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Notre Dame Journal of Formal Logic
Volume XX, Number 3, July 1979
NDJFAM

GENERALIZED RESTRICTED GENERALITY

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Introduction When we write $(\forall u).X(u) \supset Y(u)$ in predicate calculus, we require that $X(u) \supset Y(u)$ makes sense for all u in the range of quantification. This requirement for every pair of unary predicates X and Y in the calculus may impose a strong restriction on the range of quantification of the system. Hence there may well be Xs, and Ys in the system for which $X(u) \supset Y(u)$ holds or at least makes sense for one or more us not in the range of quantification.

This problem, for unary predicates, is overcome by the use of Curry's restricted generality Ξ (see [1]) which has the rule:

Rule Ξ $\qquad \Xi XY, XU \vdash YU,$

This rule does not restrict the Us we use to any particular range. (Note that we write XU instead of $X(U)$, also we will usually write $Xu \supset_u Yu$ for ΞXY.)

If, however, X and Y are binary predicates we find that the problem arises again. If we want to represent "Whenever XUV holds, YUV holds" using Ξ, the best we can do is what is suggested in [2], i.e., to write:

$$X_1 u \supset_u (X_2 uv \supset_v Yuv),$$

where X_1 is a range of quantification. Taking A for X_1 as a common range of quantification for all such Xs and Ys may well be as inappropriate as it was above and finding an X_1 and X_2 may not be possible for each X, so it seems reasonable to introduce a generalized version of Ξ. If we introduce a Ξ^2 such that

$$\Xi^2 XY, XU_1U_2 \vdash YU_1U_2$$

and similarly $\Xi^3, \ldots \Xi^n \ldots$ all such problems are solved. If we now want to represent whenever XUV and YUV hold, ZUV holds we can use

$$Xuv \wedge Yuv \supset_{u,v} Zuv$$

provided we have the conjunction $\wedge$. If, however, we want to leave open the

Received July 27, 1977

possibility of defining $\wedge$ in terms of Ξ and other notions (as in [4]) we have to have some other way of representing this.

We therefore introduce a version ${}^k\Xi^n$ of Ξ that generalizes it in two ways. These are brought out in the following rule:

Rule ${}^k\Xi^n$ $\quad {}^k\Xi^n X_1 \ldots X_k Y, X_1 U_1 \ldots U_n, \ldots, X_k U_1 \ldots U_n \vdash YU_1 \ldots U_n.$

where k and n are non-negative integers.

We show below that with axioms similar to those given for Ξ in [3], we can prove a Deduction Theorem for ${}^k\Xi^n$ similar to that proved for Ξ in [3].

Rule ${}^k\Xi^n$ and the Deduction Theorem for ${}^k\Xi^n$ We should note that as it stands we have not only generalized Rule Ξ (which is Rule ${}^k\Xi^n$ with $k = n = 1$) to cases where $k \geq 1$ and $n \geq 1$, but also to ${}^0\Xi^n$, a generalized universal generality (${}^0\Xi^1$ corresponds to Π in [1]) and to ${}^k\Xi^0$, a generalized implication (${}^1\Xi^0$ corresponds to $\mathbf{P}$ in [1]). In most systems we will not need a Rule ${}^k\Xi^n$ for each k, $n \in N$. If we have Rule ${}^k\Xi^n$ for k and n sufficiently large we can define:

$${}^i\Xi^{j-1} = \lambda x_1 \ldots \lambda x_i \lambda y\ {}^i\Xi^j(\mathbf{K}x_1) \ldots (\mathbf{K}x_i)(\mathbf{K}y)^1$$

and

$${}^{i-1}\Xi^j = \lambda x_1 \ldots \lambda x_{i-1} \lambda y\ {}^i\Xi^j x_1 \ldots x_{i-1}(\mathbf{K}(\ldots(\mathbf{K}T)\ldots))y$$

where there are j $\mathbf{K}$s in $(\mathbf{K}(\ldots(\mathbf{K}T)\ldots))$ and where T is any theorem.

These with Rule ${}^k\Xi^n$ will give us Rule ${}^i\Xi^j$ for $i \leq k$ and $j \leq n$.

