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Concerning Some Cylindric Algebra Versions of the Downward Löwenheim-Skolem Theorem

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1 Introduction The theory of cylindric algebras (CA's) is the algebraic theory of first-order logics. Several ideas about logic are easier to formulate in the frame of CA theory. Examples are some concepts of Abstract Model Theory (cf. [1]-[4], [8], [11]-[13], [22], and [24]), as well as considerations about relationships between several axiomatic theories of different similarity types (cf. [5]-[7], [9], [14], and [22]). (This second topic is sometimes mentioned under the name "Theory Morphisms", "the category of theories and theory morphisms", or "interpretations".) For these reasons, certain branches of theoretical computer science are based on algebraic logic instead of pure logic (cf. e.g. [9], [14]). For applications of CA theory in computer science, see e.g. [9], [10], [15], [20], and [21].

The connection between logic and CA theory is elaborated in [1], [3], [11], [12], [16], [19], and [22]. The connection between model theory and CA theory is elaborated in [19], [22], [23]. For example, it is proved in [23] that the simple algebraic property of a class K of CA's that all epimorphisms in K are surjective is equivalent to a definability theoretic property of first-order logics (more precisely, model theories) associated to K.

It was found that in general it is the classes Crs_α and Gs_α^{reg} of cylindric set algebras that provide the fundamental link between model theory and CA theory. The CA-theoretic counterparts of the model theoretic notions are usually the fundamental notions of Crs_α (and Gs_α^{reg}) theory. (It is shown in [22] that Crs_α's which are not Gs_α^{reg}'s arise from nonclassical and unusual model theories when the usual process of algebraization is applied to them.) CA theory is much more "algebraic abstract model theory" than "algebraic classical first-order logic". This helps to explain the fact that frequently CA counterparts of classical results are harder to prove than those results: the CA counterparts say that

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a certain property holds for a wide class of logics, not just classical logic. For example, by I.3.18 of [18] we know that the finitary logic ${}_cL_F^t$ of infinitary structures as elaborated in Section VI on p. 36 of [1] (first introduced by L. Henkin in 1955) enjoys the Löwenheim-Skolem-Tarski property and by II.7.14 (p. 254), I.7.25, and II.7.12 of [18] it has this property in the upward direction also.

*2 **The results*** Here we extensively use the notation and terminology of the fundamental textbook [18], or equivalently [19], of cylindric set algebras and their generalizations. Because [18] consists of two parts, the Second Part beginning at p. 131, we shall refer to item n in the First Part of [18] by "I.n" and to item n in the Second Part by "II.n". For example, "I.3.18 of [18]" means item 3.18 in the First Part of [18] (on p. 47) and "II.8.17 of [18]" means item 8.17 in the Second Part of [18] (on p. 289).

Theorem 1 and Proposition 4 below give an exhaustive answer to Problem 5 on p. 310 of [18]. Hence they improve: (I) the algebraic downward Löwenheim-Skolem-Tarski theorem (Theorem I.3.18 of [18], p. 47) and (II) the discussion of this theorem which is item I.3.19 of [18], pp. 50–54. The problems raised in 3 of II.7.13 and in 3 and 5 of II.7.16 of [18] also receive a partial answer in the present paper. Conjecture II.3.9 of [18], p. 175, turns out to be true for Cs_α^{reg} and false for Ws_α, by Theorem 1 below.

Before stating our theorem, we recall Theorem I.3.18 of [18] (or equivalently, Theorem 3.1.45 of [19]) as Theorem 0.

Theorem 0 (Andréka-Monk-Németi) *Let $\mathfrak{A}$ be a Crs_α with unit element V and base U. Let κ be an infinite cardinal such that $|A| \le \kappa$ and $\kappa \le |U|$. Assume $S \subseteq U$ and $|S| \le \kappa$. Then there is a W with $S \subseteq W \subseteq U$ such that $|W| = \kappa$ and:*

(i) *Each of the following conditions (a)–(c) implies that $\langle X \cap^\alpha W : X \in A\rangle$ is a strong ext-isomorphism of $\mathfrak{A}$ onto a Crs_α $\mathfrak{C}$:*

- (a) *$\mathfrak{A}$ is a Ws_α;*
- (b) *$\kappa = \kappa^{|\alpha|}$; then if $\mathfrak{A}$ is a Cs_α it follows that $\mathfrak{C}$ is a Cs_α with base W;*
- (c) *$\mathfrak{A}$ is a regular Gs_α, and $\kappa = \Sigma_{\mu<\lambda}\kappa^\mu$ where λ is the least infinite cardinal such that $|\Delta X| < \lambda$ for all $X \in A$; then $\mathfrak{C}$ is a regular Gs_α with base W, and is a Cs_α if $\mathfrak{A}$ is a Cs_α.*

(ii) *If $\mathfrak{A}$ is a Gws_α, then $\mathfrak{A}$ is ext-isomorphic to a Gws_α with base W.*

(iii) *If $|\alpha| \le \kappa$, then $\mathfrak{A}$ is ext-isomorphic to a Crs_α with base W.* ■

Consider the following two properties (*) and (**) of a Crs_α $\mathfrak{A}$ and a cardinal κ.

(*) $(\exists G \subseteq A)[A = Sg\, G$ and $|G| \le \kappa]$.

(**) $(\exists G \subseteq A)[A = Sg\, G$ and $|G| \le \kappa$ and $(\forall x \in A)\kappa^{|\Delta x|} = \kappa]$.

Note that (*) says that $\mathfrak{A}$ can be generated by $\le\kappa$ elements. Theorem 1, together with Remark 1 below, says that:

(***) The condition $|A| \le \kappa$ can be replaced with (*) in Theorem 0 iff we omit statements (i)(a) and (ii) from the conclusion of Theorem 0.

This cannot be improved by using (**) instead of (*), because the quoted two conclusions also remain false under the stronger assumption (**).

Theorem 1

(I) *Let* $\mathfrak{A} \in Gs_\alpha^{reg}$ *and let* κ *be an infinite cardinal such that* $\kappa \leq |base(\mathfrak{A})|$. *Let* $S \subseteq base(\mathfrak{A})$ *with* $|S| \leq \kappa$. *Assume* $(**)$. *Then there is a* W *with* $S \subseteq W \subseteq base(\mathfrak{A})$ *such that* $|W| = \kappa$ *and* $rl({}^\alpha W): \mathfrak{A} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{C} \in Gs_\alpha^{reg}$ *for some* $\mathfrak{C}$.

(II) *Let* $\alpha \geq \omega^+$. *Then there are* $\mathfrak{A} \in Ws_\alpha \cap Mn_\alpha$ *and* $\kappa = |\kappa| \geq \omega$ *satisfying both* $(*)$ *and* $(**)$ *such that for all* W *if* $|W| \leq \kappa$ *then the conclusions* (i)(a) *and* (ii) *of Theorem 0 fail.*

Remark 1: Before proving Theorem 1, we indicate here how statement $(***)$ above follows from Theorem 1: Conclusion (i)(c) of Theorem 0 trivially follows from Theorem 1(I). Cases (i)(b) and (iii) are easy, since in these cases $\kappa = \kappa^{|\alpha|}$ is assumed and hence $\kappa \geq \alpha$ which by $(*)$ and by $\kappa \geq \omega$ implies $\kappa \geq |A|$. Then, by Theorem 0, we are finished.

Proof of Theorem 1: In the proof we shall need the following notation and Lemmas 2, 3:

Notation: (Recalled from [17]) $x \subseteq| \in y \underset{df}{\Leftrightarrow} \exists z(x \subseteq z \text{ and } z \in y)$.

Lemma 2 (Algebraic version of the Vaught criterion for elementary submodels) *Let* $\alpha \geq 2$, $\mathfrak{A} \in Gs_\alpha^{reg}$ *and* $W \subseteq base(\mathfrak{A})$. *Let* $V = 1^{\mathfrak{A}}$. *Then* $rl({}^\alpha W) \in Hom(\mathfrak{A}, \mathfrak{Sb}(V \cap {}^\alpha W))$ *iff the following Condition* (V) *holds.*

Condition (V) $(\forall x \in A)(\forall i \in \Delta x)(\forall q \in {}^{((2 \cup \Delta x) \sim \{i\})}W)\,[q \subseteq| \in x \Rightarrow q \subseteq| \in (x \cap {}^\alpha W)]$.

