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A SEMANTICAL ACCOUNT OF THE VICIOUS CIRCLE PRINCIPLE

PHILIP HUGLY and CHARLES SAYWARD

Russell claimed that statements about *all* propositions are meaningless (Russell [4], p. 63 and Russell and Whitehead [5], p. 37). Here we attempt to give formal expression of Russell's view by developing a semantical account of propositional quantification that has the vicious circle principle as a consequence. According to the account we give the vicious circle principle bears an interesting relation to the view that there are no extraordinary sets (sets which are members of themselves).

1 We focus on a particular formal system, **F**, which contains these symbols: **I** (a binary connective to be read 'The proposition that . . . is the same as the proposition that _____'), Π (the universal quantifier), sentence variables (p_0, p_1, p_2, . . .), sentence constants (q_0, q_1, q_2 . . .). **F** contains as formulas p_n, q_k, $\mathbf{I}\phi\psi$ and nonvacuous quantifications $\Pi p_n\psi(p_n)$. A sentence is a formula with no free occurrences of variables.

Philosophical considerations count against interpreting Π substitutionally. Consider, for example, that all instances of 'Some sentence of English says that p' are true, but its universal quantification seems false. It seems false because for any language there are more truths and falsehoods (e.g., about numbers) than can be expressed in that language. So a model of **F** has to be so defined that, for some model $\mathfrak{M}$, $\Pi p_n\psi(p_n)$ is false in $\mathfrak{M}$ while $\psi(\alpha)$ is true in $\mathfrak{M}$ for every admissable substituend α of the variable p_n. Such a model must consist in part of a set of propositions such that each of the α's get assigned a proposition but some of the propositions get assigned to no α.

Next a semantical account of **F** must provide a way of evaluating identities

$$\mathbf{I}q_0q_1\ ,$$
$$\mathbf{I}q_0\Pi p_0p_0\ ,$$
$$\mathbf{I}\Pi p_0p_0\Pi p_1p_1\ ,$$

and so on. One way of dealing with this is to structure the domain so that it

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consists of three sorts of elements: those propositions expressed by the constants (elementary propositions), those expressed by the identities, those expressed by the quantifications. (If **F** contained the truth functional connectives, the domain would have to contain negative propositions that get assigned to negation sentences, disjunctive propositions that get assigned to disjunctive sentences, and so on.)

On the basis of such considerations we define a domain as follows:

A domain D is a union of sets $D_0, D_1, \ldots, D_i, \ldots$ D_0 is a set consisting of at least three objects. D_{n+1} is D_n plus all nonempty subsets of D_n.

A domain consists of three different sorts of elements: (*i*) basic elements belonging to D_0 (think of these as elementary propositions); (*ii*) one or two membered sets (think of these as identity propositions); (*iii*) sets consisting of at least three elements (think of these as quantification propositions). The elementary propositions are to be assigned to the sentence constants; identity propositions get assigned to identity sentences; quantification propositions get assigned to quantification sentences.

A second component of a model is a function that does this assigning. We call such a function a proposition function. Formally a proposition function f maps $\{q_0, q_1 \ldots\}$ into D_0 (the elementary propositions) and is defined for compound sentences as follows:

(fi) $f(\mathbf{I}\phi\psi) = \{f(\phi), f(\psi)\}$
(fii) $f(\Pi p_n\psi(p_n)) = \{f'(\psi(q_k))$: q_k is the first constant not occurring in $\psi(p_n)$ and f' agrees with f for all constants except possibly $q_k\}$.

Now a sentence ϕ will be true (false) in a model just in case the proposition ϕ expresses in the model, $f(\phi)$, is true (false). So a third component is a function that assigns truth values to the propositions comprising a domain. An evaluation function v maps D_0 into $\{1, 0\}$ and is defined for complex propositions as follows:

(vi) If x and y are elements of D, $v(\{x, y\}) = 1$ if $x = y$; otherwise $v(\{x, y\}) = 0$.
(vii) If A is a three or more membered set that is an element of D, $v(A) = 1$ if $v(x) = 1$ for all $x \epsilon A$; otherwise $v(A) = 0$.

