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# THE POSSIBILITY OF A CONDITIONAL LOGIC

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The current interest in relevant logics suggests the importance of reexamining early work in this area. Routley and Montgomery [2] attempt to prove that no "connective" logic containing either Aristotle's thesis, $\sim(\sim A \rightarrow A)$, or Beothius' thesis, $(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$, can have a suitable interpretation as a conditional[1] logic. One of the systems criticized is that of Angell [1], who presents a formalized logic, shows it consistent, and attempts to show it useful under interpretation as a conditional logic.

This article attempts to show that the criticisms offered by Routley and Montgomery of Angell's system, $\mathbf{P}_{A1}$, in particular, are ineffective.

Let us first examine $\mathbf{P}_{A1}$. The primitive symbols are: parentheses and brackets as grouping devices; $\sim, \cdot, \rightarrow$ as connectives; and $A, B, C, D, A_1, B_1, C_1, \ldots$ as variables. There are standard recursive rules of formation, and standard definitions of $\vee$, $\supset$, and $\equiv$ as abbreviations. The axioms are:

$A_1$ $(B \rightarrow C) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$
$A_2$ $(A \rightarrow B) \rightarrow [(C \cdot A) \rightarrow (B \cdot C)]$
$A_3$ $[A \rightarrow \sim(B \cdot C)] \rightarrow [(B \cdot A) \rightarrow \sim C]$
$A_4$ $[A \cdot (B \cdot C)] \rightarrow [B \cdot (A \cdot C)]$
$A_5$ $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
$A_6$ $\sim\sim A \rightarrow A$
$A_7$ $(A \rightarrow B) \rightarrow \sim(A \cdot \sim B)$
$A_8$ $\sim[(A \cdot B) \cdot \sim A]$
$A_9$ $\sim[A \cdot \sim(A \cdot A)]$
$A_{10}$ $(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$

In the axioms the outer parentheses have been dropped. Such abbreviatory practices will be continued in what follows. Finally, Angell includes four rules of inference:

$R_1$ If $\vdash S$ and $\vdash(S \rightarrow S')$, then $\vdash S'$
$R_2$ If $\vdash S$ and $\vdash S'$, then $\vdash(S \cdot S')$
$R_3$ If $\vdash S$ and if x is a propositional variable occurring in S, then if $S'$ is obtained by replacing all occurrences of x in S by any wff, T, then $\vdash S'$.

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$R_4$ If $\vdash$S and S′ is obtained by replacing any part, or all of S, by an expression equivalent through rules of abbreviation; then $\vdash$S′.

There are two parts to the criticisms offered to conditional interpretations of connective logics by Routley and Montgomery.

The first part shows that various extensions of a system they refer to as strong normal implication, $\mathbf{Z}_1$, are inconsistent. This part causes no difficulty for $\mathbf{P}_{A1}$, however, since Angell has discovered a set of matrices which prove the consistency of $\mathbf{P}_{A1}$.

| $\sim A$ | |
|---|---|
| 3 | 0 |
| 2 | 1 |
| 1 | 2 |
| 0 | 3 |

| $A \cdot B$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 0 | 3 | 2 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 3 | 2 | 3 | 2 |
| 3 | 2 | 3 | 2 | 3 |

| $A \rightarrow B$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 2 |
| 1 | 2 | 1 | 2 | 3 |
| 2 | 1 | 2 | 1 | 2 |
| 3 | 2 | 1 | 2 | 1 |

Using 0 and 1 as designated values, all axioms take on a designated value under every possible assignment, and the rules of inference preserve designated values. Since $\sim$ is the symbol for negation we can easily see that Angell's $\mathbf{P}_{A1}$ is simply consistent since if $\vdash \sim A$, $\sim A$ has the value 0 or 1 and $A$ must then have the value 3 or 2 indicating that it is not a theorem.

The second part of their argument depends upon the claim that the following are necessary conditions for any adequate conditional interpretation of a sentential logic:

Condition I The logic must avoid a sufficiently large class of paradox principles.

Condition II The logic must allow alternative incompatible antecedent conditions for the truth of some subjunctive conditionals.

Condition III Conjunctions must reduce as exemplified by the requirement that $(A \cdot A) \rightarrow A$ is a theorem if and only if $A \rightarrow A$ is a theorem.