To prove Lemma 2, we shall use the following more general lemmas:

Lemma 2.1 (Generalized Vaught criterion) *Let* α *and* $\mathfrak{A} \in Crs_\alpha$ *be arbitrary. Let* $Z \subseteq 1^{\mathfrak{A}}$. *Then* (i) *and* (ii) *below are equivalent.*

(i) $rl_Z \in Hom(\mathfrak{A}, \mathfrak{Sb}Z)$

(ii) $(\forall g \in Z)(\forall x \in A)(\forall i \in \Delta x)\,[\exists a(g_a^i \in x) \Rightarrow \exists a(g_a^i \in Z \cap x)]$.

Proof: Let $V \stackrel{d}{=} 1^{\mathfrak{A}}$. Note that (ii) is equivalent to $Z \cap C_i^V x \subseteq C_i^V(Z \cap x)$ which is the same as $rl_Z(C_i^V x) \subseteq C_i^V rl_Z(x)$ for all x and $i \in \Delta x$. Hence (i) $\Rightarrow$ (ii) is obvious. Assume (ii). Let $i \in \Delta x$. Then $rl_Z(C_i^V x) \subseteq Z \cap C_i^V rl_Z(x) = C_i^Z rl_Z(x)$. Thus $rl_Z(C_i^V x) \subseteq C_i^Z rl_Z(x)$ is proved for $i \in \Delta x$. If $i \notin \Delta x$ then $rl_Z(C_i^V x) = rl_Z(x) \subseteq C_i^Z rl_Z(x)$. Thus $(\forall i \in \alpha) rl_Z(C_i^V x) \subseteq C_i^Z rl_Z(x)$ is proved. The other inclusion $(\supseteq)$ always holds obviously. We proved that rl_Z preserves c_i for all $i \in \alpha$. It is known from BA-theory that rl_Z preserves all the other operations. ■

Lemma 2.2 *Let* $\alpha \geq 2$, $\mathfrak{A} \in (Gws_\alpha^{norm})^{reg}$, $W \subseteq base(\mathfrak{A})$, *and* $V = 1^{\mathfrak{A}}$. *Assume that Condition* (V) *of Lemma* 2 *holds for* $\mathfrak{A}$ *and* W. *Then* $rl({}^\alpha W) \in Hom(\mathfrak{A}, \mathfrak{Sb}(V \cap {}^\alpha W))$.

Proof: Assume the hypotheses. Let $Z \stackrel{d}{=} V \cap {}^\alpha W$. To prove that condition (ii) of Lemma 2.1 holds, let $x \in A$, $i \in \Delta x$, and $g \in Z$ with $g_a^i \in x$. Let $D \stackrel{d}{=} (2 \cup \Delta x) \sim \{i\}$ and $q \stackrel{d}{=} D \upharpoonright g$. Then $q \in {}^D W$ and $q \subseteq| \in x$, hence by Condition (V), $q \subseteq f \in x \cap Z$ for some f. Let $b \stackrel{d}{=} f(i)$. Then $g_b^i \in V$ since V is a Gws_α^{norm}-unit and $f \cap g \supseteq q \neq 0$. Hence $g_b^i \in x$ since x is regular in V and $1 \cup \Delta x \upharpoonright g_b^i \subseteq q \cup \{\langle i, b\rangle\} \subseteq f \in x$. By $b \in W$ and $g \in Z$, $g_b^i \in Z$ proving condition (ii) of Lemma 2.1. Then Lemma 2.1 completes the proof. ■

Proof of Lemma 2: Lemma 2.2 proves the "if" part of Lemma 2 (since $Gs_\alpha \subseteq Gws_\alpha^{norm}$). To prove the "only-if" part, assume the hypotheses of Lemma 2, let $Z \stackrel{d}{=} V \cap {}^\alpha W$ and assume $rl_Z \in Hom(\mathfrak{A}, \mathfrak{Sb}Z)$. To prove Condition (V), let $x \in A$, $i \in \Delta x$, $D \stackrel{d}{=} (2 \cup \Delta x) \sim \{i\}$, $q \in {}^D W$, and $q \subseteq p \in x$. Let $g = q \cup \langle q_0 : j \in \alpha \sim D\rangle$. Then $g \in {}^\alpha W$. By $Rg(g) \subseteq Rg(p)$ and $p \in V$, we have $g \in V$ since V is a Gs_α-unit. Thus $g \in Z$. By $\Delta(C_i^V x) \subseteq D$ and $p \in C_i^V x$, we have $g \in C_i^V x$ by regularity of A. By Lemma 2.1 then $g_b^i \in Z \cap x = x \cap {}^\alpha W$ for some b. Since $i \notin D$, we conclude $q = D \upharpoonright g \subseteq g_b^i \in x \cap {}^\alpha W$ proving the conclusion of Condition (V), as desired. ■

Lemma 3 *Let $\mathfrak{A} \in Gs_q^{reg}$ with base U and let κ be an infinite cardinal such that $\kappa \le |U|$. Assume* (i) *and* (ii) *below.*

(i) $(\exists G \subseteq A)[A = Sg\, G$ *and* $|G| \le \kappa]$.

(ii) $\kappa = \Sigma\{\kappa^\mu : \mu < \lambda\}$ *where λ is the least infinite cardinal such that $|\Delta x| < \lambda$ for all $x \in A$.*

Then for every $S \subseteq U$ with $|S| \le \kappa$ there is a $W \subseteq U$ such that $S \subseteq W$, $|W| = \kappa$ and $rl({}^\alpha W) : \mathfrak{A} \twoheadrightarrow \mathfrak{C} \in Gs_\alpha^{reg}$ for some $\mathfrak{C}$.

Proof: Assume the hypotheses of Lemma 3. Then $(\forall x \in A)|\Delta x| < \kappa$ follows from (ii). Let $H \stackrel{d}{=} \cup \{\Delta x : x \in G\}$. Then $|H| \le \kappa$ by $|G| \le \kappa$.

To see that we may assume $H \subseteq \kappa$, let $\rho: \alpha \rightarrowtail\!\!\!\!\twoheadrightarrow \alpha$ such that $\rho^{-1*}H \subseteq \kappa$. For any $Y \in A$ let $fY = \{y \circ \rho : y \in Y\}$. Then by I.8.1 and I.8.4 of [18], $f: \mathfrak{Rd}^{(\rho)}\mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{B} \in Gs_\alpha^{reg}$. Then f^*G generates $\mathfrak{B}$, and for any $x \in A$, $\Delta^{(\mathfrak{B})} fx = \rho^{-1}\Delta^{(\mathfrak{A})}x$. Hence $\cup\{\Delta^{(\mathfrak{B})} fx : x \in G\} \subseteq \kappa$. Assume that we have established our result for $\mathfrak{B}$. Then $g = rl({}^\alpha W) : \mathfrak{B} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{C} \in Gs_\alpha^{reg}$. Let $f'Y = \{y \in {}^\alpha W : y \circ \rho \in Y\}$ for all $Y \in C$. Then again by I.8.1 and I.8.4 of [18], $f' : \mathfrak{Rd}^{(\rho^{-1})}\mathfrak{C} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{D} \in Gs_\alpha^{reg}$. It is routine to check that $g: \mathfrak{A} \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{D}$ and $f' \circ (B \upharpoonright g) \circ f = A \upharpoonright g$. Hence our result follows. This means that we may indeed assume $H \subseteq \kappa$ (without loss of generality).

If $|\alpha| \le \kappa$ then $|G| \le \kappa$ implies $|A| \le \kappa$ and hence we are finished by (the original) Theorem 0.

