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Tense Trees: A Tree System for K_t

B. J. COPELAND

In this paper Jeffrey's elegant and simple decision procedure for the classical propositional calculus is extended to yield a decision procedure for Lemmon's minimal tense logic K_t. Familiarity with Jeffrey [1] is assumed.

The syntax used is that of McArthur ([3], p. 17), who takes as primitive a stock of present tensed statements, the connectives $\sim$ and $\supset$, the future tense operator F ("it will be the case that"), and the past tense operator P ("it has been the case that"). The operator G ("it will always be the case that") is defined as $\sim F \sim$, and the operator H ("it has always been the case that") as $\sim P \sim$. Letters A, B, C are used to represent arbitrary wffs.

Preamble concerning the axiomatic system K_t Various formulations of Lemmon's system K_t exist; the following is taken from McArthur ([3], p. 18).

Axioms

all truth functional tautologies
$G(A \supset B) \supset (GA \supset GB)$
$H(A \supset B) \supset (HA \supset HB)$
$A \supset HFA$
$A \supset GPA$
GA if A is an axiom
HA if A is an axiom

Rule

modus ponens on $\supset$

K_t is a *minimal* tense logic—a tense logic involving no assumptions concerning the physical properties of time. Logics which do make such assumptions may be obtained by the addition of further axioms to K_t. For example, the addition of the following axioms yields a logic for infinite linear time (Scott [6], p. 2):

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$GA \supset FA$ (forwards infinity); $HA \supset PA$ (backwards infinity); $FFA \supset FA$ (forwards transitivity); $PPA \supset PA$ (backwards transitivity); $PFA \supset (A \vee FA \vee PA)$ (forwards connectedness); $FPA \supset (A \vee FA \vee PA)$ (backwards connectedness). Various other extensions of K_t have been investigated, for example the logics of finite time, of dense time, of relativistic causal time, of discrete time (vide either [4] or [5] for a comprehensive survey). There is a well-known connection between the minimal tense logic K_t and the minimal modal logic T. If we define $\Box A$ as $A \wedge GA$ (the so-called Diodorian definition of necessity) then the theorems of K_t containing no logical symbols other than $\Box$ and truth functional connectives are precisely the theorems of T. (Adopting the so-called Aristotelian definition of necessity, $HA \wedge A \wedge GA$, enlarges the set of such theorems to precisely the theorems of the Brouwersche modal system, itself an extension of T.)

Description of the tree system TK_t The trees of TK_t differ from Jeffrey's propositional truth trees in that every formula occurring in a tree has an index assigned to it. The notation A/i will be used to indicate that formula A carries the index i. Informally A/i may be thought of as asserting that A is true at *time i*. The inference rules of TK_t are as follows.

$(\sim\sim)$
$$\begin{array}{c} \sim\sim A/i \\ | \\ A/i \end{array}$$

$(\supset)$
$$\begin{array}{ccc} & A \supset B/i & \\ \swarrow & & \searrow \\ \sim A/i & & B/i \end{array}$$

$(\sim\supset)$
$$\begin{array}{c} \sim(A \supset B)/i \\ | \\ A/i \\ \sim B/i \end{array}$$

(F)
$$\begin{array}{c} FA/i \\ | \\ A/j\ (i < j) \end{array}$$

(P)
$$\begin{array}{c} PA/i \\ | \\ A/j\ (j < i) \end{array}$$

j must be an index new to the tree. Notice that the strings $(i < j)$, $(j < i)$ are actually part of the conclusions of these rules. They may be thought of informally as recording the stipulation that time i is earlier than (respectively, later than) time j. Strings of this sort will be called *markers*.

$(\sim F)$
$$\begin{array}{c} \sim FA/i \\ | \\ \sim A/j \end{array}$$

$$
\begin{array}{ll}
(\sim P) & \sim PA/i \\
 & \quad | \\
 & \sim A/j
\end{array}
$$

In $(\sim F)$ j is any index for which the marker $(i < j)$ appears in the path to which the addition will be made, in $(\sim P)$ j is any index for which the marker $(j < i)$ appears in the path to which the addition will be made.

