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SCOTT'S MODELS AND ILLATIVE COMBINATORY LOGIC

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Introduction In most work on illative combinatory logic the equality predicate is assumed to have properties that include:

(ξ) *If* $X = Y$, *then* $\lambda uX = \lambda uY$.

and

(η) *If* u *is not free in* X, $\lambda u(Xu) = X$,

which together are equivalent to the extensionality property

(ζ) *If* $Xu = Yu$ *for all* u *not free in* X *and* Y, *then* $X = Y$.

Scott's model $\mathfrak{D}_\infty$ for pure combinatory logic (see [8]) satisfies all these properties, but his graph model in [10] does not satisfy (η) (or (ζ)), (ξ) is satisfied but Scott seems to have some doubt about it.

In [7] where Kleene sets up a formal system in which intuitionistic mathematics can be developed, (ξ) and (η) are not used. Kleene defines λuX only if X is a "term", λuX is then a "functor" (which is not a term). In his formal system equality is defined only over terms so that (ξ) and (η) are meaningless.

It is therefore of some interest to see to what extent combinatory logic can be applied if it lacks these rules. We show in this paper that the results in [1]-[5]—the basing of propositional and (higher order) predicate calculus and set theory on combinatory logic—go through with only minor modifications if a weak equality (one without (ξ) and (η)) is used. We also investigate how much of the illative system developed in these papers can be interpreted in models such as Scott's model in [9]. Seldin has done this for a similar system which since then, he has proved to be inconsistent.

A system of rules If we take as primitive rules:

Rule Eq *If* $X = Y$, *then* $X \vdash Y$.

Ξ $\Xi XY, XU \vdash YU$.

DTΞ *If* Δ, $XU \vdash YU$ *where* U *is an indeterminate not free in* Δ, X *or* Y, *then* Δ, $\mathbf{L}X \vdash \Xi XY$.

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H $\quad X \vdash \mathbf{H}X$

HΞ $\quad \mathbf{F}X\mathbf{H}Y,\ \mathbf{L}X \vdash \mathbf{H}(\Xi XY),$

where $\mathbf{L}$ is $\mathbf{FAH}$ or $\mathbf{FHH}$,[1] and we take as axioms

LH $\quad \vdash \mathbf{FHHH}$

LA $\quad \vdash \mathbf{FAHA},$

where the = in Rule Eq is the weak equality, we can develop propositional and predicate calculus and set theory as in [2] and [3] if we amend a few of the proofs. As an illustration we amend the proof of Theorem 5 of [2]: We have by HΞ

$$\mathbf{F}(\mathbf{K}X)\mathbf{H}(\mathbf{K}Y),\ \mathbf{L}(\mathbf{K}X) \vdash \mathbf{H}(\Xi(\mathbf{K}X)(\mathbf{K}Y))$$

so to prove

$$X \supset \mathbf{H}Y,\ \mathbf{H}X \vdash \mathbf{H}(X \supset Y)$$

we need

$$\mathbf{H}X,\ X \supset \mathbf{H}Y \vdash \mathbf{F}(\mathbf{K}X)\mathbf{H}(\mathbf{K}Y) \tag{1}$$

and

$$\mathbf{H}X \vdash \mathbf{L}(\mathbf{K}X). \tag{2}$$

Now by Rule Eq,

$$\mathbf{H}X,\ \mathbf{A}U \vdash \mathbf{BH}(\mathbf{K}X)U$$

so an application of DTΞ gives (2).

$X \supset \mathbf{H}Y$ is $\Xi(\mathbf{K}X)(\mathbf{K}(\mathbf{H}Y))$, so

$$X \supset \mathbf{H}Y,\ X \vdash \mathbf{H}Y$$

and by Rule Eq

$$X \supset \mathbf{H}Y,\ \mathbf{K}XU \vdash \mathbf{BH}(\mathbf{K}Y)U$$

so (1) follows by (2) and DTΞ.

A finite formulation If however, we use a finite formulation as in [1] (with $\mathbf{L} = \mathbf{FAH}$ or $\mathbf{FHH}$), one vital step in the proof of DTΞ from the axioms fails if we do not have (η) and (ξ)[2] and in fact it is possible to prove theorems with DTΞ (for example $\vdash \Xi\mathbf{A}(\mathbf{BA}(\mathbf{SKS}))$) which are not provable using the axioms without one of (ξ) and (η). The proof of DTΞ in [1] proceeds by induction on the number of steps in the proof of YU from Δ and XU and we reconsider the case where YU is obtained by Rule Eq. We have by the inductive step:

$$\Delta \vdash \Xi XZ,$$

we have $ZU = YU$ and we want to conclude

$$\Delta \vdash \Xi XY$$

and we use (ξ), (μ) (*If* $X = Y$, *then* $VX = VY$) and (Eq). If we add the extra rule:

$$\textit{If } YU = ZU, \textit{ then } \mathbf{L}X,\ \Xi XZ \vdash \Xi XY,^3$$

this problem is overcome, but we have to then consider the case where this new rule is used in the proof of YU from Δ and XU. We then require:

$$\textit{If } YU = ZU, \textit{ then } \mathbf{L}X,\ \mathbf{L}X_1,\ \Xi X_1(\lambda u_1(\Xi XZ)) \vdash \Xi X_1(\lambda u_1(\Xi XZ)).$$

etc.