Assume therefore $|\alpha| > \kappa$. Then $\beta \stackrel{d}{=} \kappa + \omega \in \alpha$. Let $\mathfrak{N} \stackrel{d}{=} \mathfrak{Nr}_\beta\mathfrak{A}$. Then $G \subseteq N$ by $H \subseteq \kappa \subseteq \beta$. Let $\mathfrak{M} \stackrel{d}{=} \mathfrak{Sg}^{(\mathfrak{N})}G$. Then $|M| \le \kappa$ by $|\beta| = \kappa$. Let $rs_\beta \stackrel{d}{=} \langle\{\beta \upharpoonright q : q \in x\} : x \in M\rangle$. By II.8.17 of [18] (p. 289) then $rs_\beta \in Is(\mathfrak{M}, \mathfrak{B})$ for some $\mathfrak{B} \in Gs_\beta^{reg}$. Clearly, $|B| \le \kappa$ and base$(\mathfrak{B}) = U$. By Theorem 0 then $rl({}^\alpha W) \in Is(\mathfrak{B}, \mathfrak{R})$ for some $\mathfrak{R} \in Gs_\beta^{reg}$ and W with $S \subseteq W \subseteq U$ and $|W| = \kappa$. Then we have

(1) $rl({}^\beta W) \in Ism(\mathfrak{B}, \mathfrak{Sb}(1^{\mathfrak{B}} \cap {}^\beta W))$.

We show that $rl({}^\alpha W) \in Ism(\mathfrak{A}, \mathfrak{Sb}(1^{\mathfrak{A}} \cap {}^\alpha W))$.

Let $x \in A$ be arbitrary. Since $x \in Sg^{(\mathfrak{A})}G$, there exists a finite $L \subseteq \alpha$ such that

(2) $\Delta x \subseteq \beta \cup L$ and $x \in Sg^{(\mathfrak{Rd}(\beta \cup L)^{\mathfrak{A}})}G$.

There exists a permutation $\xi: \alpha \rightarrowtail\!\!\!\!\twoheadrightarrow \alpha$ of α such that $\xi^*\beta = \beta \cup L$ and $\kappa \upharpoonright \xi \subseteq Id$. Then

(3) $x \in Sg^{(\mathfrak{Rd}_\beta \mathfrak{Rd}^\xi \mathfrak{A})}G$.

Let $k \stackrel{d}{=} \langle f \circ \xi : f \in {}^\alpha U\rangle$. Let $V \stackrel{d}{=} 1^{\mathfrak{A}}$. Then $k: {}^\alpha U \rightarrowtail\!\!\!\!\twoheadrightarrow {}^\alpha U$ and $k^*V = V$ by $\mathfrak{A} \in Gs_\alpha$. Let $h \stackrel{d}{=} k^*$. Then $h: SbV \to SbV$ and especially $h: A \to SbV$. By I.8.1 of [18] we have:

(4) $h \in Ism(\mathfrak{Rd}^\xi\mathfrak{A}, \mathfrak{Sb}V)$.

Clearly, $G \restriction h \subseteq Id$ since if $y \in G$ then $\Delta y \restriction \xi \subseteq Id$ and y is regular in a Gs_α. By (3) and (4) then $h(x) \in Sg^{(\mathfrak{Rd}_\beta \mathfrak{Sb} V)} G = Sg^{(\mathfrak{Rd}_\beta A)} G = Sg^{(\mathfrak{N})} G = M$. Let $y \stackrel{d}{=} rs_\beta h(x)$. Then $y \in B$ and $\Delta^{\mathfrak{B}} y = (\xi^{-1})^* \Delta^{\mathfrak{A}} x$, by (4).

To prove that Condition (V) of Lemma 2 is satisfied, let $i \in \Delta x$, $D \stackrel{d}{=} (2 \cup \Delta x) \sim \{i\}$, $q \in {}^D W$, and $p \in x$ such that $q \subseteq p$. Let $j \stackrel{d}{=} \xi^{-1}(i)$ and $E \stackrel{d}{=} (\xi^{-1})^* D$. Then $j \in \Delta^{\mathfrak{B}} y$ and $E = (2 \cup \Delta y) \sim \{j\}$. Further,

(5) $\quad q \circ \xi \in {}^E W$ and $q \circ \xi \subseteq (\beta \restriction (p \circ \xi)) \in rs_\beta h(x) = y$.

Since $\mathfrak{B} \in Gs_\beta^{reg}$, by (1) we can apply Lemma 2 to derive from (5) that $(\exists f \in y \cap {}^\beta W) q \circ \xi \subseteq f$. Then $f \subseteq g \in h(x)$ for some g, by $y = rs_\beta h(x)$. Let $t \stackrel{d}{=} f \cup \langle f_0 \colon i \in \alpha \sim \beta \rangle$. Then $t \in V \cap {}^\alpha W$ because V is a Gs_α-unit. Since $1 \cup \Delta h(x) \restriction t \subseteq f \subseteq g \in h(x)$ and $h(x)$ is regular in $h^*(\mathfrak{Rd}^\xi \mathfrak{A})$ (by I.8.4 of [18]), we have $t \in h(x) \cap {}^\alpha W$ by $t \in V$. Then $t = d \circ \xi$ for some $d \in x \cap {}^\alpha W$. By $q \circ \xi \subseteq f \subseteq t = d \circ \xi$ and $Dog \subseteq \alpha = Dod = Rg\xi$, we conclude $q \subseteq d \in x \cap {}^\alpha W$. By this we proved (V) of Lemma 2. Hence by the choice of x, we conclude that Condition (V) of Lemma 2 is satisfied by $\mathfrak{A}$ and W (for all x). Hence by Lemma 2 we have that $rl({}^\alpha W) \in Hom(\mathfrak{A}, \mathfrak{Sb}(V \cap {}^\alpha W))$.

To prove that $rl({}^\alpha W)$ is one-one on A, assume that $x \neq 0$. Thus $rs_\beta h(x) \neq 0$. By (1) then ${}^\beta W \cap rs_\beta h(x) \neq 0$. Hence there exists a $q \in rs_\beta h(x) \cap {}^\beta W$. Let $t \stackrel{d}{=} q \cup \langle q_0 \colon i \in \alpha \sim \beta \rangle$. Then $t \in {}^\alpha W$. Since V is a Gs_α-unit and $q \subseteq f \in h(x) \subseteq V$ for some f, we have $t \in V \cap {}^\alpha W$. Since $1 \cup \Delta h(x) \subseteq \beta$ (by (2)) and $h(x)$ is regular in V and $\beta \restriction t = q \subseteq f \in h(x)$, we conclude that $t \in h(x)$. Then $t = g \circ \xi$ for some $g \in x$. Since $t \in {}^\alpha W$, also $g \in {}^\alpha W$, hence $g \in x \cap {}^\alpha W$. We have proved that $rl({}^\alpha W)\, x \neq 0$.

So far we have proved $rl({}^\alpha W) \in Ism(\mathfrak{A}, \mathfrak{Sb}(V \cap {}^\alpha W))$. Then there is $\mathfrak{C} \subseteq \mathfrak{Sb}(V \cap {}^\alpha W)$ such that $rl({}^\alpha W) \in Is(\mathfrak{A}, \mathfrak{C})$. By I.3.16 of [18] we have $\mathfrak{C} \in Gs_\alpha^{reg}$. ■

We turn to the proof of Theorem 1: (I) is proved as Lemma 3.

Proof of (II): Let $\alpha \geq \omega^+$. Let $p \stackrel{d}{=} \alpha \restriction Id$. Let $V \stackrel{d}{=} {}^\alpha\alpha^{(p)}$. Let $\mathfrak{A} \stackrel{d}{=} \mathfrak{Mn}(\mathfrak{Sb} V)$. Then $\mathfrak{A} \in Ws_\alpha \cap Mn_\alpha$ and $A = Sg\{0\}$. Let $\kappa \stackrel{d}{=} \omega$. Then $\mathfrak{A}$ and κ satisfy both (*) and (**) above the formulation of Theorem 1. Let W be arbitrary but such that $|W| \leq \kappa$. Then $V \cap {}^\alpha W = 0$ since $(\forall f \in V)|Rgf| \geq |Rgp| = |\alpha| > \kappa \geq |W|$. By $V \neq 0$ then we conclude $rl({}^\alpha W) \notin Is(\mathfrak{A})$, moreover $(\forall y \subseteq {}^\alpha W)\ rl_Y \notin Is(\mathfrak{A})$. By $\mathfrak{A} \in Ws_\alpha$, the assumptions of (i)(a) and (ii) are satisfied. Hence (i)(a) and (ii) fail as was desired. ■

Proposition 4 below completes the discussion (I.3.19 in [18]) of the conditions of Theorem 0. It says, roughly, that statement (iii) cannot be improved.