To construct a TK_t tree for a given formula, index the formula with an arbitrary time, apply the appropriate rule to the formula, and then to the resulting formulas, and so on. The order in which formulas are dealt with is immaterial. When applying a rule to a formula through which there exist more than one path, write the conclusion of the rule at the bottom of each of these paths *except* in the case of $(\sim F)$ and $(\sim P)$ where the conclusion may be written only in such of these paths as already contain the index occurring in the conclusion. On applying a rule other than $(\sim F)$ or $(\sim P)$ to a formula, place a tick ($\surd$) to the left of the formula. On applying $(\sim F)$ or $(\sim P)$ to a formula write the index occurring in the conclusion of the rule to the left of the formula and tick the index.

A path is *fully grown* iff: (i) the only unticked formulas in the path are sentence letters or negations of sentence letters or of the form $\sim PA$ or $\sim FA$, (ii) every entry in the path of the form $\sim FA/i$ has ticked to its left every index j for which a marker $i < j$ appears in the path, and (iii) every entry in the path of the form $\sim PA/i$ has ticked to its left every index j for which a marker $j < i$ appears in the path. (Notice that condition (ii) is satisfied when no indices are ticked to the left of an entry $\sim FA/i$ provided there are no markers of the form $i < j$ in the path; and similarly for condition (iii).) A tree is fully grown iff all paths in it are fully grown. Notice that it will always require only a finite number of applications of rules to produce a fully grown tree for a formula. A path is *closed* iff it contains a formula and its negation both with the same index. A tree is closed iff every path in it is closed.

Example: To show that $(A \supset \sim A) \supset \sim F\sim P\sim A$ is a theorem of K_t:

$$
\begin{array}{rcl}
\surd & \sim((A \supset \sim A) \supset \sim F\sim P\sim A)/t_0 & \\
 & | & \\
\surd & (A \supset \sim A)/t_0 & \\
\surd & \sim\sim F\sim P\sim A/t_0 & \\
 & | & \\
\surd & F\sim P\sim A/t_0 & \\
 & | & \\
t_0\ \surd & \sim P\sim A/t_1 & (t_0 < t_1) \\
 & | & \\
\surd & \sim\sim A/t_0 & \\
 & | & \\
 & A/t_0 & \\
 & \diagup \quad \diagdown & \\
 & \sim A/t_0 \qquad \sim A/t_0\,. &
\end{array}
$$

Here the rules were applied in the order: $(\sim\supset)$, $(\sim\sim)$, (F) (introducing the index t_1), $(\sim P)$, $(\sim\sim)$, $(\supset)$. Both paths in the tree are closed.

Adequacy A proof of the following is outlined: A is a theorem of K_t (henceforward, $\vdash A$) iff a fully grown TK_t tree commencing with $\sim A$ (henceforward, a $\sim A$-tree) is closed. (We state without proof that if one $\sim A$-tree is closed, all are.)

We utilise the formulation of K_t given by McArthur ([3], p. 18), which has modus ponens as the sole rule of inference.

The negation of every axiom of K_t has a closed tree. So to prove the theorem from left to right it must be shown that if $\sim A$ and $\sim(A \supset B)$ have closed trees so does $\sim B$. To this end consider the rule:

$$(\sim) \qquad \begin{array}{c} \sim B/i \\ \hline \sim A/i \qquad\qquad \sim(A \supset B)/i. \end{array}$$

(It is stipulated that $(\sim)$ may be applied to at most one formula in a path.) A routine induction establishes that if a formula has a closed tree containing applications of $(\sim)$, then the formula has a closed tree containing no application of $(\sim)$.

To prove the theorem from right to left the methods of Kripke [2] are utilised. The stage at which the negation of the formula to be tested is written down and indexed is called the initial stage of the construction; the stage at which the m^{th} rule has been applied (counting downwards from the root of the tree) is the $m + 1^{\text{th}}$ stage. (Thus every stage of a tree is a subtree of the tree, but not vice versa: the $m + 1^{\text{th}}$ stage is the result of making *all* the additions to the m^{th} stage called for by the m^{th} rule.) The index written down at the initial stage is called the initial index.