Given a small number of axioms for Ξ, and either $\mathbf{H}$ ("$\mathbf{H}X$" represents "X is a proposition") or $\mathbf{L}$ ("$\mathbf{L}X$" represents "X is a first order predicate"), Rule Ξ can be reversed as follows, (see [3] and [5]):

The Deduction Theorem for Ξ. *If Δ, $XU \vdash YU$ where Δ is any sequence of* obs *and U is an indeterminate not free in Δ, X, or Y, then Δ, $\mathbf{L}X \vdash \Xi XY$.*

If we write "$\mathbf{L}_n X$" for "X is an n-ary predicate", we can set up similar axioms to prove the following Deduction Theorem for ${}^k\Xi^n$:

The Deduction Theorem for ${}^k\Xi^n$. *If $\Delta, X_1 U_1 \ldots U_n, \ldots, X_k U_1 \ldots U_n \vdash YU_1 \ldots U_n$ where $U_1, \ldots, U_n$ are indeterminates not free in $\Delta, X_1, \ldots, X_k$ or Y and $\Delta \vdash \mathbf{L}_n X_i$ for $1 \leq i \leq k$, then $\Delta \vdash {}^k\Xi^n X_1 \ldots X_k Y$.*

The axioms required for the proof of this (numbered as in [3]) are:

Axiom 2 $\vdash \mathbf{L}_n x_1 \supset_{x_1} \ldots \mathbf{L}_n x_k \supset_{x_k} x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} x_i u_1 \ldots u_n$[2]

1. $\mathbf{K}$ is a combinator with the property $\mathbf{K}XY = X$ for all X and Y.

2. For expressions involving $\supset$, $\supset_{x_i}$, $\supset_{u_1,\ldots u_n}$ etc. we assume association to the right.

Axiom 3 $\vdash \mathbf{L}_n x_1 \supset_{x_1} \ldots \mathbf{L} x_k \supset_{x_k} \mathbf{H} y \supset_y y \supset x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1 \ldots u_n} y$

Axiom 4 $\vdash \mathbf{L}_n x_1 \supset_{x_1} \ldots \mathbf{L}_n x_k \supset_{x_k} (x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} w_1 u_1 \ldots u_n v_1 \ldots v_q, \ldots, w_t u_1 \ldots u_n v_1 \ldots v_q \supset_{v_1,\ldots,v_q} y u_1 \ldots u_n v_1 \ldots v_q) \supset_{w_1,\ldots,w_t,y} [\{x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} w_1 u_1 \ldots u_n (t_1 u_1 \ldots u_n) \ldots (t_q u_1 \ldots u_n)\}, \ldots, \{x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} w_t u_1 \ldots u_n (t_1 u_1 \ldots u_n) \ldots (t_q u_1 \ldots u_n)\} \supset_{t_1,\ldots,t_q} (x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} y u_1 \ldots u_n (t_1 u_1 \ldots u_n) \ldots (t_q u_1 \ldots u_n)]$

Axiom 6 $\vdash x \supset_x \mathbf{H} x$.

In [3] and [5] we also needed a universal class **E** to express these axioms, here this is not necessary and so **E** and **Q** for equality (in [3] we defined **E** to be **WQ**) become optional extras. If **Q** were included we could add the following axioms to the above:

$$\vdash \mathbf{WQK}, \vdash \mathbf{WQS}, \vdash \mathbf{WQ}\,{}^k\Xi^n, \vdash \mathbf{WQQ}, \vdash \mathbf{WQL}_n,$$

Axiom 1 $\vdash \mathbf{WQ} x \supset_x \mathbf{WQ} y \supset_y \mathbf{WQ}(xy)$.

Axiom 5 $\vdash \mathbf{L}_n x_1 \supset_{x_1} \ldots \mathbf{L}_n x_k \supset_{x_k} x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} \mathbf{WQ} u_i$, *for* $1 \leqslant i \leqslant k$.

The following theorems follow from Axioms 2, 3, and 4:

Theorem 1 $\mathbf{L}_n x_1, \ldots, \mathbf{L}_n x_k \vdash x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} x_i u_1 \ldots u_n$, *for* $1 \leqslant i \leqslant k$.

Theorem 2 $\mathbf{L}_n x_1, \ldots, \mathbf{L}_n x_k, Y \vdash x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} Y$.