*CONDITION* I Condition I is well taken; the truth of conditionals does not depend solely on the truth or falsity of their components, so certainly the paradoxes of material implication should be avoided. The paradoxes of strict implication also offend our intuitions about the nature of conditional statements; otherwise we would not call them paradoxes. $\mathbf{P}_{A1}$ is exemplary in this respect; the paradox principles just cited are indeed excluded.

When the material implication sign is used, a false antecedent will lead to any conclusion, but $\sim A \rightarrow (A \rightarrow B)$ is shown not to be a theorem of $\mathbf{P}_{A1}$ by using the consistency matrices and assigning the values 0 and 1 to $A$ and $B$ respectively. The expression then takes on the value 2 which is not a designated value. Therefore, the expression cannot be derived from the axioms of $\mathbf{P}_{A1}$, which all take on designated values, using only the rules of $\mathbf{P}_{A1}$, under which designated values are inheritable. The mere fact that the antecedent of a conditional is false does not ensure the truth of that conditional in $\mathbf{P}_{A1}$.

Similarly $A \rightarrow (B \rightarrow A)$ can be shown not to be a theorem by assigning 0 to both $A$ and $B$. Under these conditions the expression takes on the value 2, which is not a designated value, showing the expression not to be a theorem of $\mathbf{P}_{A1}$. But with the material conditional sign, the truth of the consequent does ensure the truth of the entire statement.

An examination of the conjunction matrix will show that the expression $A \cdot \sim A$ will take on the value 2 under all conditions; from this $(A \cdot \sim A) \rightarrow B$ can be shown not to be a theorem by assigning $B$ either the value 1 or 3. Therefore, the paradox that an impossible statement implies any statement, which holds in both the system of material implication and of strict implication, does not hold for $\mathbf{P}_{A1}$.

Similarly, $\sim(A \cdot \sim A)$ will always have the value 1 and $B \rightarrow \sim(A \cdot \sim A)$ can be shown not to be a theorem by assigning $B$ either the value 0 or 2. Clearly a conditional statement with a necessary statement as its conclusion can be false in $\mathbf{P}_{A1}$, avoiding this paradox which applies to systems of both strict implication and material implication.

With respect to the list of common paradoxes, then, $\mathbf{P}_{A1}$ avoids them all. Indeed the only possible paradox principle which Routley and Montgomery were able to suggest was $(A \cdot A) \rightarrow [(B \cdot B) \rightarrow (A \cdot A)]$, which is hardly the most memorable of paradox principles. At that, the principle is only a "possible" principle in that it is not excluded from the set of theorems of $\mathbf{P}_{A1}$ by the particular set of matrices used by Angell to prove the consistency of the axiom system. No proof exists that it is a theorem of the system and even if it were, it seems clear that $\mathbf{P}_{A1}$ would still meet Condition I by avoiding "a sufficiently wide class of paradox principles". Thus $\mathbf{P}_{A1}$ meets Condition I.

*CONDITION* II Condition II is also well-taken, but in their discussion of it Routley and Montgomery make some serious errors in logical interpretation. They state that their general case is that a logic which has $(A \rightarrow B) \rightarrow \sim(\sim A \rightarrow B)$ as a theorem "undermines" $C \rightarrow B$ whenever $C \rightarrow \sim A$. At least two problems must be cleared up here. First, what has this principle to do with $\mathbf{P}_{A1}$? It is not Boethius' theses (which appears as $A_{10}$ in $\mathbf{P}_{A1}$) though it does look similar. Is it indeed a theorem of $\mathbf{P}_{A1}$? That it is, can be shown as follows:[2]

*RMa* $[(A \rightarrow B) \rightarrow \sim(\sim B \rightarrow A)] \rightarrow [(\sim B \rightarrow A) \rightarrow \sim(A \rightarrow B)]$
$[A_5, A/A \rightarrow B, B/\sim B \rightarrow A]$

*RMb* $(\sim B \rightarrow A) \rightarrow \sim(A \rightarrow B)$
$[*90; RMa; R_1]$

*RMc* $[(\sim A \rightarrow B) \rightarrow (\sim B \rightarrow A)] \rightarrow [(\sim A \rightarrow B) \rightarrow \sim(A \rightarrow B)]$
$[A_1, A/\sim A \rightarrow B, B/\sim B \rightarrow A, C/\sim(A \rightarrow B); RMb; R_1]$