A model is a triple, $\langle \mathrm{D}, f, v\rangle$, where D is a domain, f a proposition function and v an evaluation function. A sentence ϕ is true in a model, $\langle \mathrm{D}, f, v\rangle$, if and only if $v(f(\phi)) = 1$. Validity and the consequence relation are defined in the usual way.

2 The sentence

$$\Pi p_0 \mathbf{I} p_0 p_0$$

says all elementary propositions are self identical; it does not say anything about all propositions. Generally, a universal quantification $\Pi p_n\psi(p_n)$ says something about all elementary propositions.

Changes are possible which permit one to go beyond talk of elementary propositions. For example, we might introduce into **F** sentence variables of different levels:

$$\begin{array}{l} p_0^0,\ p_1^0,\ \ldots \\ p_0^1,\ p_1^1,\ \ldots \\ p_0^2,\ p_1^2,\ \ldots \\ \quad \cdot \quad \cdot \quad \cdot\cdot\cdot \\ \quad \cdot \quad \cdot \quad \cdot\cdot\cdot \end{array}$$

Similarly, for constants:

$$\begin{array}{l} q_0^0,\ q_1^0,\ \ldots \\ q_0^1,\ q_1^1,\ \ldots \\ q_0^2,\ q_1^2,\ \ldots \\ \quad \cdot \quad \cdot \quad \cdot\cdot\cdot \\ \quad \cdot \quad \cdot \quad \cdot\cdot\cdot \end{array}$$

The idea would be that

$$\Pi p_n^h \psi(p_n^h)$$

should say something about all elements of D_h (all h-level propositions), and that a constant q_k^h should express some particular proposition in D_h (some particular h-level propositions).

The only changes needed would be in the definition of proposition function f.

(fi)* $f(q_k^h) \in D_h$
(fii)* $f(\mathbf{I}\phi\psi) = \{f(\phi), f(\psi)\}$
(This is the same as before.)
(fiii)* $f(\Pi p_n^h \psi(p_n^h) = \{f'(\psi(q_k^h))\colon$ q_k^h is the first sentence constant of level h not in ψ and f' agrees with f except possibly for what is assigned to $q_k^h\}$

Take an example. The sentence

$$\Pi p_0^1 p_0^1$$

says all 1-level propositions are true. The proposition this expresses, f('$\Pi p_0^1 p_0^1$'), is a second level proposition (a member of D_2). In fact, f('$\Pi p_0^1 p_0^1$') $= D_1$ and $D_1 \in D_2$.

So we can talk about all elementary propositions, all 1-level propositions, all 2-level propositions, etc. Can we talk about all propositions? Is it possible to effect changes such that a quantification says something about all members of D?

Evidently not. Suppose variable β is introduced to range over all elements of D so that $\Pi\beta\beta$ expressed the proposition that all members of D are true. This very proposition—$f(\Pi\beta\beta)$—would itself be a member of D. Thus D would have to be a member of itself. Similarly, in order for $\Pi\beta\mathbf{I}\beta\beta$ to say *all* propositions are self identical D would have to contain an extraordinary set as member.

Our definition of a domain permits a domain to be neither an extraordinary set nor to contain extraordinary sets as members. Thus, from the perspective we have sought to express formally, the cost of construing a quantification as expressing a proposition about all propositions would require changing the definition of a domain so as to allow the existence of extraordinary sets. Such sets are radically counterintuitive; this is a point stressed by such authors as Fränkel and Suppes, and, we think, evident in itself. (See Fränkel [2], pp. 425-426, Suppes [6], pp. 53-55).

3 Were our semantical account of propositional quantification the only one available, we feel a strong case would have been made for Russell. But there does exist an alternative account given by Grover in [3] using ideas from Belnap in [1]. On the Belnap-Grover account talk about all propositions would be possible. So our account does not settle matters. Nonetheless (we think) an advance has been made, since it is now possible to discuss whether Russell was right in a more definite way than before, namely, by comparing two formal theories.

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University of Nebraska-Lincoln
Lincoln, Nebraska