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THE THEORY OF HOMOGENEOUS SIMPLE TYPES AS A SECOND ORDER LOGIC

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In its original form the theory of simple types, hereafter called **ST**, is a theory of predication and not, or at least not primarily, a theory of membership. With that original form in mind we construct in this paper* a second order counterpart of **ST** which we call **ST***. We briefly compare **ST*** with an alternative extension of second order logic, viz., the author's system **T***(*) of [1], which was proposed as characterizing the original (and yet consistent!) logistic background of Russell's paradox of predication. In [2], the author showed the completeness of **T****, plus an extensionality axiom (Ext*), relative to a Fregean interpretation of subject-position occurrences of predicates, viz., that such occurrences of predicates denote individuals correlated with the properties (or "classes") designated by predicate-position occurrences of the same predicates. It is observed here that when the semantical Fregean frames characterized satisfy **ST***'s stratified comprehension principle instead of **T****'s general comprehension principle, then the same Fregean interpretation yields a completeness theorem for monadic **ST*** + (Ext*) as well. It has been found convenient, on the other hand, to consider (monadic) **ST** as a theory of membership rather than a theory of predication when axioms of extensionality are included in its characterization. So considered, Quine proposed his system **NF** as a first order counterpart of **ST**, though of course, as is well-known, **NF** far exceeds **ST** in deductive powers. We show here *per contra* that while (monadic) **ST*** + (Ext*) is motivated in its construction along lines followed by Quine in the construction of his first order counterpart **NF**, viz., the reduction of **ST**'s metatheoretic feature of typical ambiguity to a stratified comprehension principle, our system, unlike **NF**, is equiconsistent with **ST**. This, along with the fact that the non-abstract individuals (or "urelements") of **ST** are retained unmodified in **ST***, indicates that

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ST*, *as a theory of predication*, is to be preferred to **NF**, *as a theory of membership*, in the interpretation which each gives to **ST**'s metatheoretic feature of typical ambiguity. We show in addition that if to (monadic) **ST*** + (Ext*) we add the assumption that whatever is a value of an individual variable is also (or, on the Fregean interpretation, is correlated with) a value of a (monadic) predicate variable, i.e., the assumption that every individual is a "class", then the resulting system is equiconsistent with **NF**. We refer to monadic **ST*** + (Ext*) as **NFU*** and show that it contains Jensen's system **NFU** as well.

1 *Syntax* We assume as given an enumerable infinity of individual variables 'x', 'y', 'z', . . ., and, for each natural number n, an enumerable infinity of n-place predicate variables 'F^n', 'G^n', 'H^n', ... (We shall drop the superscript when the context makes clear the addicity of the predicate variable in question.) For convenience, we shall use 'α', on the one hand, and 'π', 'σ', on the other, as metasyntactical variables referring to individual and predicate variables, respectively. When referring to individual and predicate variables collectively, we use 'μ' and 'η'.

As primitive *logical constants* we take $\wedge$, $\sim$, and $\rightarrow$ for the universal quantifier sign, the negation sign, and the (material) conditional sign, respectively. Other logical particles, e.g., $\bigvee$, $\leftrightarrow$, $\boldsymbol{\wedge}$, $\boldsymbol{\vee}$ are understood as having been defined in the usual way as metalinguistic abbreviations. We use parentheses and brackets as auxiliary signs and represent concatenation by juxtaposition. By an *atomic wff* we understand an expression of the form: $\pi(\mu_0, \ldots, \mu_{n-1})$, where π is an n-place predicate variable and $\mu_0, \ldots, \mu_{n-1}$ are any variables, predicate or individual, whatsoever. (If $n = 0$, this is presumed to be π itself.) The indicated occurrences of $\mu_0, \ldots, \mu_{n-1}$ within the pair of parentheses of such an atomic wff are said to occupy its *subject-positions* whereas the initial occurrence of π is said to occupy its predicate-position. We note that while a predicate variable can occupy both a subject- and a predicate-position, an individual variable can occupy only a subject-position.

We understand the set of wffs to be defined as the smallest set K containing all of the atomic wffs and such that $\sim\varphi$, $(\varphi \rightarrow \psi)$, $\wedge\mu\varphi$ are in K whenever φ, ψ are in K and μ is a variable. We shall use 'φ' and 'ψ' as metasyntactical variables referring to wffs in general. We understand the subject- and predicate-positions of an arbitrary wff φ to be the subject- and predicate-positions of its atomic subwffs. When only individual variables occupy the subject-positions of φ, we say that φ is a *standard second order wff*. Finally, we say that a wff φ is *stratified* if there exists an assignment $\mathfrak{t}$ of natural numbers to the variables occurring in φ such that for each atomic subwff $\pi(\mu_0, \ldots, \mu_{n-1})$ of φ,

$$\mathfrak{t}(\pi) = 1 + \max[\mathfrak{t}(\mu_0), \ldots, \mathfrak{t}(\mu_{n-1})].$$

For convenience, we abbreviate *indiscernibility* as follows:

$$\mu \equiv \eta =_{df} \wedge\pi[\pi(\mu) \rightarrow \pi(\eta)]$$

where π is the first 1-place predicate variable distinct from both μ and η.

2 *Two minimal systems* For heuristic purposes we shall use a *-label for a system or a principle that explicitly involves subject-position occurrences of predicate variables. We propose to motivate our axiomatic descriptions by first characterizing two minimal systems which are of special interest when restricted to standard second order wffs. We assume throughout that *modus ponens* is our only *primitive* inference rule. Others mentioned or presupposed, e.g., the rule of alphabetic rewrite of bound variables or the rule of (universal) generalization are easily seen to be derivable in the usual way.

By an axiom of the (minimal) system **M*** we understand any tautologous wff and any generalization of a wff of one of the following forms:

(A1) $\wedge\mu(\varphi \rightarrow \psi) \rightarrow (\wedge\mu\varphi \rightarrow \wedge\mu\psi)$
(A2) $\varphi \rightarrow \wedge\mu\varphi$, where μ does not occur free in φ,
(A3) $\mathbf{V}\alpha(\mu \equiv \alpha)$, where α is an individual variable and μ is any variable whatsoever,
(A4) $\mathbf{V}\pi(\sigma \equiv \pi \wedge \wedge\alpha_0 \ldots \wedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \sigma(\alpha_0, \ldots, \alpha_{n-1})])$, where π, σ are both n-place predicate variables and $\alpha_0, \ldots, \alpha_{n-1}$ are pairwise distinct individual variables,
(A5) $\mu \equiv \eta \rightarrow (\varphi \rightarrow \psi)$, where φ, ψ are atomic wffs and ψ is obtained from φ by replacing *a subject-position occurrence* of η by a subject-position occurrence of μ.

By means of essentially the same proofs as occur in [2], section **2**, these axioms yield:

(1) Leibniz' law for subject-position occurrences of variables, i.e.,

(LL*) $\vdash_{\overline{\mathbf{M}^*}} \mu \equiv \eta \rightarrow (\varphi \leftrightarrow \psi)$

where ψ is obtained from φ by replacing one or more free subject-position occurrence of η by a free occurrence of μ;
(2) the principle of universal instantiation of any variable, predicate or individual, for an individual variable, i.e.,

(UI*) $\vdash_{\overline{\mathbf{M}^*}} \wedge\alpha\varphi \rightarrow \varphi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}$

where $\varphi\begin{bmatrix}\alpha\\ \mu\end{bmatrix}$ is exactly like φ except for containing free occurrences of μ wherever φ contains free occurrences of the individual variable α; and
(3) the principle of universal instantiation of any n-place predicate variable for a generalized n-place predicate variable, i.e.,

(UI_n*) $\vdash_{\overline{\mathbf{M}^*}} \wedge\pi\varphi \rightarrow \varphi\begin{bmatrix}\pi\\ \sigma\end{bmatrix}$

where π, σ are both n-place predicate variables and $\varphi\begin{bmatrix}\pi\\ \sigma\end{bmatrix}$ is obtained from φ by replacing free occurrences of π (in predicate- or subject-position) by free occurrences of σ.

Now it is easily seen, by (1)-(3) above, that a standard second order wff is a theorem of **M*** iff it is true in every second order frame, standard or non-standard. Later, in terms of the Fregean semantics of section **6** below, we shall note that a parallel result applies to non-standard wffs as well. We extend **M*** now to include a minimal *comprehension principle*, viz., one restricted to standard second order wffs:

(CP) $\mathbf{V}R \wedge x_0 \ldots \wedge x_{n-1}[R(x_0, \ldots, x_{n-1}) \leftrightarrow \varphi]$

where φ is a standard second order wff in which 'R' does not occur (free) but which contains 'x_0', . . ., 'x_{n-1}' among its free individual variables.[1] Again it is clear by known results that a standard second order wff is a theorem of **M*** + (CP) iff it is true in every *general model*, i.e., every second order frame which satisfies (CP). This last class of wffs constitutes the theorems of standard second order logic.