*RMd* $(\sim A \rightarrow B) \rightarrow \sim(A \rightarrow B)$
$[*19, A/B, B/A; RMc; R_1]$

*RMe* $(A \rightarrow B) \rightarrow \sim(\sim A \rightarrow B)$
$[A_5, A/\sim A \rightarrow B, B/A \rightarrow B; RMd; R_1]$

The question of its relation to $\mathbf{P}_{A1}$ now being clear, we can proceed to the second problem. This problem is what precisely is "undermined" by this relationship. The mere fact that both the antecedent and consequent are true by no means ensures that there is a conditional relationship between them. The fact that my desk lamp is now on and the sun is shining does not mean that if my desk lamp is on, then the sun is shining. Thus $C$ could be true and therefore $B$ also true and $A$ false, and yet $\sim(\sim A \rightarrow B)$ could be true. It is therefore not clear how this relationship "undermines" anything. Unfortunately, the examples which they offer to clarify this situation contain errors of their own.

Routley and Montgomery offer the following sentences:

*S1* If Hitler had invaded England in 1940 ($P$), then Germany would have won the Second World War ($R$).

*S2* If Hitler had not invaded England in 1940, but had dropped atomic bombs on England in 1941 ($Q$), then Germany would have won the Second World War ($R$).

*S3* It is not the case that if Hitler had not invaded England in 1940, then Germany would have won the Second World War.

They suggest that *S3* follows from *S1* by $\sim[(A \rightarrow B) \cdot (\sim A \rightarrow B)]$ which is derivable in system $\mathbf{P}|_{A1}$. Thus:

*S1* is $P \rightarrow R$
*S2* is $Q \rightarrow R$
*S3* is $\sim(\sim P \rightarrow R)$

which follows from *S1* by $\sim[(A \rightarrow B) \cdot (\sim A \rightarrow B)]$ (and DeMorgan and Disjunctive Syllogism). But, the argument goes, *S3* conflicts with *S2* since *S2* is $Q \rightarrow R$ and *S3* is $\sim(\sim P \rightarrow R)$. Since the conflict is not immediately obvious, let us examine further.

Remember that *S1* and *S2* were written so that $P \rightarrow \sim Q$ would hold (it was intended by Routley and Montgomery to demonstrate Condition II dealing with incompatible antecedent conditions); but was that intention fulfilled? We might represent $P$ as $H$ (Hitler had invaded) and $Q$ as $\sim H \cdot D$ (Hitler did not invade but dropped atomic bombs). The claim that $P \rightarrow \sim Q$ now becomes the claim that $H \rightarrow \sim(\sim H \cdot D)$. But this is of the form $\sim A \rightarrow \sim(A \cdot B)$ which is *not* a theorem of $\mathbf{P}|_{A1}$, as Angell points out [1], p. 341. Thus the attack on $\mathbf{P}|_{A1}$ as failing Condition II is cut short before it even starts.

One might want to claim, however, that $\sim A \rightarrow \sim(A \cdot B)$ *ought* to be a theorem of an adequate conditional logic, because it catches a significant feature of the way we use conditionals. Nothing could be further from the truth. In fact, the conditionals cited as counterexamples are examples of what Nelson Goodman calls semi-conditionals, which assert, not a *connection* between antecedent and consequent, but the *lack* of a connection. If a match would light whether we struck it or not (The temperature was at its flash point), we would not say "If you were to strike that match it would light"; we would rather say "Even if you didn't strike that match it would light", thereby denying any connection between striking the match and its lighting. In the case of Germany, we would not say "If Hitler had the A-bomb and didn't invade England, he would have won the war", but would rather say "If Hitler had the A-bomb he would have won the war even if he didn't invade England''. But let us go further and imagine that if Hitler didn't drop the atomic bomb and invaded England, he would win; and if he did drop the atomic bomb and didn't invade, he would win; but for some reason if he dropped atomic bombs *and* invaded England, he would lose. Notice that in this case the person who declared ''If Hitler had invaded England, he would have won'' could properly be challenged unless it was understood that Hitler didn't drop A-bombs. If the question were raised in this context the conditionals would have to be:

*S4* If Hitler invaded England in 1940 *and did not drop atomic bombs on England* in 1941 ($S$), then Germany would have won the Second World War ($R$).

*S5* If Hitler had not invaded England in 1940 and dropped atomic bombs on England in 1941 ($Q$), then Germany would have won the Second World War ($R$).