The (P) rule and the (F) rule will be called the index introducing rules. The n^{th} index of a path at a particular stage is the index introduced by the n^{th} application encountered of the index introducing rules, counting upwards from the bottom of the path at that stage. We describe how to eliminate the n^{th} index of a path at a particular stage (the result of doing this will be set Π_n of formula/index pairs). Let Π_0 be the set of all formula/index pairs occurring in the path at the stage in question. To obtain Π_n from Π_{n-1} make the following changes in Π_{n-1} (supposing the n^{th} index to have been introduced by an application of (F) (alternatively, (P)) to a formula carrying an index i): firstly form the conjunction of all formulas in Π_{n-1} indexed by the n^{th} index; secondly prefix this conjunction by F (alternatively, P); thirdly add the resultant formula to Π_{n-1} and index it by i; fourthly delete from Π_{n-1} all entries bearing the n^{th} index.

Where a path at a particular stage contains m applications of the index introducing rules, only the initial index will occur in the result of eliminating the m^{th} index of the path at that stage. The conjunction of the formulas in this result will be called the characteristic formula (*cf*) of the path at that stage. Finally we define the characteristic formula of a stage as $D_1 \vee \ldots \vee D_k$, where $D_1, \ldots, D_k$ are the characteristic formulas of all the paths at that stage. cf_m will be written for the *cf* of the m^{th} stage. In what follows we will abstract from the order of conjuncts (disjuncts) within conjunctions (disjunctions) occurring in characteristic formulas.

Lemma *Let C be the cf of the initial stage of any TK_t tree, and let C' be the cf of any stage of the tree. Then $\vdash C \supset C'$.*

Proof: The proof proceeds by induction. For illustration we detail just one of the seven cases in the proof of $\vdash cf_n \supset cf_{n+1}$, namely where the n^{th} rule is $(\sim F)$. Call the path in which the upper formula of the n^{th} rule stands at the n^{th} stage Q, let the upper formula be $\sim FA$, and let the upper and lower formulas be indexed by i and j, respectively. Thus the marker $(i < j)$ occurs in Q, and either (i) j was introduced by an application of (F) to a formula indexed by i, or (ii) i was introduced by an application of (P) to a formula indexed by j. (i) Where i is the m^{th} index of Q, the result of eliminating the $m - 1^{\text{th}}$ index of Q will contain a formula FX, say, indexed by i and obtained as a result of eliminating j from Q. Then if cf_n is --- $FX \wedge \sim FA$ ---, cf_{n+1} is --- $F(X \wedge \sim A) \wedge \sim FA$ ---. Hence by substitution of equivalents $\vdash cf_n \supset cf_{n+1}$. (ii) Where $X_1, \ldots, X_k, \sim FA$ are all the formulas indexed by i occurring in the result of eliminating the $m-1^{\text{th}}$ index, and cf_n is --- $P(X_1 \wedge \ldots \wedge X_k \wedge \sim FA)$ ---, then cf_{n+1} is --- $\sim A \wedge P(X_1 \wedge \ldots \wedge X_k \wedge \sim FA)$ ---. Again by substitution of equivalents $\vdash cf_n \supset cf_{n+1}$.

To complete the proof of the theorem, let $D_1, \ldots, D_k$ be the cfs of all the paths through the $\sim A$-tree. Since the tree is closed, each D_i is of the form --- $(B \wedge \sim B)$ ---, where $B \wedge \sim B$ occurs in the scope of nothing but $\wedge$, P, F. An induction establishes that $\vdash D_i \equiv . B \wedge \sim B$ for $1 \leqslant i \leqslant k$. By the lemma $\vdash \sim A \supset . D_1 \vee \ldots \vee D_k$. Whence $\vdash A$ (utilising lemmata 1(f) and 2(b) of McArthur ([3], pp. 67-68)).

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Department of Philosophy
The Queens University of Belfast
Belfast BT7 1NN
Northern Ireland