3 *Two alternative extensions of* (CP) We wish now to extend (CP) so as to apply to all the wffs, those with predicate variables in subject-positions as well as those without. The first and most natural alternative for such an extension is the principle (CP*) which is exactly like (CP) except for dropping the restriction that no predicate variable occupies a subject-position in the comprehending wff φ. The system **M*** + (CP*) is essentially the system **T**** of [2], and we shall so refer to it hereafter.[2]

Now it is noteworthy that Russell's paradox (of predication) is not derivable in **T****. Thus we observe on the one hand that the expression

$$\mathbf{V}F \wedge x[F(x) \leftrightarrow \sim x(x)]$$

is not well-formed, having as it does an individual variable in a predicate-position. On the other hand, while

$$\mathbf{V}F \wedge G[F(G) \leftrightarrow \sim G(G)]$$

is well-formed, it is not an instance of (CP*), since that principle posits properties or relations only through conditions that apply to all the individuals, be these individuals themselves properties, relations or otherwise. The above wff fails in this by positing a property through a condition which applies only to arguments that are themselves properties.[3] Despite its consistency, **T**** is not without its (apparent) oddities. Thus while (CP*) does not warrant positing the existence of "the Russell property," it does posit the related property of being an individual which is indiscernible from a property which that individual does not have:

$$\vdash_{\overline{\mathbf{T}^{**}}} \mathbf{V}F . \wedge x[F(x) \leftrightarrow \mathbf{V}G(x \equiv G \wedge \sim G(x))]$$

The existence in **T**** of this last property, while it does not yield a contradiction, shows us that

(Ind*) $\wedge F \wedge G(F \equiv G \rightarrow \wedge x[F(x) \leftrightarrow G(x)])$

is refutable in **T****, i.e., that ($\sim$Ind*) is a theorem of **T****. And what this shows us is that co-extensivity between properties (or n-ary relations in general, for $n \geqslant 1$) is not a proper relation in **T****, i.e.,

$$\vdash_{\mathbf{T^{**}}} \sim \bigvee R \wedge F \wedge G(R(F, G) \leftrightarrow \wedge x[F(x) \leftrightarrow G(x)])$$

Other related "curiosities" of **T**** are that in general there is no property of being a property of a given individual, i.e.,

$$\vdash_{\mathbf{T^{**}}} \sim \wedge x \bigvee F \wedge G[F(G) \leftrightarrow G(x)]$$

and, consequently, that predication is not a relation:

$$\vdash_{\mathbf{T^{**}}} \sim \bigvee R \wedge F \wedge x[R(F, x) \leftrightarrow F(x)]$$

As a formal representative of an ontology in which properties (and relations) are individuals, these facts about **T**** show it to be of no little philosophical interest, especially since (CP*) seems to be the most natural way of extending (CP) once predicate variables are allowed to be substituends of individual variables, i.e., once predicate variables are allowed to occupy subject- as well as predicate-positions.[4] Nevertheless, notwithstanding the naturalness of (CP*) in this regard, an alternative way of extending (CP) along lines motivated by the theory of simple types and which allows properties (or relations) to be specified through conditions that apply only to properties (or relations) is the following *stratified comprehension principle*:

(SCP*) $\bigvee R \wedge \mu_0 \ldots \wedge \mu_{n-1}[R(\mu_0, \ldots, \mu_{n-1}) \leftrightarrow \varphi]$

where φ is a stratified wff containing free occurrences of the (predicate or individual) variables $\mu_0, \ldots, \mu_{n-1}$ and in which 'R' does not occur (free). Like (CP*), (SCP*) includes (CP) since every standard second order wff is stratified. We understand **ST*** to be the system obtained by adding all universal generalizations of instances of (SCP*) to **M***.

It is noteworthy that while (Ind*) is refutable in **T****, it is provable in **ST***:

(Ind$_n$*) $\vdash_{\mathbf{ST^*}} \wedge F^n \wedge G^n(F \equiv G \rightarrow \wedge x_0 \ldots \wedge x_{n-1}[F(x_0, \ldots, x_{n-1}) \leftrightarrow G(x_0, \ldots, x_{n-1})])$

and the obvious reason why is that unlike its status in **T****, co-extensivity, by (SCP*), is a theoretically projected relation of the ontology represented by **ST***:

$$\vdash_{\mathbf{ST^*}} \bigvee R \wedge F^n \wedge G^n(R(F,G) \leftrightarrow \wedge x_0 \ldots \wedge x_{n-1}[F(x_0, \ldots, x_{n-1}) \leftrightarrow G(x_0, \ldots, x_{n-1})])$$

Consequently, since by (SCP*) every property or relation has the relational-property of being co-extensive with itself, a property or relation will be co-extensive with any property or relation which has all the properties which it has. Similarly, the other mentioned "curiosities" of **T**** also fail in **ST***. For by (SCP*) predication is (or "stands for") a relation in the ontology of **ST***, in which case, again by (SCP*), there exists in that ontology the relational-property of being a property of any given individual.

4 *Properties as classes and membership as predication* We have thus far referred to the theoretically projected values of monadic predicate variables as properties and not as classes. We have done so because it is with theories of predication and not with theories of membership that we are here principally concerned. Nevertheless, tradition does allow our identifying properties with classes if the following extensionality principle is presupposed:

(Ext*) $\bigwedge F \bigwedge G(\bigwedge x[F(x) \leftrightarrow G(x)] \rightarrow F \equiv G)$

Identifying properties with classes does not mean, however, that predication is hereby identified or reduced to membership. Indeed, it is quite the opposite that is the case. Consider in this regard the following definition:

$$\mathrm{E}(x, y) =_{df} \bigvee F[y \equiv F \wedge F(x)]$$

The existence of such a relation as is hereby defined, let us note, is warranted by both (CP*) and (SCP*). In **T****, however, while

$$\vdash_{\mathbf{T^{**}}} F(x) \rightarrow \mathrm{E}(x, F)$$

the converse fails as a consequence of the failure there of (Ind*). This fact, incidentally, indicates something quite significant about membership in **T****, viz., that it is not and cannot be a perfect mirror-image of predication. In (**ST***), on the other hand, (Ind*) is provable, and therefore:

$$\vdash_{\mathbf{ST^*}} F(x) \leftrightarrow \mathrm{E}(x, F)$$

Moreover, the stratified comprehension principle for membership as defined above is also a theorem schema of **ST***:

(CP-NF*) $\vdash_{\mathbf{ST^*}} \bigvee y \bigwedge x[\mathrm{E}(x, y) \leftrightarrow \varphi]$

where φ is a stratified wff in which 'y' does not occur (free). Thus where neither 'F' nor 'G' occur (free) in φ, then by (Ind*), elementary logical transformations and the derived rule of generalization:

$$\vdash_{\mathbf{ST^*}} y \equiv G \wedge \bigwedge x[G(x) \leftrightarrow \varphi] \rightarrow \bigwedge x(\bigvee F[y \equiv F \wedge F(x)] \leftrightarrow \varphi)$$

and consequently, by similar application of the same rules,

$$\vdash_{\mathbf{ST^*}} \bigvee y \bigvee G(y \equiv G \wedge \bigwedge x[G(x) \leftrightarrow \varphi]) \rightarrow \bigvee y \bigwedge x[\mathrm{E}(x, y) \leftrightarrow \varphi]$$

But by (SCP*) and (A3),

$$\vdash_{\mathbf{ST^*}} \bigvee y \bigvee G(y \equiv G \wedge \bigwedge x[G(x) \leftrightarrow \varphi])$$

where φ is stratified, from which (CP-NF*) follows.

Nevertheless, even in **ST*** + (Ext*) membership does not suffice for all the purposes of predication. In particular, without the assumption that every individual is a class, the extensionality principle for membership (as defined above), i.e.,

(Ext) $\bigwedge x \bigwedge y(\bigwedge z[\mathrm{E}(z, x) \leftrightarrow \mathrm{E}(z, y)] \rightarrow x \equiv y)$

is not provable in **ST*** + (Ext*), which is as it should be if **ST*** (with or without (Ext*)) is to be considered the second order counterpart of the theory of simple types. Indeed, as we now show, (Ext) is equivalent in **ST*** + (Ext*) to the assumption that every individual is a class, i.e.,

(A) $\vdash_{\mathbf{ST^*}+(\mathrm{Ext}^*)} \bigwedge x \bigvee F(x \equiv F) \leftrightarrow (\mathrm{Ext})$

For the proof of the left-to-right direction of (A), note first that by elementary logical transformations

$$\vdash_{\mathbf{ST^*}} x \equiv F \wedge y \equiv G \rightarrow (\bigwedge z[x \equiv F \wedge F(z) \leftrightarrow y \equiv G \wedge G(z)] \rightarrow \bigwedge z[F(z) \leftrightarrow G(z)])$$