*S6* It is not the case that if Hitler had either not invaded England in 1940 or had dropped atomic bombs on England in 1941, then Germany would have won the Second World War.

Note that *S6* is correctly worded given our hypothesis, since its antecedent would be fulfilled by Hitler both invading England in 1940 and dropping atomic bombs in 1941 which, by our hypothesis, would result in the loss of the war. Thus the three conditionals are seen to be not at all contradictory.

In an attempt to strengthen their case, Routley and Montgomery make another instructive observation. They point out that the two statements:

1. If the match had been scratched, it would have lighted.
2. If the match had not been scratched, it would have lighted.

are not conflicting, logically incompatible statements. That (in the sense of a strict impossibility) is of course true; they are not logically incompatible statements in either a logic of material or strict implication; they are incompatible in $\mathbf{P}_{A1}$ and in our ordinary usage of conditional statements in English. When I assert that ''If the match had been scratched, it would have lighted'', the counter-assertion ''If the match had not been scratched, it would have lighted'' is equivalent to the semifactual, ''Even if the match had not been scratched, it would have lighted'' which is a denial of conditional connection. This observation is instructive because it points out the necessity of keeping a clear distinction between different senses of ''logical''. We must of course always observe context. Indeed with respect to two similar though different statements, ''If the match had been scratched, it would have lighted'' and ''If the match had been scratched, it would not have lighted'', Angell observes that they are logically incompatible statements. But that is true only if we are talking in terms of $\mathbf{P}_{A1}$. The use of ''logically incompatible'' is too flexible in common usage to be of much help. The specifications of technical use must in this case create order, not follow it. Thus Condition II causes no problems for $\mathbf{P}_{A1}$.

*CONDITION* III In order to evaluate the claim that Condition III is necessary for an adequate conditional logic, it is important first to examine one of the features of $\mathbf{P}_{A1}$.

$F_1 \sim[(A \rightarrow B) \cdot \sim([A \cdot C] \rightarrow B)]$

can be shown *not* to be a theorem of $\mathsf{P}_{A1}$ by assigning $A = 0$, $B = 0$ and $C = 0$ with the result that the expression takes on the value 2 which is not a designated value. The importance of this fact is perhaps more clearly seen if we use definition 3 to rewrite the expression as

$F_2\ (A \rightarrow B) \supset [(A \cdot C) \rightarrow B]$

Clearly if such a theorem existed in $\mathsf{P}_{A1}$ we could add any condition to the antecedent of a true conditional statement and the result would be another true conditional statement. Such a principle not only permits the antecedent to contain superfluous and irrelevant conditions, it permits actual contradictions within the antecedent.

Take the conditional

$C_1$ If I strike this match, it will light.

If we assume that conditional is true, we could then assert (assuming the existence of the principle above)

$C_2$ If I strike this match and I don't strike it, it will light.

or even worse

$C_3$ If I strike this match and it doesn't light, it will light.

Both of these "inferences" from $C_1$ are statements which would not ordinarily be accepted were it not for the vagaries of ordinary extensional logic. It is interesting to note that even if we attach additional restrictions to ensure that $C$ is merely superfluous and does not imply either the denial of $A$ nor the denial of $B$, the proposition is still not a theorem. Thus

$F_3 \sim[([\sim(C \rightarrow \sim B) \cdot \sim(C \rightarrow \sim A)] \cdot [A \rightarrow B]) \cdot \sim([A \cdot C] \rightarrow B)]$

is shown not to be a theorem by assigning $A = 0$, $B = 0$ and $C = 0$ which results in the nondesignated value 2.

This exclusion of merely superfluous additions to the antecedent is important. As an example consider that you are a member of a baseball team, it's the last of the ninth inning, your team is losing four runs to five, two outs, you're next batter after the present batter. You say to your coach, "I hope I get up, because I'm hot now. If I get up, I'll hit a home run." If he replies, "If you get up and someone helps you swing, you'll get a home run.", he is not agreeing with you. The general use of conditionals does not permit the addition of superfluous material to the antecedent.