The following generalized version of Rule Eq encompasses all such rules:

Rule EqΞ *If* $YU = ZU$ *where* U *is not free in* X_1, Y, *or* Z, *then for all* $n \geq 0$ *and all terms* $X_1 \ldots X_n$ *and variables* $v_1 \ldots v_{n-1}$ $\mathbf{L}X$, $\mathbf{L}X_1 \ldots \mathbf{L}X_n$, $\Xi X_n(\lambda v_{n-1}(\Xi X_{n-1}(\lambda v_{n-2} \ldots \lambda v_1(\Xi X_1 Y) \ldots) \vdash \Xi X_n(\lambda v_{n-1}(\Xi X_{n-1}(\lambda v_{n-2} \ldots \lambda v_1(\Xi X_1 Z) \ldots)$.

Note that for $n = 0$ this is Rule Eq. Rule EqΞ is implied by Rules Eq and (ξ) or by Rules Eq and DTΞ and it satisfies the model of [10].

Interpretations in Scott's model in [9] Now we look at the interpretation of our illative system in this model. If we interpret ΞXY as $\forall u(\sim u \in X \vee u \in Y)$ (in Scott's notation, using Scott's quantifiers and connectives which have far stronger properties than those in [2]) and interpret $\mathbf{A}$ as a universal class we have an interpretation similar to that developed by Seldin. Thus Rules Eq, Ξ, DTΞ, HΞ (both for $\mathbf{L} = \mathbf{FAH}$)[4] and H and Axiom LA hold in the model. DTΞ and HΞ for $\mathbf{L} = \mathbf{FHH}$ however clearly do not hold in the model as they would require $\mathbf{VH}$ and so $\forall u . \mathbf{H}u \vee \sim \mathbf{H}u$, which leads to a contradiction as in [6].

Of the work in [2] we therefore cannot interpret the sections dealing with conjunction and disjunction as these are defined in terms of quantification over $\mathbf{H}$. The standard false proposition $\Xi\mathbf{HI}$ of [2] must be interpreted as $\forall u . u$ which is not a proposition in the sense of [8] (i.e., it is not such that $\forall u . u \vee \sim \forall u . u$). The alternative standard false proposition $\mathbf{Q}(\mathbf{KI})(\mathbf{BI})$ of [2] (this assumes the introduction of an equality $\mathbf{Q}$ with appropriate properties) is a proposition in the sense of [9] if $\mathbf{Q}xy$ is interpreted as $X = Y$. Thus the strong negation of [2] can be interpreted in the model. As Seldin shows that Peirces law holds in the model, we have that the full classical system of [2] can be interpreted in the model, while the intuitionistic subsystem cannot.

In the development of the Zermelo-Fraenkel and Bernays set theories in [3], $\mathbf{A}$ and $\mathbf{M}$ (the class of sets) are identified, so as we had to identify $\mathbf{A}$ and $\mathbf{E}$ to interpret the predicate calculus, it is clear that no substantial part of either system can fit the model. For the Gödel set theory developed in [3], however, $\mathbf{A}$ and $\mathbf{M}$ were not identified and a number of axioms used there hold in the model. We would, however, require an interpretation of $\mathbf{M}$ in the model for a substantial part of the set theory to hold in the model.

NOTES

1. In [1] and [2] quantification over **H** was possible by means of the Axiom $\vdash$**FAHH**, however this is inconsistent if **A** = **E** (a universal class) or with certain other axioms. In [4] the use of $\vdash$**FAHH** in the proof of DTΞ was avoided by taking **L** as primitive however this approach does not allow the derivation of the properties of conjunction and distinction as they are defined in terms of Ξ in [2]. The approach used here is that of [5].

2. Using only (ξ) we can prove:

   If $\Delta, X \vdash Y$, *then if* u *is not free in* Δ, $\Delta, \mathbf{L}(\lambda uX) \vdash \Xi(\lambda uX)(\lambda uY)$.

   When u is not free in X or Y this gives us the deduction theorem for **P** (implication) as in [1], but without (η) this is of little use when u is free in X or Y.

3. A stronger version of this:

   If $XU = YU$ *and* $ZU = VU$, *then* $\Xi XZ \vdash \Xi YV$,

   suggested by J. P. Seldin lead the author to this rule and eventually to Rule EqΞ below.

4. As for **A** = **E**, **V** and **L** can be identified, Scott's deduction theorem for **F** in [8] is virtually identical to that in [2], which was first published in the authors dissertation in 1969.

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