Given (Ext*), however, and the fact that indiscernibility is an equivalence relation:

$$\vdash_{\mathbf{ST^*}+(\mathrm{Ext}^*)} x \equiv F \wedge y \equiv G \rightarrow (\bigwedge z[F(z) \leftrightarrow G(z)] \rightarrow x \equiv y)$$

and therefore by tautologous transformations,

$$\vdash_{\mathbf{ST^*}+(\mathrm{Ext}^*)} x \equiv F \wedge y \equiv G \rightarrow (\bigwedge z[x \equiv F \wedge F(z) \leftrightarrow y \equiv G \wedge G(z)] \rightarrow x \equiv y)$$

Consequently, by the derived rule of generalization, a commutation and confinement law for quantifiers and elementary transformations:

$$\vdash_{\mathbf{ST^*}+(\mathrm{Ext}^*)} \bigvee F(x \equiv F) \wedge \bigvee G(y \equiv G) \rightarrow (\mathrm{Ext})$$

But, it is clear that

$$\vdash_{\mathbf{ST^*}} \bigwedge x \bigvee F(x \equiv F) \rightarrow \bigvee F(x \equiv F) \wedge \bigvee G(y \equiv G)$$

from whence follows the left-to-right direction of (A). For the converse direction, note that by definition of membership:

$$\vdash_{\mathbf{ST^*}} \bigvee z\, \mathrm{E}(z, x) \rightarrow \bigvee F(x \equiv F)$$

On the other hand,

$$\vdash_{\mathbf{ST^*}} \sim \bigvee z\, \mathrm{E}(z, x) \rightarrow (\sim \bigvee z\, \mathrm{E}(z, F) \rightarrow \bigwedge z[\mathrm{E}(z, x) \leftrightarrow \mathrm{E}(z, F)])$$

and therefore

$$\vdash_{\mathbf{ST^*}} \sim \bigvee z\, \mathrm{E}(z, F) \rightarrow (\sim \bigvee z\, \mathrm{E}(z, x) \rightarrow [(\mathrm{Ext}) \rightarrow x \equiv F])$$

Accordingly, by generalization and elementary transformations,

$$\vdash_{\mathbf{ST^*}} \bigvee F \sim \bigvee z\, \mathrm{E}(z, F) \rightarrow (\sim \bigvee z\, \mathrm{E}(z, x) \rightarrow [(\mathrm{Ext}) \rightarrow \bigvee F(x \equiv F)])$$

But by (SCP*),

$$\vdash_{\mathbf{ST^*}} \bigvee F \sim \bigvee z F(z)$$

and therefore, by (Ind*),

$$\vdash_{\mathbf{ST^*}} \bigvee F \sim \bigvee z\, \mathrm{E}(z, F)$$

Consequently, by tautologous transformations,

$$\vdash_{\mathbf{ST^*}} \bigvee z\, \mathrm{E}(z, x) \vee \sim \bigvee z\, \mathrm{E}(z, x) \rightarrow [(\mathrm{Ext}) \rightarrow \bigvee F(x \equiv F)]$$

whence follows the converse direction of (A).

For convenience, let us refer to the assumption that every individual is a class as the assumption (Q*) (for *Quine's assumption*), i.e.,

(Q*) $\bigwedge x \bigvee F(x \equiv F)$

Then, where **NFU*** is *monadic* **ST*** + (Ext*), we take **NF*** to be monadic **ST*** + (Ext*) + (Q*). We observe that although (Ext) is not a theorem of **NFU***, nevertheless, Jensen's modification of (Ext) is:

(Ext′) $\vdash_{\mathbf{NFU^*}} \bigvee z\, \mathrm{E}(z,x) \wedge \bigwedge z[\mathrm{E}(z,x) \leftrightarrow \mathrm{E}(z,y)] \rightarrow x \equiv y$

Consequently, **NFU*** contains a representation of Jensen's system **NFU** ("New Foundations with Urelements" in [4]). Similarly, by (A), **NF*** contains a representation of Quine's system **NF**.

Theorem (a) *If* **NFU*** *is consistent, then so is* **NFU**.
(b) *If* **NF*** *is consistent, then so is* **NF**.

5 *Axioms of infinity and choice in* **NFU*** In pure standard second order logic the expression of an axiom of infinity or an axiom of choice requires quantifiers binding n-place predicate variables, for some $n \geqslant 2$. Once subject-position occurrences of predicate variables are allowed, however, we can express such axioms in terms strictly of monadic wffs:

(Inf*) $\bigvee F[\bigvee xF(x) \wedge \bigwedge x(F(x) \rightarrow \bigvee Gx \equiv G) \wedge \bigwedge H(F(H) \rightarrow \bigvee G[\bigwedge x(H(x) \rightarrow G(x)) \wedge \sim\bigwedge x(G(x) \rightarrow H(x)) \wedge F(G)])]$
(AC*) $\bigwedge F[\bigwedge x(F(x) \rightarrow \bigvee Gx \equiv G) \wedge \bigwedge G(F(G) \rightarrow \bigvee xG(x)) \wedge \bigwedge G \bigwedge H(F(G) \wedge F(H) \rightarrow \sim\bigvee x[G(x) \wedge H(x)]) \rightarrow \bigvee G(\bigwedge x[G(x) \leftrightarrow \bigvee H(F(H) \wedge H(x))] \wedge \bigwedge H[F(H) \rightarrow \bigvee y(G(y) \wedge H(y) \wedge \bigwedge z[G(z) \wedge H(z) \rightarrow z \equiv y])])]$

It is clear by obvious transformations of **ST*** that these axioms yield the following axioms of infinity and choice in terms of membership (as defined in section **4**):

(Inf) $\bigvee x[\bigvee y\, \mathrm{E}(y,x) \wedge \bigwedge y(\mathrm{E}(y,x) \rightarrow \bigvee z[y \subseteq z \wedge z \not\subseteq y \wedge \mathrm{E}(z,x)])]$
(AC) $\bigwedge x[\bigwedge y(\mathrm{E}(y,x) \rightarrow \bigvee z\, \mathrm{E}(z,y)) \wedge \bigwedge y \bigwedge w(\mathrm{E}(y,x) \wedge \mathrm{E}(w,x) \rightarrow \sim\bigvee z[\mathrm{E}(z,y) \wedge \mathrm{E}(z,w)]) \rightarrow \bigvee y \bigwedge w(\mathrm{E}(w,x) \rightarrow \bigvee z[\mathrm{E}(z,y) \wedge \mathrm{E}(z,w) \wedge \bigwedge u(\mathrm{E}(u,y) \wedge \mathrm{E}(u,w) \rightarrow u \equiv z)])]$

Now because (AC) is provable for finite classes, it follows that (AC) is refutable only if (Inf) is provable. Specker in [6], however, has shown that (AC) is refutable in **NF** and that therefore (Inf) is provable in **NF**. Obviously, the same result applies to **NF***. Jensen in [4], on the other hand, has shown that (AC) is not refutable in **NFU** and that **NFU** (+(Inf), +(AC)) is consistent relative to weak Zermelo set theory[5] (+(Inf), +(AC)), which in turn is equiconsistent with the theory of simple types (+(Inf), +(AC)). In section **9** below we show that the theory of simple types (+(Inf), +(AC)) is consistent relative to **NFU*** (+(Inf*), +(AC*)). Nevertheless, independently of that result we already have the following as a consequence of the implication of (Inf) and (AC) by (Inf*) and (AC*).

Theorem *If* **NFU*** +(Inf*), +(AC*) *is consistent, then so is* **NFU** +(Inf), +(AC).

6 *A Fregean semantics* It is well-known that Frege viewed properties, which he also called (first-level) concepts, as *unsaturated* (*ungesättigt*) entities. What this means, at least in part, is that properties (and relations) cannot themselves be logical subjects of predication in the same sense in which individuals in general are. Nevertheless, Frege did allow that expressions such as 'the concept Horse' or 'the property of being a horse' do denote individuals, though of course, since no concept or property is itself an individual, these expressions do not denote what they purport to denote, viz., a concept or property. Instead, such expressions, according to Frege, denote an individual which is correlated with the concept or property in question. This interpretation of Frege's was adopted and applied in [2] to subject-position occurrences of predicate variables. Aside from generating a completeness theorem for **T**** +(Ext*), this interpretation, so applied, shows us that the failure of (Ind*) in **T**** is in effect merely a variant of Cantor's theorem. We note in what follows that the same Fregean interpretation also generates a completeness theorem for **ST*** +(Ext*), though of course we must replace (CP*)-normalcy conditions for Fregean frames by conditions suitable to (SCP*) instead. Naturally, since (Ind*) is provable in **ST***, the Fregean correlations of the frames so altered do not conform to Cantor's theorem but are instead one-to-one, which also is as it should be if **ST*** is really the second order counterpart of the theory of simple types. Incidentally, we might note here that while Frege distinguished (unsaturated) properties, i.e., (first-level) concepts, from (saturated) classes (*Begriffsumfangen* which he later identified with certain abstract individuals he called *Wertverläufe*), he nevertheless understood properties to be "identical" when they were co-extensive. The difficulty for Frege, however, was that in his *Begriffsschrift* this "identity" could not be formulated since identity there meant indiscernibility and the latter in this case requires allowing subject-position occurrences of predicate variables. This difficulty of course is overcome once we formally allow such occurrences of predicate variables but so restricted in their interpretation as to conform to Frege's own informal proposal for nominalized predicate expressions. Thus (Ext*) under such an interpretation appears to say precisely what Frege intended but could not say in the original form of his *Begriffsschrift*. Accordingly, the validity of (Ext*) in the Fregean frames characterized below is just as it should be were we to adopt an ontology similar to Frege's.