Consider the following exchange. Reporter: "Now Professor, you say that if a person who has warts rubs them with 2-diarsenoethane by the light of a full moon, the warts will go away." Professor: "Certainly not, I said nothing about a full moon; the chemical removes the warts; the moon has nothing to do with it." Reporter: "But the moon makes it sound more traditional and what harm will it do to include it—the warts will go away, won't they?" Professor: "Of course they will go away. If you rub warts with the 2-diarsenoethane, they'll go away even if you hop naked on your left leg under a battery of klieg lights; but the point is that your statement *implies* that the moon has something to do with it; that if you didn't apply the ointment under a full moon, the warts wouldn't go away, and that's not true.'' Reporter: ''Thank you for your help, Professor; let's see, apply ointment, naked while hopping . . . ''

Let us consider this dialogue. We may want to suggest that the Professor's use of ''implies'' is an informal usage—indeed the Professor himself might revise his wording were it pointed out to him. Still it is clear that if someone told us our warts would be cured by applying a certain ointment by the light of a full moon, we would assume that at least the person giving us instructions thought the moon had something to do with the effect. If a friend in giving instructions for operating his gas oven tells us, ''If you turn the oven dial to the temperature you want and keep pressure firmly against the dial, the oven will light,'' our natural question would be, ''What's wrong with your oven?'' If he tells us the oven will light even if we don't press the dial firmly, we think he has been making a joke at our expense.

These examples are sufficient to show that the addition of irrelevant conditions to a conditional statement is at least eccentric, but does it make those conditionals false? First we must be careful. We have all been living in the shadow of the material conditional. We are accustomed to shrugging our shoulders and saying the addition doesn't matter because the result still occurs—that is, we have assumed that conditionals are truth-functional. But are they? Consider the examples given and others you can imagine yourself. Typically, the immediate response of the professor or the oven lighter is, ''But there's no connection between . . . '' A conditional is not simply a truth-function; it points to a connection.

Consider another typical response, ''But even if . . . ''. Typically a statement which begins this way is used to challenge a related conditional assertion. The denial of a conditional is the denial of a connection between the antecedent (or *part* of the antecedent) and its consequent. The warts will disappear even if the ointment is rubbed on at some place or time not providing the light of a full moon. The stove will light when the dial is turned on even if you don't press against the dial. These are not contradictory to the original statements for both could be false (the ointment ineffective on warts, the pilot light out), they are rather contraries—both can be false but both cannot be true. In order to obtain a contradictory we would of course have to deny the efficacious connection of the entire antecedent to the consequent—''Even if you rub the ointment on in the light of a full moon, the warts won't disappear''. (A more cautious individual would use ''may not'' in place of ''won't'', but the concern here is simply with the reaction to irrelevant additions.)

The point of this examination is to point out that the addition of irrelevancies is not acceptable in conditional statements in English. Just because $A \rightarrow B$ it does not follow that $(A \cdot C) \rightarrow B$.

In Condition III Routley and Montgomery criticize $\mathbf{P}_{A1}$ for not accepting

"If the match had been scratched and the match had been scratched, it would have lighted" when and only when "If the match had been scratched, it would have lighted" is accepted, for they claim that both antecedents specify exactly the same conditions. However, we have seen above that normal usage rejects irrelevant additions to the antecedents of conditionals and the fact that $\mathsf{P}'_{A1}$ formalizes this feature is a powerful argument in its favor. The addition even of $A$ is irrelevant to $A \rightarrow B$ and thus should not be accepted.

One could claim of course that $A \cdot A$ makes no new specification while $A \cdot C$ does, but the point is that the modification in both cases is irrelevant. There is no brief to support the contention that irrelevant additions which make no new specifications must be admitted, and clear evidence that in general irrelevant additions should be rejected.

Therefore, $\mathsf{P}'_{A1}$ escapes all the problems suggested by Routley and Montgomery either by meeting the conditions laid down or by a better analysis of conditionals than that which created the conditions.

## NOTES

1. It should be noted that both Routley and Montgomery, and Angell wrote of subjunctive conditional logics. The requirement that antecedent and consequent be relevant was one reason a new logic was believed necessary.

2. Note that '$A_1$', '$R_1$', '$D_1$', and '*43' in justifications refer respectively to that numbered axiom, rule, definition, or theorem which appears in [1].

## REFERENCES

[1] Angell, R. B., "A propositional logic with subjunctive conditionals," *The Journal of Symbolic Logic*, vol. 27 (1962), pp. 325-343.

[2] Routley, R. and H. Montgomery, "On systems containing Aristotle's thesis," *The Journal of Symbolic Logic*, vol. 33 (1968), pp. 82-98.

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