Proposition 4 *The condition* $|\alpha| \leq \kappa$ *is needed in* (iii). *In fact: Let* $\kappa = |\kappa| \geq \omega$ *and assume* $|\alpha| \nleq \kappa$. *Then there is* $\mathfrak{A} \in Crs_\alpha^{reg}$ *such that* $(\forall W)\,[rl_W \in Is(\mathfrak{A}) \Rightarrow |base(W)| > \kappa]$ *and* $|A| < \kappa < |base(\mathfrak{A})|$ *and* $(\forall x \in A)\ \Delta x = 0$.

Proof: Let $\omega \leq |\kappa| = \kappa < |\alpha|$ and $V = \{\alpha \restriction Id\}$. Let $\mathfrak{A} \stackrel{d}{=} \mathfrak{Sb} V$. Then $|A| = 2 < \kappa$ and $\mathfrak{A} \in Crs_\alpha^{reg}$. By $A = \{0, V\}$, we have $(\forall W \subseteq V)\,[rl_W \in Is(\mathfrak{A}) \Rightarrow V = W]$. Hence $\alpha \subseteq base(W)$ if $rl_W \in Is(\mathfrak{A})$. ■

Problem 5 What are the necessary conditions for Theorem 0 to remain true if we replace the word "ext-isomorphic" with the word "ext-homomorphic"? That is: Let $\mathfrak{A} \in Crs_\alpha$, $\kappa = |\kappa| \geq \omega$, and $S \subseteq base(\mathfrak{A})$ with $|S| \leq \kappa \leq |base(\mathfrak{A})|$. What are the necessary conditions for (I) or (II) below?

(I) $(\exists W \subseteq V)[|base(W)| = \kappa$ and $rl_W \in Hom(\mathfrak{A}, \mathfrak{Sb}\, W)]$

(II) $(\exists W)[|W| = \kappa$ and $rl({}^\alpha W) \in Hom(\mathfrak{A}, \mathfrak{Sb}[V \cap {}^\alpha W])]$.

In this connection we note that in the counterexample on pp. 51_7–52 of [18], the $\mathfrak{A} \in Gws_\alpha^{reg} \cap Lf_\alpha$ with $|A| \leq |\alpha| \leq |base(\mathfrak{A})|$ is such that $[\forall W \subseteq base(\mathfrak{A})]\,(\forall \mathfrak{B} \in Crs_\alpha)\,[|W| = \alpha \Rightarrow rl({}^\alpha W) \notin Hom(\mathfrak{A}, \mathfrak{B})]$. This might suggest that perhaps the only improvement will be that the condition $|A| \leq \kappa$ can be replaced with $\alpha < \kappa^+$.

3 ***Discussion*** Lemmas 2.1, 2.2, and 2 are algebraic versions of the well-known model theoretic Vaught criterion for elementary submodels. Since one of the main motivations for CA theory and Crs_α theory is *to do algebraic logic for first-order logics*, it might be worth reflecting upon these results briefly.

We consider whether the conditions of these three lemmas are needed.

Proposition 6 *For $\alpha \geq 3$, Lemma 2.2 becomes false if we replace Gws_α^{norm} by Crs_α in it. That is, there are $\mathfrak{A} \in Crs_\alpha^{reg}$ and $W \subseteq 1^{\mathfrak{A}}$ satisfying Condition* (V) *of Lemma 2 such that $rl({}^\alpha W) \notin Hom(\mathfrak{A}, \mathfrak{Sb}(V \cap {}^\alpha W))$.*

Proof: Let $\alpha \geq 3$. Let $\bar{n} \overset{d}{=} \langle i\colon i \in \alpha\rangle$ for all n. Let $V \overset{d}{=} \{\bar{1}^{0}_{2}, \bar{1}^{01}_{20}, \bar{3}^{0}_{2}, \bar{3}^{01}_{24}\}$ (see Figure 1). Let $x \overset{d}{=} \{\bar{1}^{01}_{20}, \bar{3}^{01}_{24}\}$ and $W \overset{d}{=} 4$. Let $\mathfrak{B} = \mathfrak{Sb}\, V$ and $\mathfrak{A} \overset{d}{=} \mathfrak{Sg}^{(\mathfrak{B})}\{x\}$. Then $A = \{0, x, V \sim x, V\}$ and $\Delta^{[V]}(x) = \{1\}$. Further, $(\forall y \in A)\Delta(y) \subseteq \{1\}$. It is easy to check that Condition (V) is satisfied by $\mathfrak{A}$ and W. But $rl({}^\alpha W) \notin Hom(\mathfrak{A}, \mathfrak{Sb}(V \cap {}^\alpha W))$ since $C_1^V(x) = V$ but $x \cap {}^\alpha W = \{\bar{1}^{01}_{20}\}$ and $C_1\{\bar{1}^{01}_{20}\} = \{\bar{1}^{0}_{2}, \bar{1}^{01}_{20}\} \neq V \cap {}^\alpha W$. Clearly, $\mathfrak{A}$ is *regular* by $1 \in \Delta x$ and by $|A| = 4$. Hence $\mathfrak{A} \in Crs_\alpha^{reg}$. ∎

Problem 7 Does Lemma 2.2 remain true if we replace Gws_α^{norm} with Gws_α in it?

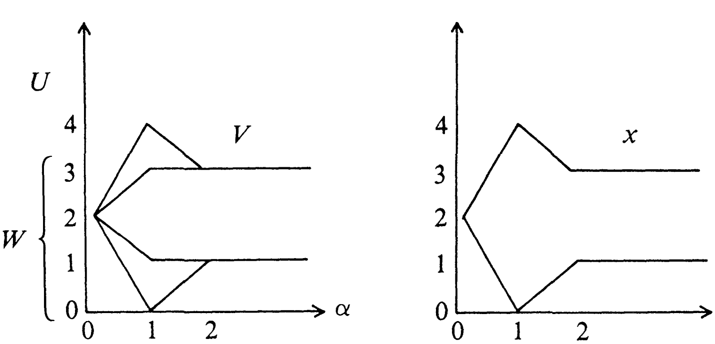

Figure 1.

Proposition 8 *Lemma 2.2 becomes false if we replace* $(Gws_\alpha^{norm})^{reg}$ *by* $Cs_\alpha \cap Lf_\alpha$ *in it. Thus regularity is needed in 2.2 even if we restrict ourselves to the* Cs_α*-case.*

Proof: Let $\alpha \geq \omega$ and $U \overset{d}{=} \omega + \omega$. Let $\bar{n} \overset{d}{=} \langle n: i \in \alpha \rangle$ for all n. Let $Q_n \overset{d}{=} {}^{\alpha}U^{(\bar{n})}$ for all $n \in \omega$. Let $E \overset{d}{=} \{2 \cdot n: n \in \omega\} \cup \{\omega + (2 \cdot n): n \in \omega\}$. $(\forall n \in \omega \sim 1) Y_n \overset{d}{=} \{q \in Q_n: n < q_0 \in E\}$. $Y_0 \overset{d}{=} \{q \in Q_0: \omega < q_0 \in E\}$. $Q_\omega \overset{d}{=} ({}^{\alpha}U) \sim \bigcup\{Q_n: n \in \omega\}$. $Y_\omega \overset{d}{=} \{q \in Q_\omega: q_0 \in E\}$. $x \overset{d}{=} \bigcup\{Y_i: i \in \omega + 1\}$. Let $\mathfrak{B} = \mathfrak{Sb}({}^{\alpha}U)$ and $\mathfrak{A} = \mathfrak{Sg}^{(\mathfrak{B})}\{x\}$. Then $\mathfrak{A} \in Cs_\alpha \cap Lf_\alpha$. Let $W \overset{d}{=} \omega$, $Z \overset{d}{=} {}^{\alpha}W$ and $V \overset{d}{=} {}^{\alpha}U$.