Let us proceed then with the definition of what we take a *Fregean frame* to be, viz., an indexed triple $\mathfrak{A} = \langle D, A_n, f\rangle_{n\in\omega}$ where (1) D is a non-empty domain, (2) A_n is an ω-indexed family such that for $n \in \omega$, A_n is a non-empty subset of $\mathcal{P}(D_n)$, i.e., A_n is a non-empty set every member of which is a set of n-tuples whose constituents are drawn from D, and (3) f is a function whose domain is $D \cup \bigcup_{n\in\omega} A_n$ and such that for all $n \in \omega$, for all $X \in A_n$, $f(X) \in D$ and for all $x \in D$, $f(x) = x$. (We include in f the identity function on D only for convenience so as to simplify our definition below of satisfaction in $\mathfrak{A}$.)[6]

By an *assignment* in a Fregean frame $\mathfrak{A}$ of values to variables we understand a function $\mathfrak{a}$ which has the set of individual and predicate variables as its domain and which is such that for each individual variable α, $\mathfrak{a}(\alpha) \in D$, and, for $n \in \omega$, for each n-place predicate variable π, $\mathfrak{a}(\pi) \in A_n$. We take $\mathfrak{a}\binom{\mu}{y}$ to be that assignment which is identical to $\mathfrak{a}$ in all respects except (at most) in its assigning y to μ.

Where $\mathfrak{A} = \langle D, A_n, f\rangle_{n\in\omega}$ is a Fregean frame and $\mathfrak{a}$ is an assignment in $\mathfrak{A}$, then we recursively define the *satisfaction in* $\mathfrak{A}$ *by* $\mathfrak{a}$ of a wff as follows:

(1) $\mathfrak{a}$ *satisfies* an atomic wff $\pi(\mu_0, \ldots, \mu_{n-1})$ in $\mathfrak{A}$ iff $\langle f(\mathfrak{a}(\mu_0)), \ldots, f(\mathfrak{a}(\mu_{n-1}))\rangle \in \mathfrak{a}(\pi)$;
(2) $\mathfrak{a}$ *satisfies* $\sim\varphi$ in $\mathfrak{A}$ iff $\mathfrak{a}$ does not satisfy φ in $\mathfrak{A}$;
(3) $\mathfrak{a}$ *satisfies* $(\varphi \rightarrow \psi)$ in $\mathfrak{A}$ iff either $\mathfrak{a}$ does not satisfy φ in $\mathfrak{A}$ or $\mathfrak{a}$ satisfies ψ in $\mathfrak{A}$;
(4) where α is an individual variable, $\mathfrak{a}$ *satisfies* $\bigwedge\alpha\varphi$ in $\mathfrak{A}$ iff for all $x \in D$, $\mathfrak{a}\binom{\alpha}{x}$ satisfies φ in $\mathfrak{A}$;
(5) where $n \in \omega$ and π is an n-place predicate variable, $\mathfrak{a}$ *satisfies* $\bigwedge\pi\varphi$ in $\mathfrak{A}$ iff for all $X \in A_n$, $\mathfrak{a}\binom{\pi}{X}$ satisfies φ in $\mathfrak{A}$.

As usual, we understand a wff to be *true in* a Fregean frame if it is satisfied by every assignment in that frame. In addition, we say that $\mathfrak{A}$ is a *general Fregean frame* if every instance of (CP*) is true in $\mathfrak{A}$. On the other hand, we shall say that $\mathfrak{A}$ is a *stratified Fregean frame* if every instance of (SCP*) is true in $\mathfrak{A}$.

As already indicated, in [2] we showed that a wff φ is a theorem of **T**** +(Ext*) iff φ is true in every general Fregean frame. Actually, the proof given in [2] suffices for strong completeness as well, i.e., a wff or set of wffs is consistent relative to **T**** + (Ext*) iff it is (simultaneously) satisfiable in some general Fregean frame. Naturally we shall not repeat the proof here. Nevertheless, it is noteworthy that essentially the same proof suffices for **ST*** + (Ext*) as well, the only difference being the relativization of *maximal consistency* (and the use thereof in the application of Lindenbaum's Lemma) to **ST*** + (Ext*) rather than to **T**** + (Ext*). Indeed, essentially the same proof also suffices for the subsystems **M*** + (Ext*) and **M*** + (CP) + (Ext*), where the Fregean frame constructed for the former need satisfies none of our comprehension principles while in that for the latter all instances of (CP) must be true.

Theorem *A set of wffs is consistent relative to* **ST*** + (Ext*) *iff it is simultaneously satisfiable in some stratified Fregean frame.*

Corollary *A wff is a theorem of* **ST*** + (Ext*) *iff it is true in every stratified Fregean frame.*

7 *The equiconsistency of* **NF** *with* **NF*** We have already shown that **NF** is consistent if **NF*** is. In this section we utilize the preceding semantical theorem to show that the converse also holds and that consequently **NF** and **NF*** are equiconsistent.

Theorem **NF** *is equiconsistent with* **NF***

Proof: Since **NF** is a first order theory its consistency implies the existence of a model $\langle D, \epsilon'\rangle$ which satisfies extensionality and the stratified comprehension principle for membership in **NF**. Let g be that function with D as domain and such that for $x \epsilon D$, $g(x) = \{y \epsilon D \mid y \epsilon' x\}$. We note that because of **NF**'s principle of extensionality, g is one-to-one, i.e., for $x, z \epsilon D$, if $g(x) = g(z)$, then $x = z$. Now let $A = \{g(x) \mid x \epsilon D\}$ and let f be that function from $D \cup A$ into D such that for $x \epsilon D$, $f(x) = x = f(g(x))$. Finally, we set $\mathfrak{A} = \langle D, A, f\rangle$ and note that $\mathfrak{A}$ is a Fregean frame (restricted to interpreting only *monadic* wffs). Consequently, by the completeness of **M*** + (Ext*), $\mathfrak{A}$ satisfies all the (monadic) axioms of **M*** + (Ext*). In addition, since by definition $x = f(g(x))$ for all $x \epsilon D$, then $\mathfrak{A}$ satisfies (Q*) as well. It remains then only to show that $\mathfrak{A}$ satisfies every monadic instance of (SCP*).

Toward showing this we assume a bijection from the set of monadic predicate and individual variables onto the individual variables. (Such a bijection must exist since both sets are of cardinality $\aleph_0$.) Where μ is a monadic predicate or individual variable, we take $\overline{\mu}$ to be the associated individual variable. We now recursively define a function which transforms each monadic wff into a first order wff of **NF**. Such a function in effect replaces each predicate (and individual) variable by its associated individual variable and interprets predication as **NF**'s membership:

(i) $\mathrm{s}(\pi(\mu)) = \overline{\mu} \epsilon \overline{\pi}$
(ii) $\mathrm{s}(\sim\varphi) = \sim \mathrm{s}(\varphi)$
(iii) $\mathrm{s}(\varphi \rightarrow \psi) = (\mathrm{s}(\varphi) \rightarrow \mathrm{s}(\psi))$
(iv) $\mathrm{s}(\bigwedge\mu\varphi) = \bigwedge\overline{\mu}\,\mathrm{s}(\varphi)$

We now show by induction on the structure of monadic wffs that if $\mathfrak{a}$ is an assignment in $\mathfrak{A}$ and $\mathfrak{b}$ is an assignment in $\langle D, \epsilon'\rangle$ such that $\mathfrak{b}(\overline{\mu}) = f(\mathfrak{a}(\mu))$ for all monadic predicate or individual variables μ, then $\mathfrak{a}$ satisfies a monadic wff φ in $\mathfrak{A}$ iff $\mathfrak{b}$ satisfies $s(\varphi)$ in $\langle D, \epsilon'\rangle$. In the atomic case we note that by definition $\mathfrak{a}$ satisfies $\pi(\mu)$ in $\mathfrak{A}$ iff $f(\mathfrak{a}(\mu)) \epsilon \mathfrak{a}(\pi)$, and that $\mathfrak{b}$ satisfies $\mathrm{s}(\pi(\mu))$ in $\langle D, \epsilon'\rangle$ iff $f(\mathfrak{a}(\mu)) \epsilon' f(\mathfrak{a}(\pi))$. However, since $\mathfrak{a}(\pi) = g(x)$ for some $x \epsilon D$ and $f(g(x)) = x$ for all $x \epsilon D$, then $f(\mathfrak{a}(\mu)) \epsilon \mathfrak{a}(\pi)$ iff $f(\mathfrak{a}(\mu)) \epsilon' f(\mathfrak{a}(\pi))$; and therefore $\mathfrak{a}$ satisfies $\pi(\mu)$ in $\mathfrak{A}$ iff $\mathfrak{b}$ satisfies $\mathrm{s}(\pi(\mu))$ in $\langle D, \epsilon'\rangle$. The cases for when φ is a negation or conditional follow trivially by the inductive hypothesis. So too when φ is of the form $\bigwedge\alpha\psi$ or $\bigwedge\pi\psi$, though in the latter case we must utilize the fact that g is one-to-one. We conclude accordingly that if $\mathrm{s}(\varphi)$ is true in $\langle D, \epsilon'\rangle$, then φ is true in $\mathfrak{A}$. Consequently, since the s-transform of every monadic instance of (SCP*) is an instance of **NF**'s stratified comprehension principle in terms of membership and therefore true in $\langle D, \epsilon'\rangle$, then each such instance of (SCP*) is true in $\mathfrak{A}$.