Theorem 3 $\mathbf{L}_n x_1, \ldots, \mathbf{L}_n x_k, (x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} w_1 u_1 \ldots u_n (t_1 u_1 \ldots u_n) \ldots (t_q u_1 \ldots u_n), \ldots, (x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} w_t u_1 \ldots u_n (t_1 u_1 \ldots u_n) \ldots (t_q u_1 \ldots u_n)), [x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} \{w_1 u_1 \ldots u_n v_1 \ldots v_q, \ldots, w_t u_1 \ldots u_n v_1 \ldots v_q \supset_{v_1,\ldots,v_q} y u_1 \ldots u_n v_1 \ldots v_q\}]$
$\vdash x_1 u_1 \ldots u_n, \ldots, x_k u_1 \ldots u_n \supset_{u_1,\ldots,u_n} y u_1 \ldots u_n (t_1 u_1 \ldots u_n) \ldots (t_q u_1 \ldots u_n)$.

(With $n = k = 1$ these are identical to Theorems 1, 2, and 3 of [3]).

To prove Theorem 1 from Axiom 2 we require only Rule ${}^1\Xi^1$, to prove Theorem 2 we need Rule ${}^1\Xi^1$, Rule ${}^1\Xi^0$ (i.e., Rule **P**) and Axiom 6, but to prove Theorem 3 we need Rules ${}^1\Xi^1$, ${}^1\Xi^{t+1}$, and ${}^t\Xi^q$.

Proof of the Deduction Theorem for ${}^k\Xi^n$*:* Let there be p steps $Y_1 U_1 \ldots U_n, \ldots, Y_p U_1 \ldots U_n = Y U_1 \ldots U_n$ in the proof of $Y U_1 \ldots U_n$ from Δ and $X_1 U_1 \ldots U_n, \ldots, X_k U_1 \ldots U_n$. We show by induction on m that provided

$$\begin{aligned} &\Delta \vdash \mathbf{L}_n X_i, \text{ for } 1 \leqslant i \leqslant k, \\ &\Delta \vdash {}^k\Xi^n X_i \ldots X_k Y_m, \text{ for } 1 \leqslant m \leqslant p. \end{aligned} \qquad (1)$$

There are five cases to consider (assuming $\vdash \mathbf{WQ} U$ is included):

1. Y_m is X_i for some $1 \leqslant i \leqslant k$,
2. $Y_m U_1 \ldots U_n$ is a constant (wrt $U_1, \ldots, U_n$), i.e., an axiom or a part of Δ,
3. $Y_m U_1 \ldots U_n$ is $\mathbf{WQ}U_i$,
4. $Y_m U_1 \ldots U_n$ is obtained from $Y_i U_1 \ldots U_n$ by Rule Eq.,
5. $Y_m U_1 \ldots U_n$ is obtained from $Y_{i_1} U_1 \ldots U_n, \ldots, Y_{i_t} U_1 \ldots U_n$ and $Y_j U_1 \ldots U_n$ by Rule ${}^t\Xi^n$ where $i_1 \ldots i_t, j < m$.

Cases 1, 2, and 3 involve no inductive hypotheses and so take care of the $m = 1$ step, but they are also applicable when $m > 1$. In the inductive step the theorem is assumed for Y_t with $t < m$. Cases 1 and 2 are given directly by Theorems 1 and 2 and Case 3 follows from Axiom 5 by applying Rule ${}^1\Xi^1$ k times.

Case 4: If $\Delta \vdash Y_m U_1 \ldots U_n$ follows from $\Delta \vdash Y_l U_1 \ldots U_n$ and

$$Y_m U_1 \ldots U_r = Y_l U_1 \ldots U_r \; (0 \leqslant r \leqslant n),$$

then it follows that $Y_m = Y_l$ so that:

$$X_1 u_1 \ldots u_n, \ldots, X_k u_1 \ldots u_n \supset_{u_1 \ldots u_n} Y_l u_1 \ldots u_n = X_1 u_1 \ldots u_n, \ldots, X_k u_1 \ldots u_n \supset_{u_1 \ldots u_n} Y_m u_1 \ldots u_n;$$

the result follows.