Claim 1 $rl_Z \notin Hom(\mathfrak{A}, \mathfrak{Sb}Z)$.

Proof: Clearly, $\bar{0} \in C_0(x)$ by $\bar{0}_n^0 \notin Y_0 \subseteq x$. Hence $\bar{0} \in rl_Z(C_0 x)$. But $\bar{0} \notin C_0^Z(Z \cap x)$ since $(\forall n \in W)\bar{0}_n^0 \notin x$, by $\bar{0}_n^0 \notin Q_0$ for all $n \in \omega$.

Claim 2 $\mathfrak{A}$ *and* W *satisfy Condition (V).*

Proof: Let $y \in A$ be arbitrary. Then $|\Delta y| < \omega$. Let $i \in \Delta y$ and $D = (2 \cup \Delta y) \sim \{i\}$ as in the formulation of Condition (V). Then $|D| < \omega$. Let $q \in {}^{D}W$ and $q \subseteq p \in y$ be arbitrary. Then $(\exists n \in \omega) q \in {}^{D}n$ by $|D| < \omega$. Let this n be fixed. Let $L \overset{d}{=} 2 \cup \Delta y$ and $k \overset{d}{=} L \upharpoonright p$.

Case 1. Assume $p \in Q_m$ for some $m \in \omega + 1$ with $m > 0$. Now $Q_m \in Zd\mathfrak{B}$, so $rl_{Q_m}: \mathfrak{A} \twoheadrightarrow \mathfrak{Rl}_{Q_m}\mathfrak{A}$ is a homomorphism. Now $\mathfrak{Rl}_{Q_m}\mathfrak{A}$ is generated by $\{x \cap Q_m\} = \{Y_m\}$ and Y_m is regular with $\Delta Y_m = 1$, so by [18] I.4.1, $\mathfrak{Rl}_{Q_m}\mathfrak{A}$ is regular. Let $t = \langle m: \kappa \in \alpha \sim L \rangle$. Since $p \in Q_m \cap y$ and $\Delta(Q_m \cap y) \subseteq \Delta y$ with $\Delta y \upharpoonright p \subseteq k \cup t \in Q_m$, we infer that $k \cup t \in Q_m \cap y$. Note that $(k \cup t)j \in W$ for all $j \neq i$. If $(k \cup t)i \in W$, we have $q \subseteq k \cup t \in y \cap {}^{\alpha}W$ as desired. So suppose $(k \cup t)i \notin W$. Thus $(k \cup t)i = ki \geq \omega$. If $m < \omega$, let $s = max(m,n)$ and let g be the permutation (ki,s) of $\omega + \omega$. Then the base-authomorphism $\tilde{g}$ of $\mathfrak{Sb}Q_m$ fixes Y_m and hence pointwise fixes $\mathfrak{Rl}_{Q_m}\mathfrak{A}$. We have $q \subseteq k_s^i \cup t \in y \cap {}^{\alpha}W$, as desired. The case $m = \omega$ is similar, but one has to distinguish whether ki is in E or not.

Case 2. Assume $p \in Q_0$. Let $f \overset{d}{=} k \cup \langle 0: i \in \alpha \sim L \rangle$. Then $f \in y$ since $rl_{Q_0}: \mathfrak{A} \twoheadrightarrow \mathfrak{Rl}_{Q_0}\mathfrak{A} \in Ws_\alpha = Ws_\alpha^{reg}$. Let $\mathfrak{R}_m \overset{d}{=} \mathfrak{Rl}_{Q_m}\mathfrak{A}$ for all $m \in \omega$. Then $(\forall m \in \omega)\mathfrak{R}_m = \mathfrak{Sg}\{Y_m\} \subseteq \mathfrak{Sb}(Q_m) \in Ws_\alpha$. There is a base-isomorphism $\tilde{b}: \mathfrak{R}_0 \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{R}_n$ induced by some $b: U \rightarrowtail\!\!\!\!\twoheadrightarrow U$ such that $\tilde{b}(Y_0) = Y_n$ and $(n \sim 1) \upharpoonright b \subseteq Id$ (hint: $b^*\{r \in E: r > \omega\} = \{r \in E: r > n\}$, $b_0 = n$, $b_n = 0$, $(n \sim 1) \upharpoonright b \subseteq Id$, $b^*(\{r \in U \sim E: n < r\} \cup \{r \in E: n < r \subseteq \omega\}) = \{r \in U \sim E: n < r\})$. Then the diagram (Figure 2) commutes since $\tilde{b}(rl_{Q_0}(x)) = \tilde{b}(Y_0) = Y_n = rl_{Q_n}(x)$. Hence $\tilde{b}(y \cap$

$$
\begin{tikzcd}
 & \mathfrak{A} \arrow[dl, twoheadrightarrow, "rl(Q_0)"'] \arrow[dr, twoheadrightarrow, "rl(Q_n)"] & \\
\mathfrak{R}_0 \arrow[rr, tail, twoheadrightarrow, "b"] & & \mathfrak{R}_n
\end{tikzcd}
$$

Figure 2.

$Q_0) = y \cap Q_n$. By $f \in y \cap Q_0$, then $b \circ f \in y \cap Q_n \subseteq y$. Thus $(b \circ k) \cup \langle n: i \in \alpha \sim L\rangle = b \circ f \in y \cap Q_n \subseteq y$. Let $F: R_n \rightarrowtail Sb({}^\alpha U)$ be defined such that $(\forall z \in R_n)F(z) \stackrel{d}{=} \{h \in {}^\alpha U: \bar{n}[\Delta z/h] \in z\}$. Since $\mathfrak{R}_n \in Ws_\alpha \cap Lf_\alpha$, there is $\mathfrak{C} \in Cs_\alpha^{reg} \cap Lf_\alpha$ such that $F \in Is(\mathfrak{R}_n,\mathfrak{C})$ and $1^{\mathfrak{C}} = {}^\alpha U$, by II.3.14(iii) of [18] (p. 182). Then $\mathfrak{C} = \mathfrak{Sg}\{F(Y_n)\}$ and $F(Y_n) = \{h \in {}^\alpha U: n < h_0 \in E\}$. Let $d: U \rightarrowtail\!\!\!\!\twoheadrightarrow U$ be such that $(U \sim \{0,n\}) \upharpoonright d \subseteq Id$, $d_0 = n$, and $d_n = 0$. Then $\tilde{d}(FY_n) = FY_n$. Hence $C \upharpoonright \tilde{d} \subseteq Id$. Thus $\tilde{d}(F(y \cap Q_n)) = F(y \cap Q_n)$. By $(b \circ k) \cup \langle n: i \in \alpha \sim L\rangle \in y \cap Q_n \subseteq F(y \cap Q_n)$ we conclude that for some $u \in U$ we have $f_u^i = k_u^i \cup \langle 0: i \in \alpha \sim L\rangle = (d \circ [(b \circ k) \cup \langle n: i \in \alpha \sim L\rangle]) \in F(y \cap Q_n)$ and thus by $\Delta(y \cap Q_n) \subseteq \Delta y \subseteq L$ we have $\bar{n}[L/k_u^i] = \bar{n}[L/f_u^i] \in y \cap Q_n$. Then $k_u^i \cup \langle n: i \in \alpha \sim L\rangle = \bar{n}[L/k_u^i] \in y \cap Q_n \subseteq y$. Since $n > 0$, exactly as it was proved in Case 1, we conclude that $(\exists a \in W)[k_a^i \cup \langle n: i \in \alpha \sim L\rangle] \in (y \cap Q_n) \cap {}^\alpha W$.

Since one of Cases 1 and 2 above always holds (by $p \in \bigcup\{Q_m: m \in \omega + 1\}$) we conclude from $q \subseteq k_a^i$ that $q \subseteq f \in y \cap {}^\alpha W$ for some f. By the choice of y, i, and q, then Condition (V) is proved to hold.