Q.E.D.

8 *A relative consistency proof for* **NFU*** We are unable to modify the proof of the last theorem suitably so as to show the consistency of **NFU*** relative to **NFU**. Such a modification is possible on the other hand for a system closely related to **NFU** and which we shall call **NFU′**. **NFU′** is like **NFU** except for containing an individual constant '$\bar{o}$' for the null class along with an axiom to that effect:

($\bar{o}$) $\sim \mathbf{V}x(x \in \bar{o})$

NFU′ retains the qualified extensionality axiom of **NFU** but modifies the stratified comprehension principle as follows:

(CP-NFU′) $\mathbf{V}y[(\mathbf{V}x(x \in y) \vee y \equiv \bar{o}) \wedge \bigwedge x(x \in y \leftrightarrow \varphi)]$

where φ is a stratified formula of **NFU′** in which 'y' does not occur free.

Theorem *If* **NFU′** (+(Inf), +(AC)) *is consistent, then so is* **NFU*** (+(Inf*), +(AC*)).

Proof: As in the relative consistency proof of **NF***, the consistency of **NFU′** implies the existence of a model $\langle D, \epsilon', \underline{o}\rangle$ which satisfies the axioms of **NFU′** and where $\underline{o}$ is the element of D assigned to '$\bar{o}$'. Consequently, there is no $x \in D$ such that $x \epsilon' \underline{o}$. We define the function g as in our earlier proof, i.e., for $x \in D$, $g(x) = \{y \in D \mid y \epsilon' x\}$, but note that g is no longer one-to-one since many elements of D may be ϵ'-empty. Nevertheless, by the qualified extensionality axiom, for those elements of D that are not ϵ'-empty, g is one-to-one. Again we set $A = \{g(x) \mid x \in D\}$ but modify the definition of f whose domain is $D \cup A$ as follows: for $x \in D, f(x) = x$; for $x \in D$, if for some $y \in D$, $y \epsilon' x$, $f(g(x)) = x$; for $x \in D$ such that x is ϵ'-empty, $f(g(x)) = \underline{o}$. Finally, we set $\mathfrak{A} = \langle D, A, f\rangle$ and note that $\mathfrak{A}$ is a Fregean frame (restricted to interpreting only monadic wffs). Consequently, $\mathfrak{A}$ satisfies **M*** +(Ext*). We observe that if D contains at least two elements that are both ϵ'-empty, then (Q*) fails in $\mathfrak{A}$. It remains now only to show that $\mathfrak{A}$ satisfies every monadic instance of (SCP*).

Toward showing this, we modify the definition of the transformation $\mathfrak{s}$ of the last proof by adding a special (and now different) clause for monadic predicate variables:

(v) $\mathfrak{s}(\bigwedge \pi\varphi) = \bigwedge \bar{\pi}[\mathbf{V}\alpha(\alpha \in \bar{\pi}) \vee \bar{\pi} \equiv \bar{o} \rightarrow \mathfrak{s}(\varphi)]$

As in the earlier proof, we now show by induction that if $\mathfrak{a}$ is an assignment in $\mathfrak{A}$ and $\mathfrak{b}$ is an assignment in $\langle D, \epsilon', \underline{o}\rangle$ such that $\mathfrak{b}(\bar{\mu}) = f(\mathfrak{a}(\mu))$, then $\mathfrak{a}$ satisfies a monadic wff φ in $\mathfrak{A}$ iff $\mathfrak{b}$ satisfies $\mathfrak{s}(\varphi)$ in $\langle D, \epsilon', \underline{o}\rangle$. When φ is an atomic wff $\pi(\mu)$, we note that $\mathfrak{a}(\pi) = g(x)$ for some $x \in D$. Now if $f(\mathfrak{a}(\mu)) \in \mathfrak{a}(\pi) = g(x)$, then, by definition, $f(\mathfrak{a}(\mu)) \epsilon' x$, in which case, since x is not ϵ'-empty, $f(\mathfrak{a}(\mu)) \epsilon' f(g(x))$. Consequently, if $\mathfrak{a}$ satisfies $\pi(\mu)$ in $\mathfrak{A}$, then $\mathfrak{b}$ satisfies $\mathfrak{s}(\pi(\mu))$ in $\langle D, \epsilon', \underline{o}\rangle$. The argument for the converse direction is entirely similar. The cases for when φ is a negation, conditional or of the form $\bigwedge \alpha\psi$ all follow trivially from the inductive hypothesis. Suppose finally that φ is of the form $\bigwedge \pi\psi$. Then $\mathfrak{a}$ satisfies $\bigwedge \pi\psi$ in $\mathfrak{A}$ iff for all

$x \in D$, $\mathfrak{a}\binom{\pi}{g(x)}$ satisfies ψ in $\mathfrak{A}$, and therefore by the inductive hypothesis, iff for all $x \in D$, $\mathfrak{b}\binom{\overline{\pi}}{f(g(x))}$ satisfies $s(\psi)$ in $\langle D, \epsilon', \underline{o}\rangle$, and consequently by the definition of f, iff for all $x \in D$, if either x is not ϵ'-empty or $x = \underline{o}$, then $\mathfrak{b}\binom{\overline{\pi}}{x}$ satisfies $s(\psi)$ in $\langle D, \epsilon', \underline{o}\rangle$, i.e., for all $x \in D$, $\mathfrak{b}\binom{\overline{\pi}}{x}$ satisfies $[\mathbf{V}\alpha(\alpha \in \overline{\pi}) \mathbf{v}\ \overline{\pi} \equiv \overline{o} \rightarrow s(\psi)]$. We conclude then that if $s(\varphi)$ is true in $\langle D, \epsilon', \underline{o}\rangle$, then φ is ture in $\mathfrak{A}$.

Finally, we note that if φ is a monadic instance of (SCP*), $s(\varphi)$ might fail in being an instance of (CP-NFU′) because '$\overline{o}$' occurs in $s(\varphi)$ in such a way as to require more than one type assignment. All we need do in that case is replace each except the first occurrence of '$\overline{o}$' by a different individual variable new to $s(\varphi)$. The universal closure of the result is then an instance of (CP-NFU′) from which $s(\varphi)$ is derivable. Consequently, in general if φ is an instance of monadic (SCP*), then $s(\varphi)$ is a theorem of **NFU′**, from which it follows that φ is ture in $\mathfrak{A}$. In addition it is clear that if (Inf) and/or (AC) are true in $\langle D, \epsilon', \underline{o}\rangle$,[7] then so are s(Inf*) and s(AC*), and therefore (Inf*) and/or (AC*) are true in $\mathfrak{A}$. Q.E.D.

Now it is easily seen that Jensen's proof (*cf.* [4]) of the consistency of **NFU** (+(Inf), +(AC)) relative to weak Zermelo set theory (+(Inf), +(AC)) applies to the system **NFU′** (+(Inf), +(AC)) as well. Consequently, we are able to state the following result.

Theorem *If weak Zermelo set theory* (+(Inf), +(AC)) *is consistent, then so is* **NFU*** (+(Inf*), +(AC)).

9 *The equiconsistency of* **NFU*** *with the* (*monadic*) *theory of simple types* For convenience we assume the theory of simple types, **ST**, to be given in the form associated with **NF** but where the individual variables have been assigned type-indices (as superscripts) and where atomic wffs must be of the form '$x^i \in y^{i+1}$'. Naturally, because of the restriction on the conditions for well-formedness, the comprehension principle of **ST** is automatically stratified. We include the extensionality axiom as the only other special axiom of **ST**. (The logical basis of **ST** is that of the many-sorted predicate calculus.) When speaking of axioms of infinity and choice we mean (Inf) and (AC) as formulas of **ST**, i.e., where type indices have been suitably assigned to the variables occurring therein (which is possible since the formulations given in section **5** above are easily seen to be stratified) and where the predicate for (type) membership is taken as primitive.

