in T(p)$ and $\Vdash^{p} \neg^n A$ or there is a $q \leq p$ such that $\nVdash^{q} \neg^{n+1} A$

g. $\Vdash^{p} A_1 \ldots A_n \to B_1 \ldots B_m$ iff for all q such that $p \leq q$
$\Vdash^{p} A_1$ and $\ldots$ and $\Vdash^{p} A_n$ imply $\Vdash^{p} B_1$ or $\ldots$ or $\Vdash^{p} B_m$.

Definition 2 Let $M = \langle W, \leq, T, \Vdash \rangle$

$M \Vdash A$ iff for all $p \in W$, $\Vdash^{p} A$

$M \Vdash \Pi \to \Gamma$ iff for all $p \in W$, $\Vdash^{p} \Pi \to \Gamma$

$\Vdash A$ iff for all C_ω-Kripke models M, $M \Vdash A$

$\Vdash \Pi \to \Gamma$ iff for all C_ω-Kripke models M, $M \Vdash \Pi \to \Gamma$.

Example 1: Induce M by $W = \{1, 2\}$, $1 < 2$, $T(1) = T(2) = \{\neg X\}$
$\Vdash^{1} A$ iff $A \equiv X$, $\Vdash^{2} A$ iff $A \equiv X, Y, V$ for A atomic;
$M \Vdash X \wedge \neg X$; $\nVdash^{1} (X \wedge \neg X) \supset [(Y \supset (U \vee V)) \supset ((Y \supset U) \vee V)]$.

Proposition 2 $\Vdash^{p} A$ iff for all q such that $p \leq q$ $\Vdash^{q} A$.

Proof: By induction on the length of A, in the case of $\wedge$, $\vee$, $\supset$ corresponding to the proof for the usual intuitionistic Kripke-semantics. $\Vdash^{p} \neg^1 A \Rightarrow \neg^1 A \in T(p)$ or there is a $q \leq p$ such that $\nVdash^{q} A$ $p \leq r \Rightarrow \neg^1 A \in T(r)$, or there is a $q \leq r$ such that $\nVdash^{q} A$ $(s \leq t \Rightarrow T(s) \subseteq T(t)) \Rightarrow \Vdash^{r} \neg^1 A$, the proof in case $\neg^{n+1} A$ similar using the induction hypothesis.

3 Soundness and completeness of C_ω and CGP_ω

Proposition 3 $\nVdash^{p} A \Rightarrow$ *there is no $q \leq p$ such that $\nVdash^{q} \neg A$.*

Proof: $\Vdash^{p} A$ and $(\exists q \leq p)$ $\nVdash^{q} \neg A \Rightarrow (\exists q \leq p)(\forall r \leq q)$ $\Vdash^{r} A \Rightarrow \nVdash^{p} A$ and $\Vdash^{p} A$. Contradiction.

Proposition 4

(i) $(\Vdash^{p} \Pi \to \Gamma, A$ *and* $\Vdash^{p} A, \Pi' \to \Gamma') \Rightarrow (\Vdash^{p} \Pi\Pi' \to \Gamma\Gamma')$ *(validity of the cut rule)*

(ii) $\Vdash^{p} \to A, \neg A$

(iii) $\Vdash^{p} \neg\neg A \to A$.

Proof: (i) and (ii) are obvious. (iii) proceeds as follows:

$$\nVdash^{p} A \Rightarrow \nVdash^{p} A \text{ and } \neg(\exists q \leq p) \nVdash^{q} \neg A \Rightarrow \nVdash^{p} \neg\neg A.$$

Theorem 1 $\vdash_{\overline{CGP_\omega}} \Pi \to \Gamma \Rightarrow \Vdash \Pi \to \Gamma$.

Proof: By induction on the length of the proof in CGP_ω. The case of the inference rules for $\wedge$, $\vee$, $\supset$ is treated similarly to intuitionistic logic (Takeuti [6], Section 8, proposition 8.18). The case of $\neg$-left$'$ and $\neg$-right follows from the proposition using the validity of the cut rule.

Lemma 1 *Let $\Pi \to \Gamma$ be a sequent such that Π contains B or $\neg B$ if $\neg B$ occurs as a subformula of $\Pi \to \Gamma$:*

$$\Vdash \Pi \to \Gamma \Rightarrow \vdash_{\overline{CGP_\omega}} \Pi \to \Gamma$$

without the cut rule.