Claims 1 and 2 together complete the proof of Proposition 8. ■

Proposition 9 *Lemma 2 becomes false (in the only-if direction) if we replace Gs_α by Gws_α^{comp} or by Ws_α. Regularity is also needed, even for the Cs_α case. In more detail: Let $V \stackrel{d}{=} 1^{\mathfrak{A}}$ and let $\alpha \geq \omega$. Then* (i)–(iii) *below hold.*

(i) $[\mathfrak{A} \in Ws_\alpha^{reg}$ *and* $rl({}^\alpha W) \in Hom(\mathfrak{A},\mathfrak{Sb}(V \cap {}^\alpha W))] \nRightarrow$ *Condition* (V)

(ii) $[\mathfrak{A} \in (Gws_\alpha^{comp})^{reg}$ *and* $rl({}^\alpha W) \in Ism(\mathfrak{A},\mathfrak{Sb}(V \cap {}^\alpha W))] \nRightarrow$ *Condition* (V)

(iii) $[\mathfrak{A} \in Cs_\alpha$ *and* $rl({}^\alpha W) \in Ism(\mathfrak{A},\mathfrak{Sb}({}^\alpha W))] \nRightarrow$ *Condition* (V).

Proof: Proof of (i): Let $\alpha \geq \omega$ and let $V = {}^\alpha\alpha^{(Id)}$. Let $\mathfrak{A} \stackrel{d}{=} \mathfrak{Mn}(\mathfrak{Sb}V)$. Then $\mathfrak{A} \in Ws_\alpha \cap Lf_\alpha$. Let $W \subseteq \alpha$ be such that $|W| \geq \omega \leq |\alpha \sim W|$ and $0 \in W$. Then $rl({}^\alpha W) \in Ho(\mathfrak{A},\mathfrak{Sb}0)$, since $V \cap {}^\alpha W = 0$. Let $x = V$. Then Condition (V) of Lemma 2 does not hold for this $\mathfrak{A}$, W, and x because of the following. Let $i = 1$ and $D = (2 \cup \Delta^{\mathfrak{A}}(x)) \sim \{i\}$. Then $D = 1$. Let $p = \alpha \upharpoonright Id$ and $q \stackrel{d}{=} \{\langle 0,0\rangle\}$. Then $q \subseteq p \in x$ and $q \in {}^D W$. Hence $x \cap {}^\alpha W = 0$ proves that Condition (V) is not satisfied.

Proof of (ii): Let $\alpha \geq \omega + \omega$. Let $h \stackrel{d}{=} \langle 1: i \in \omega\rangle \cup \langle 0: \omega \leq i \in \alpha\rangle$ and $k \stackrel{d}{=} \langle 2: i \in \omega\rangle \cup \langle 1: \omega \leq i \in \alpha\rangle$. Let $V \stackrel{d}{=} {}^\alpha\omega^{(h)} \cup {}^\alpha\omega^{(k)}$. Let $W \stackrel{d}{=} \omega \sim 1$. Let $\mathfrak{B} \stackrel{d}{=} \mathfrak{Sb}V$. Let $x \stackrel{d}{=} \{q \in V: (\omega \upharpoonright q \subseteq h) \text{ or } (\omega \upharpoonright q \subseteq k)\}$, $\mathfrak{A} = \mathfrak{Sg}^{\langle\mathfrak{B}\rangle}\{x\}$. Then $\Delta x = \omega$. Let $i = 0$. Let $D = (2 \cup \Delta^{\mathfrak{A}}(x)) \sim \{i\}$. Let $q \stackrel{d}{=} \langle 1: i \in \omega \sim 1\rangle$. Then $q \subseteq h \in x$ and $q \in {}^D W$ as in the hypothesis part of Condition (V) but $(\forall f \in x \cap {}^\alpha W)q \nsubseteq f$ since $x \cap {}^\alpha W \subseteq {}^\alpha\omega^{(k)}$. Hence Condition (V) fails. By II.4.6 of [18], p. 190, $\mathfrak{A}$ is regular. Hence $\mathfrak{A} \in (Gws_\alpha^{comp})^{reg}$. Let $K \stackrel{d}{=} {}^\alpha\omega^{(k)}$.

We show $rl_K \in Ism(\mathfrak{A},\mathfrak{Sb}K)$ as follows. Let $H \stackrel{d}{=} {}^\alpha\omega^{(h)}$. Let $b: \omega \rightarrowtail\!\!\!\!\twoheadrightarrow \omega$ be such that $b \circ h = k$. Then $\tilde{b}(H) = K$, hence by I.3.1 of [18], p. 33, we have $\tilde{b}: \mathfrak{Sb}H \to \mathfrak{Sb}K$. Now

(6) $\quad rl_K(x) = \tilde{b}(rl_H(x))$

is easy to see. Let $\mathfrak{A}_k \stackrel{d}{=} rl_K^*\mathfrak{A}$ and $\mathfrak{A}_h \stackrel{d}{=} rl_H^*\mathfrak{A}$. Since x generates $\mathfrak{A}$, from (6) we obtain that the diagram (Figure 3) commutes; that is,

(7) $\quad A \upharpoonright rl_K = A \upharpoonright (\tilde{b} \circ rl_H)$.

$$\begin{tikzcd} & \mathfrak{A} \arrow[dl, "rl_H"'] \arrow[dr, two heads, "rl_K"] & \\ \mathfrak{A}_h \arrow[rr, tail, two heads, "\tilde{b}"] & & \mathfrak{A}_k \end{tikzcd}$$

Figure 3.

Assume that $rl_K \notin Is\mathfrak{A}$. Then $K \cap y = 0$ for some $0 \neq y \in A$. Then $H \cap y \neq 0$ (since $1^{\mathfrak{A}} = H \cup K$). But then $\tilde{b}(rl_H(y)) \neq 0 = rl_K y$, contradicting (7). This proves $rl_K \in Is\mathfrak{A}$, which by $K \in Zd\mathfrak{Sb}V$ implies $rl_K\colon \mathfrak{A} \rightarrowtail \mathfrak{Sb}K$. We have proved $rl_K \in Ism(\mathfrak{A},\mathfrak{Sb}K)$.

Let $\mathfrak{C} \overset{d}{=} rl_K^*\mathfrak{A}$. Then what we proved is $rl_K\colon \mathfrak{A} \twoheadrightarrow\!\!\!\!\rightarrowtail \mathfrak{C} \subseteq \mathfrak{Sb}K$ where $C = Sg^{\mathfrak{C}}[x \cap K]$.

Next we prove

(8) $rl(^{\alpha}W)\colon \mathfrak{C} \rightarrowtail \mathfrak{Sb}(K \cap {}^{\alpha}W)$.

Let $2 < n < \omega$ and let $p_n \overset{d}{=} \{\langle 0,n\rangle,\langle n,0\rangle\} \cup (\omega \sim \{0,n\}) \upharpoonright Id$. Then by $\tilde{p}_n K = K$ and by Theorem I.3.1 of [18] we have $\tilde{p}_n\colon \mathfrak{Sb}K \twoheadrightarrow\!\!\!\!\rightarrowtail \mathfrak{Sb}K$. If $n > 2$ then $\tilde{p}_n(x \cap K) = x \cap K$. Therefore

(9) $C \upharpoonright \tilde{p}_n \subseteq Id$ if $n > 2$.

Let $Z \overset{d}{=} K \cap {}^{\alpha}W$ $(= {}^{\alpha}W^{(k)})$. We show that (ii) of Lemma 2.1 is satisfied for this Z and $\mathfrak{C}$. To see this, let $g \in Z$, $y \in C$, $i \in \Delta y$ be arbitrary but fixed. Assume that $a \in \omega$ is such that $g_a^i \in y$. If $a \in W$ then we are done. Assume $a \notin W$. Then $g_0^i \in y$. Let $n \in W \sim (3 \cup Rg\ g)$. Then by (9) $C \upharpoonright \tilde{p}_n \subseteq Id$. Thus $\tilde{p}_n(y) = y$. Therefore $p_n \circ g_0^i \in y$. Thus $g_n^i = p_n \circ g_0^i \in y$, by $g \in Z \subseteq {}^{\alpha}W$. This proves Lemma 2.1(ii), since $(n \in W \Rightarrow g_n^i \in Z)$. Now by Lemma 2.1

(10) $C \upharpoonright rl_Z\colon \mathfrak{C} \to \mathfrak{Sb}Z$.