Now it is well-known that the theory of simple types (+(Inf), +(AC)) as described above is equiconsistent with weak Zermelo set theory (+(Inf), +(AC)). (*Cf.* Jensen [4].) Accordingly, since we have already shown that **NFU*** (+(Inf*), +(AC)) is consistent relative to the latter system, then it follows that **NFU***(+(Inf*), +(AC*)) is consistent relative to **ST** (+(Inf), +(AC)). In the following proof we show that the converse also holds and and thereby establish the equiconsistency of **ST** (+(Inf), +(AC)) with **NFU*** (+(Inf*), +(AC*)).

Theorem **NFU*** (+(Inf*), +(AC*)) *is equiconsistent with* **ST** (+(Inf), +(AC)).

Proof: We assume accordingly that **NFU*** is consistent, from which it follows, by the semantical completeness theorem of section **6**, that there exists a stratified Fregean Frame $\mathfrak{A} = \langle D, A, f\rangle$ (restricted to interpreting only monadic wffs). We observe that by (SCP*)

$$\vdash_{\mathbf{NFU^*}} \bigvee F \bigwedge x(F(x) \leftrightarrow \sim \bigvee G[x \equiv G \wedge \bigvee y G(y)])$$

and therefore $W \epsilon A$, where W is the set of those members of D which f does not correlate with any non-empty set in A. In other words, W is the set of "urelements" of D, including the element correlated with the empty set, and therefore W is itself non-empty. (We include the element of D correlated with the empty set in A just to ensure the non-emptiness of W.) In our construction of a model of **ST** we utilize the following recursive definition:

$$U_0 = W$$
$$U_{n+1} = \{f(X) \mid X \epsilon A \text{ and } X \subseteq U_n\}$$

Between each U_i and U_{i+1} we define the following membership relation:

$$x \epsilon_i y =_{df} x \epsilon \breve{f}(y)$$

that is, $\epsilon_i = \{\langle x, y\rangle \mid x \epsilon U_i \text{ and } y \epsilon U_{i+1} \text{ and } x \epsilon \breve{f}(y)\}$. We now set $\mathfrak{B} = \langle U_i, \epsilon_i\rangle_{i \epsilon \omega}$ and show that $\mathfrak{B}$ is a model of **ST**. It suffices for this latter purpose to show that **ST**'s extensionality and comprehension principles are true in $\mathfrak{B}$.

In regard to the extensionality schema of **ST** we note that for $i \epsilon \omega$ and $X, Y \epsilon A$ such that $X, Y \subseteq U_i$, if for all $z \epsilon U_i$, $z \epsilon X$ iff $z \epsilon Y$, then $X = Y$, and therefore $f(X) = f(Y)$. In other words for all $x, y \epsilon U_{i+1}$, if for all $z \epsilon U_i$, $z \epsilon_i \breve{f}(x)$ iff $z \epsilon_i \breve{f}(y)$, then $x = y$; and, accordingly, each instance of **ST**'s extensionality schema is true in $\mathfrak{B}$.

In regard to showing that every instance of **ST**'s comprehension principle is true in $\mathfrak{B}$, we consider first the following characterization of "types" in **NFU***:

$$\mathsf{T}_0(\mu) =_{df} \sim \bigvee \sigma [\mu \equiv \sigma \wedge \bigvee \alpha \sigma(\alpha)]$$
$$\mathsf{T}_{n+1}(\mu) =_{df} \bigvee \sigma [\mu \equiv \sigma \wedge \bigwedge \alpha(\sigma(\alpha) \rightarrow \mathsf{T}_n(\alpha))]$$

where α is an individual variable and σ is the first monadic predicate variable distinct from μ.[8] We note that if φ is a stratified wff in which μ occurs free, then (by a suitable rewrite of bound variables if necessary) $[\mathsf{T}_n(\mu) \wedge \varphi]$ is also stratified. Consequently, by (SCP*),

$$\vdash_{\mathbf{NFU^*}} \bigvee \pi \bigwedge \alpha[\pi(\alpha) \leftrightarrow \mathsf{T}_n(\alpha) \wedge \bigvee \mu(\alpha \equiv \mu \wedge \varphi)]$$

where α is an individual variable and φ is a stratified wff in which μ occurs free and in which π does not occur (free). Therefore, by elementary logical transformations,

$$\vdash_{\mathbf{NFU^*}} \bigvee \pi(\bigwedge \alpha[\pi(\alpha) \rightarrow \mathsf{T}_n(\alpha)] \wedge \bigwedge \mu[\mathsf{T}_n(\mu) \rightarrow (\pi(\mu) \leftrightarrow \varphi)])$$

from which it follows that

$$\vdash_{\mathbf{NFU^*}} \mathbf{V}\pi[\mathrm{T}_{n+1}(\pi) \wedge \Lambda\mu(\mathrm{T}_n(\mu) \rightarrow [\pi(\mu) \leftrightarrow \varphi])]$$

Now let $\mathfrak{t}$ be a correlation between the type-indexed variables of **ST** and the individual and monadic predicate variables but such that $\mathfrak{t}(\alpha^0)$ is always an individual variable and $\mathfrak{t}(\alpha^n)$, for $n \geqslant 1$, is always a predicate variable. We extend $\mathfrak{t}$ into a transformation of the formulas of **ST** into wffs of **NFU*** as follows:

(i) $\mathfrak{t}(\alpha^n \epsilon\, \alpha^{n+1}) = \pi(\sigma)$, where $\mathfrak{t}(\alpha^n) = \sigma$ and $\mathfrak{t}(\alpha^{n+1}) = \pi$
(ii) $\mathfrak{t}(\sim\varphi) = \sim\mathfrak{t}(\varphi)$
(iii) $\mathfrak{t}(\varphi \rightarrow \psi) = [\mathfrak{t}(\varphi) \rightarrow \mathfrak{t}(\psi)]$
(iv) $\mathfrak{t}(\Lambda\alpha^n\varphi) = \Lambda\mu[\mathrm{T}_n(\mu) \rightarrow \mathfrak{t}(\varphi)]$, where $\mathfrak{t}(\alpha^n) = \mu$

We note that $\mathfrak{t}$ always transforms a formula of **ST** into either a stratified wff or one which by suitably rewriting bound variables is provably equivalent in **NFU*** to a stratified wff. Without loss of generality, we will assume that $\mathfrak{t}(\varphi)$ is itself always stratified. In addition, if φ is an instance of **ST**'s comprehension principle, then, by the above observation, $\mathfrak{t}(\varphi)$ is a theorem of **NFU***. Accordingly, it suffices only to show that φ is true in $\mathfrak{A}$.

Now it is easily seen by a simple inductive argument on the natural numbers that where $\mathfrak{a}$ is an assignment in $\mathfrak{A}$, $\mathfrak{a}$ satisfies $\mathrm{T}_n(\mu)$ in $\mathfrak{A}$ iff $f(\mathfrak{a}(\mu)) \epsilon\, U_n$. From this it follows for any wff ψ that (1) $\mathfrak{a}$ satisfies $\Lambda\alpha[\mathrm{T}_0(\alpha) \rightarrow \psi]$ in $\mathfrak{A}$ iff for all $x \epsilon\, U_0$, $\mathfrak{a}\binom{\alpha}{\check{f}(x)}$ satisfies ψ in $\mathfrak{A}$, and (2) $\mathfrak{a}$ satisfies $\Lambda\pi[\mathrm{T}_{n+1}(\pi) \rightarrow \psi]$ in $\mathfrak{A}$ iff for all $x \epsilon\, U_{n+1}$, $\mathfrak{a}\binom{\pi}{\check{f}(x)}$ satisfies ψ in $\mathfrak{A}$. Utilizing this data, we show by induction on the structure of a formula φ of **ST** that where $\mathfrak{b}$ is an assignment in $\mathfrak{B}$, i.e., where $\mathfrak{b}(\alpha^n) \epsilon\, U_n$, and $\mathfrak{a}$ is an assignment in $\mathfrak{A}$ such that $\mathfrak{a}(\mathfrak{t}(\alpha^n)) = \check{f}(\mathfrak{b}(\alpha^n))$ for all type-indexed variables α^n of **ST**, then: $\mathfrak{b}$ satisfies φ in $\mathfrak{B}$ iff $\mathfrak{a}$ satisfies $\mathfrak{t}(\varphi)$ in $\mathfrak{A}$. Suppose φ is an atomic formula $(\alpha^n \epsilon\, \alpha^{n+1})$, where $\mathfrak{t}(\alpha^n) = \sigma$ and $\mathfrak{t}(\alpha^{n+1}) = \pi$. Then, by definition of satisfaction, $\mathfrak{b}$ satisfies $(\alpha^n \epsilon\, \alpha^{n+1})$ in $\mathfrak{B}$ iff $\mathfrak{b}(\alpha^n) \epsilon_n \mathfrak{b}(\alpha^{n+1})$; iff $\mathfrak{b}(\alpha^n) \epsilon\, \check{f}(\mathfrak{b}(\alpha^{n+1}))$; iff $f(\check{f}(\mathfrak{b}(\alpha^n))) \epsilon\, \check{f}(\mathfrak{b}(\alpha^{n+1}))$; iff $f(\mathfrak{a}(\mathfrak{t}(\alpha^n))) \epsilon\, \mathfrak{a}(\mathfrak{t}(\alpha^{n+1}))$; iff $\mathfrak{a}$ sat $\pi(\sigma)$ in $\mathfrak{A}$. The cases for when φ is a negation or conditional follow trivially from the inductive hypothesis. Suppose finally that φ is of the form $\Lambda\alpha^n\psi$. By definition, $\mathfrak{b}$ satisfies $\Lambda\alpha^n\psi$ in $\mathfrak{B}$ iff for all $x \epsilon\, U_n$, $\mathfrak{b}\binom{\alpha^n}{x}$ satisfies ψ in $\mathfrak{B}$, and therefore, by the inductive hypothesis, iff for all $x \epsilon\, U_n$, $\mathfrak{a}\binom{\mathfrak{t}(\alpha_n)}{\check{f}(x)}$ satisfies $\mathfrak{t}(\psi)$ in $\mathfrak{A}$, and consequently, by (1) and (2) above, iff $\mathfrak{a}$ satisfies $\Lambda\mu[\mathrm{T}_n(\mu) \rightarrow \mathfrak{t}(\psi)]$ in $\mathfrak{A}$, where $\mathfrak{t}(\alpha^n) = \mu$. We conclude accordingly that where $\mathfrak{t}(\varphi)$ is true in $\mathfrak{A}$, then φ is true in $\mathfrak{B}$, which completes our proof of the consistency of **ST** relative to **NFU***. We observe as a final note, however, that if (Inf*) and/or (AC*) are also true in $\mathfrak{A}$, then so are (Inf) and (AC) which are easily seen to be related by the converse of the $\mathfrak{t}$-transformation to an infinity and choice axiom in **ST**. Therefore, **ST** +(Inf), +(AC) is consistent if **NFU*** +(Inf*), +(AC*) is. Q.E.D.