Case 5: Let $Y_m U_1 \ldots U_n$ be obtained from $Y_{i_1} U_1 \ldots U_n, \ldots, Y_{i_t} U_1 \ldots U_n$ and $Y_j U_1 \ldots U_n$ by Rule ${}^t\Xi^q$[3] (with $i_1, \ldots, i_t, j < m$, $t \leqslant k$, and $g \leqslant n$). Then $Y_j U_1 \ldots U_n$ must have the form

$$W_1 U_1 \ldots U_n v_1 \ldots v_q, \ldots, W_t U_1 \ldots U_n v_1 \ldots v_q \supset_{v_1 \ldots v_q} Z U_1 \ldots U_n v_1 \ldots v_q$$

where

$$W_p U_1 \ldots U_n V_1 \ldots V_q = Y_{i_p} U_1 \ldots U_n \text{ for some } V_1, \ldots, V_q \text{ and all } p,\ 1 \leqslant p \leqslant t,$$

(N.B. each V_r may involve $U_1 \ldots U_n$) and

$$Z U_1 \ldots U_n V_1 \ldots V_q = Y_m U_1 \ldots U_n.$$

By the inductive hypothesis we have:

$$\Delta \vdash X_1 u_1 \ldots u_n, \ldots, X_k u_1 \ldots u_n \supset_{u_1 \ldots u_n} W_p u_1 \ldots u_n V_1 \ldots V_q, \text{ for } 1 \leqslant p \leqslant t$$

and

$$\Delta_0 \vdash X_1 u_1 \ldots u_n, \ldots, X_k u_1 \ldots u_n \supset_{u_1 \ldots u_n} (W_1 u_1 \ldots u_n v_1 \ldots v_q, \ldots, W_l u_1 \ldots u_n v_1 \ldots v_q \supset_{v_1 \ldots v_q} Z u_1 \ldots u_n v_1 \ldots v_q).$$

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3. If ${}^t\Xi^q$ for $t \leqslant k$ and $q \leqslant n$ is defined in terms of ${}^k\Xi^n$ in the way suggested above, we need only consider uses of Rule ${}^k\Xi^n$ itself.

Thus taking Theorem 3 with V_r for $t_r u_1 \ldots u_n$ $(1 \leqslant r \leqslant q)$ and Z for Y we obtain (1).

Note that in proving this deduction theorem we have used Rule ${}^i\Xi^j$ only for $i \leqslant k$ and $j \leqslant n$ except perhaps in the proof of Theorem 3 where we used ${}^1\Xi^{t+1}$ where $t \leqslant k$. If we have Rule ${}^k\Xi^n$ for all non-negative integers k and n we clearly also have the Deduction Theorem for ${}^k\Xi^n$ for all k and n. If we have Rule ${}^i\Xi^j$ only for $i \leqslant k$ and $j \leqslant n$ where $k < n$ we can prove the Deduction Theorem for ${}^i\Xi^j$ for all $i \leqslant k$ and $j \leqslant n$. If, however, $k \geqslant n$ we can only prove the Deduction Theorem for ${}^i\Xi^j$ for $i \leqslant n - 1$ and $j \leqslant n$ because of our need of Rule ${}^1\Xi^{t+1}$ for $t \leqslant i$.

Also note that $\mathbf{L}_n$ has been left completely unspecified in the axioms. It could represent the class of n-ary predicates ranging over individuals, i.e., $\mathbf{L}_n = \mathbf{F}_n\mathbf{A} \ldots \mathbf{AH}$, over other predicates, e.g., $\mathbf{L}_n = \mathbf{F}_n(\mathbf{FAH})(\mathbf{F}_2\mathbf{AAH}) \ldots \mathbf{H}$, over propositions, i.e., $\mathbf{L}_n = \mathbf{F}_n\mathbf{H} \ldots \mathbf{H}$, over functions, e.g., $\mathbf{L}_n = \mathbf{F}_n(\mathbf{FAA})(\mathbf{F}_2\mathbf{AAA}) \ldots \mathbf{H}$ or over any combination of these, e.g., $\mathbf{L}_n = \mathbf{F}_n\mathbf{H}(\mathbf{FAA})(\mathbf{F}_2\mathbf{AAH}) \ldots \mathbf{H}$.

Thus as soon as we decide on a definition of $\mathbf{L}_n$ in Axioms 2, 3, 4 (and 5) we have a deduction theorem for ${}^k\Xi^n$ in terms of that $\mathbf{L}_n$. Of course certain choices of $\mathbf{L}_n$ will lead to an inconsistency (such as Curry's paradox for $\mathbf{L}_n = \mathbf{WQ}$—see [1]). It is also possible, as it was in [3], to do without Axiom 2, but this would be at the cost of complicating Axiom 3 somewhat.

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