It remains to prove that $C \upharpoonright rl(^{\alpha}W)$ is one-one. To see this, let $0 \neq y \in C$. Then $g \in y$ for some g. Let $n \in W \sim Rg\ g$ (clearly exists). Now by (9) $\tilde{p}_n(y) = y$. But $(p_n \circ g) \in Z$ (by $n \in W \sim Rg\ g$), and thus $y \cap Z \neq 0$. Thus $C \upharpoonright rl_Z$ is one-one. This and (10) together imply $C \upharpoonright rl(^{\alpha}W)\colon \mathfrak{C} \rightarrowtail \mathfrak{Sb}Z$. Since $Z = K \cap {}^{\alpha}W$, we have: $y \in C \Rightarrow y \subseteq K \Rightarrow Z \cap y = (^{\alpha}W) \cap y$. Thus $C \upharpoonright rl(^{\alpha}W) = C \upharpoonright rl_Z$. Therefore $C \upharpoonright rl(^{\alpha}W)\colon \mathfrak{C} \rightarrowtail \mathfrak{Sb}Z$. We have proved (8).

Now for every $y \in A$, $rl(^{\alpha}W) \circ rl_K(y) = {}^{\alpha}W \cap y = rl(^{\alpha}W)(y)$ (since $V \cap {}^{\alpha}W \subseteq K$). Thus $A \upharpoonright rl(^{\alpha}W) = A \upharpoonright (rl(^{\alpha}W) \circ rl_K) = (C \upharpoonright rl(^{\alpha}W)) \circ (A \upharpoonright rl_K)$. By (8) and by $rl_K \in Ism(\mathfrak{A},\mathfrak{Sb}K)$, both $C \upharpoonright rl(^{\alpha}W)$ and $A \upharpoonright rl_K$ are isomorphisms, therefore $A \upharpoonright rl(^{\alpha}W)$ is an isomorphism, too. More precisely, we have $rl(^{\alpha}W)\colon \mathfrak{A} \twoheadrightarrow\!\!\!\!\rightarrowtail \mathfrak{Sb}(K \cap {}^{\alpha}W)$. Therefore, $rl(^{\alpha}W) \in Ism(\mathfrak{A},\mathfrak{Sb}(V \cap {}^{\alpha}W))$. We proved $[\mathfrak{A} \in (Gws_{\alpha}^{comp})^{reg}$ and $rl(^{\alpha}W) \in Ism(\mathfrak{A},\mathfrak{Sb}(V \cap {}^{\alpha}W))] \not\Rightarrow$ Condition (V). The condition $\alpha \geq \omega + \omega$ can be replaced by $\alpha \geq \omega$ by the methods of Sec.I.8 and Sec.II.8 of [18].

Proof of (iii): Let $\alpha \geq \omega + \omega$. Let $h \overset{d}{=} \langle 1\colon i \in \omega\rangle \cup \langle 0\colon \omega \leq i \in \alpha\rangle$. Let $U \overset{d}{=} |\alpha|^+$, $V \overset{d}{=} {}^{\alpha}U$, and $Q \overset{d}{=} {}^{\alpha}U^{(h)}$. Let $x \overset{d}{=} \{q \in Q\colon (\forall i \in \omega)\ q_i = 1\} \cup \{q \in V \sim Q\colon (\forall i \in \omega)\ q_i = 2\}$. Let $\mathfrak{A} \overset{d}{=} \mathfrak{Sg}^{(\mathfrak{Sb}V)}\{x\}$. Then $\Delta x = \omega$. Let $W \overset{d}{=} U \sim 1$.

Claim 9.1
(I) $rl({}^{\alpha}W) \in Hom(\mathfrak{A}, {}^{\alpha}W)$
(II) $rl({}^{\alpha}W) \in Is(\mathfrak{A})$.

Proof of Claim 9.1 (I): We prove that condition (ii) of Lemma 2.1 holds for $Z \stackrel{d}{=} {}^{\alpha}W$ and $\mathfrak{A}$. To see this, let $g \in Z$, $y \in A$, $i \in \Delta y$ be arbitrary. It is enough to see that the implication $g_0^i \in y \Rightarrow (\exists b \neq 0)\ g_b^i \in Z \cap y$ holds. Assume $g_0^i \in y$. By $|Rg\ g| \leq \alpha < U = |W|$ we have $Rg\ g \subsetneqq W$ and thus $W \sim (3 \cup Rg\ g) \neq 0$. Let $b \in W \sim (3 \cup Rg\ g)$. Let $S \stackrel{d}{=} {}^{\alpha}U^{(g)}$. Recall that $p_b \stackrel{d}{=} \{\langle 0,b\rangle, \langle b,0\rangle\} \cup (U \sim \{0,b\}) \upharpoonright Id$. Then $\tilde{p}_b(S) = S$. We define $(\forall y \subseteq V)\bar{p}_b(y) \stackrel{d}{=} \tilde{p}_b(y \cap S) \cup (y \sim S)$. Next we show

(11) $\bar{p}_b\colon \mathfrak{Sb}V \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{Sb}V.$

By Theorem I.3.1 of [18], $\tilde{p}_b\colon \mathfrak{Sb}S \rightarrowtail\!\!\!\!\twoheadrightarrow \mathfrak{Sb}S$. $rl(S)$ and $rl(V \sim S)$ form a direct decomposition of $\mathfrak{Sb}V$. By this decomposition, the cone $\langle \tilde{p}_b \circ rl_S, rl(V \sim S)\rangle$ induces the endomorphism $\bar{p}_b$ of (the direct product) $\mathfrak{Sb}V$. It is easy to see that $\bar{p}_b$ is an automorphism, too. We have proved (11). Next we prove

(12) $\bar{p}_b(x) = x.$

Recall that $\bar{p}_b(x) = \tilde{p}_b(x \cap S) \cup (x \sim S)$ and $\tilde{p}_b(x \cap S) = \{p_b \circ f\colon f \in x \cap S\}$. To see $\tilde{p}_b(x \cap S) \subseteq x \cap S$, let $f \in x \cap S$ be arbitrary. Then $f \in {}^{\alpha}U^{(g)}$ from which $(p_b \circ f) \in {}^{\alpha}U^{(g)}$, further $(p_b \circ f) \notin {}^{\alpha}U^{(h)}$ $(= Q)$. Thus $\omega \upharpoonright f \subseteq \bar{2} = \langle 2\colon i \in \alpha\rangle$ and $\omega \upharpoonright (p_b \circ f) \subseteq \bar{2}$, therefore $(p_b \circ f) \in x$. Further $p_b \circ f \in S$. Thus $(p_b \circ f) \in (x \cap S)$. By this we have proved $\tilde{p}_b(x \cap S) \subseteq x \cap S$. From the latter $\tilde{p}_b(x \cap S) = x \cap S$ follows. From this (12) follows by the definition of $\bar{p}_b$.

Now, by (11) and (12) we have $A \upharpoonright \bar{p}_b \subseteq Id$. Thus $g_b^i = p_b \circ g_0^i \in \bar{p}_b(y) = y$ since $g_0^i \in S$ and $g_0^i \in y$ by hypothesis. By this we have proved that condition (ii) of Lemma 2.1 holds for ${}^{\alpha}W$ and $\mathfrak{A}$. Then, by Lemma 2.1, $rl({}^{\alpha}W) \in Hom(\mathfrak{A}, \mathfrak{Sb}({}^{\alpha}W))$.