10 *Corrigendum and addenda* (added January, 1978) Edmund Gettier has recently pointed out to the author that since predication in **ST*** corresponds to a relation and since being impredicable with respect to this relation is specifiable by a stratified wff, Russell's paradox can be reconstructed in *dyadic* **ST*** after all. It should be noted in this regard, however, that the relative consistency proof provided above is not for the full relational **ST*** system but for monadic **ST***. Indeed, all of the theorems stated above, including the semantic theorem of section **6**, continue to hold as stated.

The author's gloss in this matter was his implicit, and erroneous, assumption that by means of the Wiener-Kuratowski ordered pair construction we can prove the consistency of the full relational **ST*** system relative to that of monadic **ST***. Nevertheless, although the Wiener-Kuratowski construction fails for **ST*** it does suffice to prove the relative consistency of the system **HST*** corresponding to *homogeneous simple type theory*. **HST*** is exactly like **ST*** except for replacing (SCP*), the stratified comprehension principle, by *the homogeneously stratified comprehension principle* (HSCP*) which is itself exactly like (SCP*) described above in section **3** except for the additional restriction that the entire biconditional must be homogeneously stratified. (A wff φ is *homogeneously stratified* if there exists an assignment $\mathfrak{t}$ of natural numbers to the variables occurring in φ such that for each atomic subwff $\pi(\mu_0, \ldots, \mu_{n-1})$ of φ, (1) $\mathfrak{t}(\mu_i) = \mathfrak{t}(\mu_j)$, for all $i, j < n$, and (2) $\mathfrak{t}(\pi) = 1 + \mathfrak{t}(\mu_0)$.)

We observe that a monadic wff is stratified iff it is homogeneously stratified, and that consequently monadic **ST*** is one and the same system as monadic **HST***. Moreover, although predication is specifiable by a stratified wff it does not represent a relation in **HST*** since the biconditional:

$$\vee R \wedge F \wedge x[R(F, x) \leftrightarrow F(x)]$$

is not homogeneously stratified. Furthermore, by Russell's argument, it is provable in **HST*** that there can be no relation corresponding to predication. But other than the fact that predication is not a relation in **HST***, all of the other claims made above regarding **ST*** carry over to **HST***. We shall now sketch a proof of the consistency of the full **HST*** system relative to monadic **ST***. Toward doing so, let us identify n-**HST*** with the system **HST*** restricted to wffs in which, for all $k > n$, no k-place predicate variables occur. Then 1-**HST*** is monadic **HST***, which, as noted, is the same as monadic **ST***, and 2-**HST*** is dyadic **HST***, etc.

Theorem *If monadic* **ST*** *is consistent, then the full* **HST*** *system is also consistent.*

Proof: What we shall actually prove here is that if monadic **ST*** is consistent then so is dyadic **HST***. Essentially the same proof can be used to show that if n-**HST*** is consistent, then so is $(n + 1)$-**HST***, so that, by induction, for all $n \in \omega$, n-**HST*** is consistent; and therefore $\bigcup_{n \in \omega} (n\text{-}\mathbf{HST}^*) =$ **HST*** is consistent, if monadic **ST*** is consistent.

We abbreviate the statement that H is the *ordered pair* whose first constituent is μ and second constituent is η as follows:

$$\mathrm{OP}(\mu, \eta, H) =_{df} \bigvee F \bigvee G(\bigwedge x[F(x) \leftrightarrow x \equiv \mu] \wedge \bigwedge x[G(x) \leftrightarrow x \equiv \mu \vee x \equiv \eta] \wedge \bigwedge x[H(x) \leftrightarrow x \equiv F \vee x \equiv G])$$

Note: for the inductive ease we utilize the following abbreviation as well:

$$\mathrm{OP}^{n+1}(\mu_1, \ldots, \mu_{n+1}, H) =_{df} \bigvee G[\mathrm{OP}^n(\mu_1, \ldots, \mu_n, G) \wedge \mathrm{OP}(G, \mu_{n+1}, H)]$$

We observe now that the following wffs are readily provable in monadic **ST***:

(1) $\bigwedge x \bigwedge y \bigvee H\, \mathrm{OP}(x, y, H)$

(2) $\bigwedge x \bigwedge y \bigwedge w \bigwedge z[\mathrm{OP}(x, y, H) \wedge \mathrm{OP}(w, z, H) \rightarrow x \equiv w \wedge y \equiv z]$

and that therefore, by (2) and the fact that indiscernibility satisfies full substitutivity in monadic **ST***, the following is a theorem schema of monadic **ST***:

(3) $[\bigvee \mu \bigvee \eta(\mathrm{OP}(\mu,\eta,H) \wedge \psi) \leftrightarrow \bigvee \mu \bigvee \eta(\mathrm{OP}(\mu,\eta,H) \wedge \chi] \leftrightarrow \bigwedge \mu \bigwedge \eta[\mathrm{OP}(\mu,\eta,H) \wedge \psi \leftrightarrow \mathrm{OP}(\mu,\eta,H) \wedge \chi]$

Now let '-' (bar) be a 1-1 mapping of all predicate variables into the monadic predicate variables, and extend '-' so as to include the identity map on individual variables. We define a *translation* function $\dagger$ which translates a wff of 2-**HST*** into a wff of monadic **ST*** as follows:

(i) $\dagger(F(\mu)) = \overline{F}(\overline{\mu})$

(ii) $\dagger(R(\mu,\eta)) = \bigvee H[\mathrm{OP}(\overline{\mu},\overline{\eta}, H) \wedge \overline{R}(H)]$

(iii) $\dagger(\sim\varphi) = \sim\dagger(\varphi)$

(iv) $\dagger(\varphi \rightarrow \psi) = (\dagger(\varphi) \rightarrow \dagger(\psi))$

(v) $\dagger(\bigwedge \mu\varphi) = \bigwedge \overline{\mu}\dagger(\varphi)$

To prove our theorem, it suffices now to show that if $\vdash_{\overline{2\text{-HST*}}} \varphi$, then $\vdash_{\overline{1\text{-ST*}}} \dagger(\varphi)$, for all wffs φ of 2-**HST***. However, since modus ponens inferences are preserved under $\dagger$ and since $\dagger$ translates all axioms of 2-**HST*** other than (HSCP*), the homogeneously stratified comprehension principle, into axioms of monadic **ST***, it suffices to prove that $\dagger$ translates each instance of (HSCP*) into a theorem of monadic **ST***. Suppose then that φ is an instance of (HSCP*), i.e., that φ is (a generalization of) a wff either of the form:

(4) $\bigvee F \bigwedge \mu[F(\mu) \leftrightarrow \psi]$

or of the form:

(5) $\bigvee R \bigwedge \mu \bigwedge \eta[R(\mu,\eta) \leftrightarrow \psi]$

where the biconditionals are homogeneously stratified, etc. Now since the biconditionals are homogeneously stratified, ψ must be too; and therefore $\dagger(\psi)$ is stratified. (Note: if ψ were only stratified and not homogeneously stratified, then $\dagger(\psi)$ might not be stratified at all! This is exactly the point where the "proof" of the consistency of dyadic **ST*** relative to monadic