Proof: Define the reduction tree for $\Pi \to \Gamma$ just as in the classical case (Takeuti [6], p. 44), with the exception that $\supset$-right reduction is omitted and $\neg$-left reduction is replaced by

$\neg$-left$'$ reduction: Let $\neg\neg A_1 \ldots \neg\neg A_n$ be all forms $\neg\neg A$ in Π' in $\Pi' \to \Gamma'$ which have not been reduced:
write $A_1 \ldots A_n, \Pi' \to \Gamma'$ above $\Pi' \to \Gamma'$.

Assume $\Pi \to \Gamma$ is not provable without cuts: The reduction tree for $\Pi \to \Gamma$ contains some branch in which no sequent is provable without cuts. Let $\Pi' \to \Gamma'$ be its topmost sequent (note that $\Delta \subseteq \Delta'$ and $\Psi \subseteq \Psi'$ if $\Delta \to \Psi$ stands below $\Delta' \to \Psi'$) and let $A_1 \ldots A_n$ be all the forms $B_i \supset C_i$ in Γ'. Of course none of B_i, $\Pi' \to C_i$ is provable without cuts. Construct the reduction tree for each B_i, $\Pi' \to C_i$ and iterate this process ω times if possible. Assign to each top sequent some element p and to the lowest top sequent 0. Put $W = \{p/p$ occurs as index$\}$ and put $p \leq q$ iff $(\Pi \to \Gamma)_p \equiv \Pi_p \to \Gamma_p$ occurs below $(\Pi \to \Gamma)_q$. Put $T(p) =$

$\{\neg A / \neg A \in \Pi_p\}$ and put $\Vdash^{p} X$ iff $X \in \Pi_p$ for X atomic. (Note that $p \leq q$ implies $T(p) \subseteq T(q)$ and that $p \leq q$ and $\Vdash^{p} X$ imply $\Vdash^{q} X$.)

Proposition 5 $\quad A \in \Pi_p \Rightarrow \Vdash^{p} A,\ A \in \Gamma_p \Rightarrow \nVdash^{p} A.$

Proof: By induction on the length of A. The case of the outermost operators $\wedge$, $\vee$, $\supset$ is treated as in intuitionistic logic.

$$
\begin{aligned}
&A \equiv \neg^1 B,\ A \in \Pi_p \Rightarrow A \in T(p) \Rightarrow \Vdash^{p} A\\
&A \equiv \neg^{n+2} B,\ A \in \Pi_p \Rightarrow A \in T(p),\ \neg^n B \in \Pi_p \Rightarrow A \in T(p),\\
&\qquad \Vdash^{p} \neg^n B \Rightarrow \Vdash^{p} A\\
&A \equiv \neg B,\ A \in \Gamma_p \Rightarrow A \notin \Pi_q \text{ for } q \leq p \Rightarrow B \in \Pi_q\\
&\qquad \text{for } q \leq p \Rightarrow A \notin T(q),\ \Vdash^{q} B \text{ for } q \leq p \Rightarrow \nVdash^{p} A.
\end{aligned}
$$

This completes the proof of the lemma: If $\Pi \to \Gamma$ is not provable without cuts, there is a C_ω-Kripke model $M = \langle W, \leq, T, \Vdash\rangle$ such that $0 \in W$ and $\nVdash^{0} \Pi' \to \Gamma'$ where $\Pi' \equiv \Pi, \Delta$ and $\Gamma' \equiv \Gamma, \Psi$; i.e., $M \nVdash \Pi \to \Gamma$.

Theorem 2 $\quad \Vdash \Pi \to \Gamma \Rightarrow\ \vdash_{\overline{CGP_\omega}} \Pi \to \Gamma$ *with cuts restricted to subformulas of* $\Pi \to \Gamma$ *(alternatively: without cuts but adding the rule*

$$\text{el: } \frac{A, \Pi \to \Gamma \quad \neg A, \Pi' \to \Gamma'}{\Pi\Pi' \to \Gamma\Gamma'}$$

$\neg A$ *is a subformula of the conclusion).*

Proof: $\Vdash \Pi \to \Gamma$

$\Rightarrow \Vdash \neg B_1^{i_1} \ldots \neg B_n^{i_n} \Pi \to \Gamma$ for each tuple $\langle i_1, \ldots, i_n\rangle$ where $i_j \equiv 0, 1$, and $\neg B_i^0 \equiv B_i$, $\neg B_i^1 \equiv \neg B_i$ and where $\neg B_1 \ldots \neg B_n$ are the negated subformulas of $\Pi \to \Gamma$

$\Rightarrow \vdash_{\overline{CGP_\omega}} \neg B_1^{i_1} \ldots \neg B_n^{i_n}; \Pi \to \Gamma$ without the cut rule. Apply rule el or introduce negations and apply the restricted cut rule.