Proof of Claim 9.1(II): Now let $Z \stackrel{d}{=} V \sim Q$. Then ${}^{\alpha}W \subseteq Z$. First we show that $rl_Z \in Ism(\mathfrak{A}, \mathfrak{Sb}Z)$. Let $k \stackrel{d}{=} \langle 2\colon i \in \omega\rangle \cup \langle 1\colon \omega \leq i \in \alpha\rangle$. Let $b\colon U \rightarrowtail\!\!\!\!\twoheadrightarrow U$ be such that $b \circ h = k$. Then $\tilde{b}\colon \mathfrak{Sb}({}^{\alpha}(U)) = \mathfrak{Sb}({}^{\alpha}U)$ and $\tilde{b}(rl_Q(x)) = (x \cap Z) \cap {}^{\alpha}U^{(k)} = rl({}^{\alpha}U^{(k)})(x)$ (see Figure 4).

$$
\begin{tikzcd}
 & \mathfrak{A} \arrow[d, phantom, "\cap"] & \\
 & \mathfrak{Sb}V \arrow[dl, "rl_Q"'] \arrow[dr, "rl_Z"] & \\
\mathfrak{Sb}Q \arrow[dr, "\tilde{b}"'] & & \mathfrak{Sb}Z \arrow[dl, "rl({}^{\alpha}U^{(k)})"] \\
 & \mathfrak{Sb}({}^{\alpha}U^{(k)}) &
\end{tikzcd}
$$

Figure 4.

$$\begin{tikzcd}
 & \mathfrak{A} \arrow[dl, two heads, "rl_Q"'] \arrow[dr, two heads, "rl({}^{\alpha}U^{(k)})"] & \\
\cdot \arrow[rr, tail, two heads, "\tilde{b}"] & & \cdot
\end{tikzcd}$$

Figure 5

Thus the diagram of Figure 5 commutes; that is

(13) $A \upharpoonright (\tilde{b} \circ rl_Q) = A \upharpoonright rl({}^{\alpha}U^{(k)})$.

Now assume $rl_Z \notin Is\mathfrak{A}$. Then $(\exists y \in A \sim \{0\})\ Z \cap y = 0$. Then $Q \cap y = y \neq 0$ since $Q \cup Z = 1^{\mathfrak{A}}$. Further ${}^{\alpha}U^{(k)} \cap y = 0$. Thus $\tilde{b}(rl_Q(y)) \neq 0 = rl({}^{\alpha}U^{(k)})(y)$, contradicting (13). This proves $rl_Z \in Is\mathfrak{A}$. From this $rl_Z\colon \mathfrak{A} \rightarrowtail \mathfrak{Sb}Z$ follows since Z is zero dimensional in $\mathfrak{Sb}V$. We have proved $rl_Z \in Ism(\mathfrak{A},\mathfrak{Sb}Z)$.

Let $\mathfrak{B} \stackrel{d}{=} \mathfrak{Rl}_Z\mathfrak{A}$. Then $B = Sg^{(\mathfrak{B})}\{x \cap Z\}$. Clearly, $\mathfrak{B} \in Gws_{\alpha}^{comp} \subseteq Gws_{\alpha}^{norm}$. Let $E \stackrel{d}{=} {}^{\alpha}U^{(\bar{2})}$. We show that the conditions of Proposition 4.7 of [18] hold for $Q \stackrel{d}{=} E$ and $G \stackrel{d}{=} \{Z \cap x\}$. Condition (i) trivially holds. (Now $y = Z \cap x$). To see (ii), let $f \in y$ (then $\omega \upharpoonright f \subseteq \bar{2}$) and $q = \bar{2}$. Let $p \in {}^{\alpha}U$ and $\Gamma \subseteq_{\omega} \alpha$ be arbitrary. Now $f[\Gamma/p] \in y \Leftrightarrow (\omega \cup \Gamma) \upharpoonright p \subseteq \bar{2} \Leftrightarrow q[\Gamma/p] \in y$. Thus (ii) holds. Further, y $(= Z \cap x)$ is E-w small in Z. Now by Proposition 4.7(II) of [18] (p. 190) we have $rl_E\colon \mathfrak{B} \rightarrowtail \mathfrak{Sb}E$. Thus, by $\mathfrak{A} \overset{rl_Z}{\twoheadrightarrow} \mathfrak{B} \overset{rl_E}{\rightarrowtail} \mathfrak{Sb}E$, we have

(14) $\mathfrak{A} \overset{rl_E}{\twoheadrightarrow} \mathfrak{C} \subseteq \mathfrak{Sb}E,\ C = Sg^{\mathfrak{C}}\{E \cap x\}$.

Next we prove

(15) $(\forall y \in C)\ y \cap {}^{\alpha}W \neq 0$.

Let $g \in y \in C$. Let $b \in W \sim (3 \cup Rg\ g)$. Then $\tilde{p}_b(E) = E$, and thus by Theorem I.3.1 of [18], $\tilde{p}_b\colon \mathfrak{Sb}(E) \twoheadrightarrow \mathfrak{Sb}(E)$. Therefore $\tilde{p}_b(x \cap E) = x \cap E \Rightarrow \tilde{p}_b(y) = y \Rightarrow p_b \circ g \in y$. Thus, by $p_b \circ g \in {}^{\alpha}W$, we have $y \cap {}^{\alpha}W \neq 0$.

Now by (14) and (15) $(\forall y \in A)\ y \cap {}^{\alpha}W \neq 0$. Thus $rl({}^{\alpha}W) \in Is(\mathfrak{A})$.

By Claim 9.1 we have $rl({}^{\alpha}W) \in Ism(\mathfrak{A},\mathfrak{Sb}({}^{\alpha}W))$.

To see that Condition (V) fails, let $i = 0$ and $D = (2 \cup \Delta^{\mathfrak{A}}(x)) \sim \{i\}$. Then $D = \omega \sim 1$. Let $q = \langle 1\colon i \in D\rangle$. Then $q \in {}^{D}W$ and $q \subseteq h \in x$. Now, $(\forall f \in x \cap {}^{\alpha}W)\ q \nsubseteq f$ proves that Condition (V) is not satisfied by $\mathfrak{A}$, W, x, and i. ■

Proposition 10 *Let $\mathfrak{A} \in Crs_{\alpha}$ and $Z \in A$. Then conditions* (i)–(iii) *below are equivalent:*

(i) $rl_Z \in Hom(\mathfrak{A},\mathfrak{Sb}Z)$

(ii) $rl_Z \in Ho\mathfrak{A}$

(iii) $\Delta Z = 0$.

Proof: Immediate by Theorem 1 and Proposition 2 of [21]. ■

Problems 11 Let $\alpha \geq \omega$ be fixed. Let $\kappa \geq \beta$ be two infinite cardinals.

(1) Let $\kappa \geq 2^{|\alpha \cup \beta|}$. Is ${}_{\beta}Cs_{\alpha}^{reg} \subseteq \mathbf{I}_{\kappa}Cs_{\alpha}^{reg}$?

(2) Let $\mathfrak{A} \in {}_\beta Cs_\alpha^{reg}$, $\kappa \geq |\alpha|^+ \cup (|A| \cap 2^{|\alpha \cup \beta|})$. Is then $\mathfrak{A}$ subisomorphic to some ${}_\kappa Cs_\alpha^{reg}$?

(3) Does there exist a cardinal $\gamma \geq \beta$ such that for every cardinal $\kappa \geq \gamma$ we have ${}_\beta Cs_\alpha^{reg} \subseteq \mathbf{I}_\alpha Cs_\alpha^{reg}$?

(4) Does there exist a cardinal $\gamma \geq \beta$ such that ($\forall$cardinal $\kappa \geq \gamma$) ($\forall \mathfrak{A} \in {}_\beta Cs_\alpha^{reg}$) [$\mathfrak{A}$ is sub-isomorphic to some ${}_\kappa Cs_\alpha^{reg}$] ?

(5) Consider the logic ${}_cL_F^t$ as introduced on p. 36 of [1] with α variables where α is a limit ordinal and $Rgt \subset \alpha$. What are the exact cardinality conditions of the upward Löwenheim–Skolem–Tarski property of ${}_cL_F^t$? For example, let $\mathfrak{A}$ be a model of ${}_cL_F^t$ with $|A| = \beta$. Assume $|t| \leq \kappa$, and $\kappa \geq |\alpha|^+ \cup 2^{|\alpha \cup \beta|}$. Is there an ${}_cL_F^t$-elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ with $|B| = \kappa$? This is not settled by II.7.12(ii) of [18] because here $(\forall \gamma \in Rgt)\ \gamma + \omega \leq \alpha$! That is, the arity of each relation symbol is infinitely smaller than α. E.g. the element in the cylindric algebra of $\mathfrak{B}$ corresponding to a relation symbol $r \in Dot$ should be $t(r)$-regular (with $t(r) + \omega \leq \alpha$).

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