ST* breaks down.) Now if φ is a wff of the form (4) then $\mathrm{t}(\varphi)$ is of the form:

$$\bigvee F \bigwedge \overline{\mu}[F(\overline{\mu}) \leftrightarrow \mathrm{t}(\psi)]$$

and therefore, since $\mathrm{t}(\psi)$ is stratified, $\mathrm{t}(\varphi)$ is an instance of (SCP*), i.e., $\vdash_{\overline{1\text{-ST*}}} \mathrm{t}(\varphi)$. Suppose then that φ is a wff of the form (5). Then $\mathrm{t}(\varphi)$ is of the form:

$$\bigvee \overline{R} \bigwedge \overline{\mu} \bigvee \overline{\eta}(\bigvee H[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \overline{R}(H)] \leftrightarrow \mathrm{t}(\psi))$$

where without loss of generality we can assume that H does not occur in $\mathrm{t}(\psi)$ (otherwise rewrite). But by monadic (SCP*) we have:

$$\vdash_{\overline{1\text{-ST*}}} \bigvee \overline{R} \bigwedge H(\overline{R}(H) \leftrightarrow \bigvee \overline{\mu} \bigvee \overline{\eta}[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \mathrm{t}(\psi)])$$

and therefore, by elementary transformations,

$$\vdash_{\overline{1\text{-ST*}}} \bigvee \overline{R} \bigwedge H(\bigvee \overline{\mu} \bigvee \overline{\eta}[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \overline{R}(H)] \leftrightarrow \bigvee \overline{\mu} \bigvee \overline{\eta}[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \mathrm{t}(\psi)])$$

from which, together with theorem schema (3) above, we have:

$$\vdash_{\overline{1\text{-ST*}}} \bigvee \overline{R} \bigwedge H \bigwedge \overline{\mu} \bigwedge \overline{\eta}[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \overline{R}(H) \leftrightarrow \mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \mathrm{t}(\psi)]$$

and therefore, by elementary transformations,

$$\vdash_{\overline{1\text{-ST*}}} \bigvee \overline{R} \bigwedge \overline{\mu} \bigwedge \overline{\eta}(\bigvee H[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \overline{R}(H)] \leftrightarrow \bigvee H[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \mathrm{t}(\psi)])$$

Now by (1) and the fact that H does not occur in $\mathrm{t}(\psi)$,

$$\vdash_{\overline{1\text{-ST*}}} \mathrm{t}(\psi) \leftrightarrow \bigvee H[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \mathrm{t}(\psi)]$$

and therefore, by interchange,

$$\vdash_{\overline{1\text{-ST*}}} \bigvee \overline{R} \bigwedge \overline{\mu} \bigwedge \overline{\eta}(\bigvee H[\mathrm{OP}(\overline{\mu}, \overline{\eta}, H) \wedge \overline{R}(H) \leftrightarrow \mathrm{t}(\psi)])$$

which is what was to be shown, i.e., $\vdash_{\overline{1\text{-ST*}}} \mathrm{t}(\varphi)$. Q.E.D.

11 *Concluding remarks* It is noteworthy that in standard type theories where the typing of predicates occurs as part of the grammar of the object language we can prove the (relative) consistency of the theory of heterogeneous as well as that of homogeneous types. We now know that this does not carry over directly to the typed *-systems where the typing occurs in the metalanguage instead. Thus, whereas **HST*** and monadic **ST*** are (relatively) consistent, the full, or even just the dyadic, **ST*** system is not.

This situation, moreover, is even more pronounced when we turn to *cumulative type theory*, i.e., a theory of types in which predicates are stratified cumulatively. Again, the system in which the cumulative typing of predicates occurs as part of the grammar of the object language is known to be (relatively) consistent, whereas now even the monadic fragment of the counterpart *-system is inconsistent. To be specific: a wff φ is said to be *cumulatively stratified* if there is an assignment t of natural numbers to the variables occurring in φ such that for each atomic subwff $\pi(\mu_0, \ldots, \mu_{n-1})$ of φ, $\max[\mathrm{t}(\mu_0), \ldots, \mathrm{t}(\mu_{n-1})] < \mathrm{t}(\pi)$. Now let (CSCP*), *The cumulatively stratified comprehension principle*, be that principle which is exactly like (SCP*), the stratified comprehension principle, except for requiring that the comprehending wff in any instance of the principle need only be cumulatively stratified; and let **CST*** be that system which is exactly like **ST*** except for replacing (SCP*) by (CSCP*), i.e., **CST*** = **M*** + (CSCP*). The system **CST*** is of course the *-counterpart of cumulative type theory where the typing of predicates occurs as part of the grammar of the object language.

Now since every stratified wff is cumulatively stratified, **ST*** is a subsystem of **CST***, and therefore the dyadic or full **CST*** system is inconsistent. However, unlike monadic **ST***, monadic **CST*** is also inconsistent. For although the monadic wff

$$\bigvee F \bigwedge x[F(x) \leftrightarrow \bigvee G(x \equiv G \wedge \sim G(x))]$$

is not stratified and therefore is not an instance of (SCP*), it is cumulatively stratified and therefore an instance of (CSCP*). It is also an instance of (CP*), which imposes no stratification conditions at all, and is therefore provable in **T**** as well, as we have already pointed out in section **3**. No contradiction follows from this result in **T****, however, since the requisite thesis for deriving a contradiction in this case, viz., the thesis (Ind*) described in section **3**, is not provable, but disprovable, in **T****. On the other hand, (Ind*) is provable in monadic **ST*** and therefore also in monadic **CST***; and by (Ind*) and the above wff, Russell's paradox follows. Therefore, even monadic **CST*** is inconsistent. (The same argument also shows, incidentally, that replacing the stratified comprehension principle of Quine's system **NF** by a cumulatively stratified comprehension principle again results in a contradiction.)

Thus while the transition from the homogeneous to the heterogeneous to the cumulative theory of simple types remains consistent so long as the typing of predicates occurs as part of the grammar of the object language, the corresponding transition in the *-systems where the typing occurs in the metalanguage does not. What is of philosophical interest here, however, is that while the above transition (where the typing occurs as part of the grammar of the object language) does mark an appropriate increasing generalization on the combined roles of predicates in predicate positions and predicates in subject positions, the generalization is still inadequate since the typing of predicates as part of the grammar of the object language is itself an artificial and limiting device. The consistency of these systems is maintained, in other words, by an artificial and conceptually unmotivated device. Historically, in fact, it was only a stop-gap measure introduced so as to avoid the paradoxes.

In contrast, the appropriate and philosophically significant generalization occurs when we remove these artifically imposed grammatical constrants and turn instead to the *-systems **T****, **HST***, **ST***, and **CST***, each of which includes the comprehension principle and all the theorems of standard predicate logic. The fact that **CST*** and the full or dyadic **ST*** system are inconsistent has at least this rather fascinating philosophical result: that in the two remaining consistent systems, i.e., **T**** and **HST*** (which includes monadic **ST***), *predication does not, indeed cannot, stand for a relation.*

NOTES

1. By referring to this and other similar schemas cited below as *principles*, we mean, for each natural number n, any generalization of any instance of such a schema.

2. The only difference between $\mathbf{M}$ + (CP*) and $\mathbf{T}^{**}$ as originally described is that the latter has in place of (A4) the simpler form $\bigvee\pi(\sigma \equiv \pi)$, which together with the extensionality principle (Ext*) yields (A4).

3. *Cf.* [1] for a fuller discussion of why Russell's argument fails in $\mathbf{T}^{**}$. The system actually discussed in [1], viz., $\mathbf{T}^{*}$, is somewhat stronger than $\mathbf{T}^{**}$, but see [2] for a discussion of this.

4. *Cf.* [3] for a discussion of some of the philosophical issues involved in $\mathbf{T}^{**}$ as a formal ontology.

5. By weak Zermelo set theory we mean the theory $\mathbf{S}$ which is obtained from Zermelo set theory by replacing Zermelo's *Aussonderungsaxiom*

$$\bigvee y \wedge x[x \in y \leftrightarrow x \in z \wedge \varphi]$$

(where 'y' does not occur (free) in φ) by the *weak Aussonderungsaxiom* where only *restricted quantifiers* occur in φ.]

6. In [2] we called a Fregean frame as characterized here a quasi-Fregean model. Similarly, we referred there to the notion of a *general* Fregean frame as characterized below as a *normal* quasi-Fregean model. The present terminology, however, is to be preferred.

7. In speaking of (Inf) and (AC) as true in $\langle D, \epsilon', \bar{o}\rangle$ we mean of course their parallel first order formulations where the binary predicate for membership is taken as primitive.

8. Of course the natural characterization of type 0 is:

$$\mathsf{T}_0(\alpha) =_{df} \sim \bigvee\sigma(\alpha \equiv \sigma)\ ,$$

Our definition above is only so as to ensure the non-emptiness of $W = U_0$.

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