Corollary 1 $\quad \vdash_{\overline{C_\omega}} A \Leftrightarrow \Vdash A.$

4 C_ω and CGP_ω possess the finite model property

Definition 3 $\quad$ Let $M = \langle W, \leq, T, \Vdash\rangle$ be any C_ω-Kripke model:

$$
\begin{aligned}
\text{Set } [p] &= \{q / \text{for all subformulas } B, \neg C \text{ of } A \Vdash^{p} B \text{ iff } \Vdash^{q} B \text{ and } \neg C \in T(p)\\
&\qquad \text{iff } \neg C \in T(q)\}\\
W_A &= \{[p]/p \in W\}\\
[p] &\leq_A [q] \text{ iff there exist } p' \in [p] \text{ and } q' \in [q] \text{ such that } p' \leq q'\\
T_A([p]) &= \{B/B \in T(p) \text{ and } B \text{ is subformula of } A\}\\
&\Vdash^{[p]} X \text{ iff } \Vdash^{p} X \text{ and } X \text{ occurs in } A \text{ for } X \text{ atomic.}
\end{aligned}
$$

This clearly induces a C_ω-Kripke model $M_A = \langle W_A, \leq_A, T_A, \Vdash\rangle$ which is called the *A-filtration* of M. Note that M_A contains at most 4^n possible worlds if there exist n subformulas of A.

Proposition 6 $\quad \Vdash^{[p]} B$ *iff* $\Vdash^{p} B$ *for all subformulas B of A.*

Proof: By induction on the length of B. The only interesting case concerns subformulas $\neg C$ of A

$B \equiv \neg^1 C$: $\Vdash^{\underline{p}} \neg^1 C \Rightarrow \neg^1 C \in T(p)$ or there exist $q \leq p$ such that $\nVdash^{\underline{q}} C$
$\Rightarrow \neg^1 C \in T([p])$ or there exist $[q] \leq_A [p]$ such that $\nVdash^{\underline{[q]}} C \Rightarrow \Vdash^{\underline{[p]}} \neg^1 C$

$\Vdash^{\underline{[p]}} \neg^1 C \Rightarrow \neg^1 C \in T([p])$ or there exist $[q] \leq_A [p]$ such that $\nVdash^{\underline{[q]}} C$
$\Rightarrow \neg^1 C \in T(p)$ or there exist $q' \in [q]$ and $p' \in [p]$ such that $q' \leq p'$ and $\nVdash^{\underline{q'}} C$
$\Rightarrow \neg^1 C \in T(p)$ or $\Vdash^{\underline{p'}} \neg^1 C$ and $p' \in [p] \Rightarrow \Vdash^{\underline{p}} \neg^1 C$

$B \equiv \neg^{n+1} C$ similar using the induction hypothesis.

Corollary 2 *Assume there are n subformulas of A:*
$\Vdash A$ *iff* $M \Vdash A$ *for all M which possess at most* 4^n *possible worlds and where T is restricted to sets of subformulas of A.*

Proof: $M \nVdash A \Rightarrow$ there is a p such that $\nVdash^{\underline{p}} A \Rightarrow \nVdash^{\underline{[p]}} A \Rightarrow M_A \nVdash A$.

Corollary 3 *C_ω and CGP_ω are decidable.*

Proof: Apply Corollary 2 or Theorem 2. (Of course the necessary reduction of cuts in Theorem 2 can be obtained in a purely syntactical manner using considerations concerning the introduction of negated formulas by axioms of CGP_ω.)

This possible world semantics can easily be extended to C_ω^*, CG_ω, $C_\omega^=$. The only difference is that a reduction rule corresponding to el must be applied at each level because the objects may change from world to world. In addition, the constructed semantics can be used to demonstrate completeness of paraconsistent simple type theory relative to corresponding Henkin structures.

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Technische Universität Wien
Institut für Algebra und Diskrete Mathematik
Wiedner Hauptstrasse 8-10
A-1040 Austria