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ON FULL CYLINDRIC SET ALGEBRAS

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By a full cylindric set algebra of dimension α, full $\mathbf{CSA}_\alpha$, where α is an ordinal number, we mean a system

$$\mathfrak{A} = \langle A, \cup, \cap, \sim, 0, {}^\alpha U, \mathbf{C}_\kappa, \mathbf{D}_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$$

where U is a non-empty set, A is the power set of ${}^\alpha U$, 0 is the empty set, $\cup$, $\cap$, and $\sim$ are the set theoretic union, intersection and complement on A, and for all κ, $\lambda < \alpha$, $\mathbf{C}_\kappa$ is a unary operation on A and $\mathbf{D}_{\kappa\lambda}$ is a constant defined as follows:

$$\begin{aligned}\mathbf{C}_\kappa X = \{y\colon y \in {}^\alpha U &\text{ and for some } x \in X \text{ we have } x_\lambda = y_\lambda \text{ for all } \lambda \neq \kappa\}\\ &\text{for every } X \in A,\end{aligned}$$

and

$$\mathbf{D}_{\kappa\lambda} = \{y\colon y \in {}^\alpha U \text{ and } y_\kappa = y_\lambda\}$$

(*cf.* 1.1.5, [2]). In section **1** we give an axiom system for a subclass of cylindric algebras, which we call strong cylindric algebras, and show that $\mathfrak{A}$ is a strong $\mathbf{CA}_\alpha$, $\alpha < \omega$, if, and only if, $\mathfrak{A}$ is isomorphic to a full $\mathbf{CSA}_\alpha$.

In section **2** we restrict our attention to the theory of strong $\mathbf{CA}_2$ and show that it is definitionally equivalent to the theory of a subclass of relation algebras axiomatized by McKinsey [3].

The notation of [1] is used, and a familiarity with chapter 1 of that book is assumed.

1 *Strong cylindric algebras* We begin by introducing a piece of notation which will prove to be convenient.

Definition 1.1 If $\mathfrak{A}$ is a $\mathbf{CA}_\alpha$, $\alpha < \omega$, and $i < \alpha$, then

$$\mathbf{c}^i x = \mathbf{c}_{(\alpha\sim\{i\})} x.$$

Definition 1.2 By a strong cylindric algebra of dimension α, where α is an ordinal number less than ω, we mean a structure

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$$

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such that 0, 1 and $\mathbf{d}_{\kappa\lambda}$ are distinguished elements of A (for all $\kappa, \lambda < \alpha$), $-$ and $\mathbf{c}_\kappa$ are unary operations on A (for $\kappa < \alpha$), $+$ and $\cdot$ are binary operations on A, and such that the following postulates are satisfied for any $x, y \in A$ and any $\kappa, \lambda, \mu < \alpha$:

(C_0) $\langle A, +, \cdot, -, 0, 1\rangle$ *is a complete and atomic* **BA**
(C_1) $\mathbf{c}_\kappa 0 = 0$
(C_2) $x \leqslant \mathbf{c}_\kappa x$
(C_3) $\mathbf{c}_\kappa(x \cdot \mathbf{c}_\kappa y) = \mathbf{c}_\kappa x \cdot \mathbf{c}_\kappa y$
(C_4) $\mathbf{c}_\kappa \mathbf{c}_\lambda x = \mathbf{c}_\lambda \mathbf{c}_\kappa x$
(C_5) $\mathbf{d}_{\kappa\kappa} = 1$
(C_6) *if* $\kappa \neq \lambda, \mu$, *then* $\mathbf{d}_{\lambda\mu} = \mathbf{c}_\kappa(\mathbf{d}_{\lambda\kappa} \cdot \mathbf{d}_{\kappa\mu})$
(C_7) *if* $\kappa \neq \lambda$, *then* $\mathbf{c}_\kappa(\mathbf{d}_{\kappa\lambda} \cdot x) \cdot \mathbf{c}_\kappa(\mathbf{d}_{\kappa\lambda} \cdot \overline{x}) = 0$
(C_8) *if* $x \neq 0$, *then* $\mathbf{c}_{(\alpha)} x = 1$
(C_9) *if* $x_i \in \mathrm{At}\mathfrak{A}$, $i = 0, 1, \ldots, \alpha - 1$, *then* $\prod_i \mathbf{c}^i x_i \in \mathrm{At}\mathfrak{A}$.

In the two preceding definitions we are using the notion of generalized cylindrifications as defined in [1], pp. 205-207. That is, if $\Gamma = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is a subset of α, by $\mathbf{c}_{(\Gamma)} x$ we mean $\mathbf{c}_{\alpha_0} \mathbf{c}_{\alpha_1} \ldots, \mathbf{c}_{\alpha_n} x$. Similarly, we define the generalized diagonal elements $\mathbf{d}_\Gamma$ as

$$\prod_{i=1}^{n} \mathbf{d}_{\alpha_0 \alpha_i}.$$

Note that if $\Gamma = \{\kappa\}$, then $\mathbf{c}_{(\kappa)} = \mathbf{c}_\kappa$ and if $\Gamma = \{\kappa, \lambda\}$, then $\mathbf{d}_\Gamma = \mathbf{d}_{\kappa\lambda}$.

(C_0) through (C_7) are the standard cylindric algebra axioms with the exception of complete and atomic in (C_0). (C_8) guarantees that a strong $\mathbf{CA}_\alpha$ will be simple in the universal algebraic sense (see 2.3.14, [1]). We will show that every strong $\mathbf{CA}_\alpha$ is isomorphic to a full $\mathbf{CSA}_\alpha$. Clearly, every full $\mathbf{CSA}_\alpha$ satisfies (C_0) through (C_9).

Let $\mathfrak{A}$ be an arbitrary, but fixed, strong $\mathbf{CA}_\alpha$. We now list several fundamental results from the theory of cylindric algebras which will be used in the sequel, the proofs of which can be found in [1].

Lemma 1.3 $\mathbf{d}_{\kappa\lambda} \cdot \mathbf{c}_\kappa x = 0$ *iff* $x = 0$.

Lemma 1.4 $\mathbf{c}_\kappa x \cdot \mathbf{c}_\kappa y = \mathbf{c}_\kappa(\mathbf{c}_\kappa x \cdot \mathbf{c}_\kappa y)$.

We now let Γ and Δ be non-empty (finite) subsets of α.

Lemma 1.5 *For any sequence* $\langle \Gamma_\kappa : \kappa < \beta\rangle$ *of subsets of* α, *the structure*

$$\langle A, +, \cdot, -, 0, 1, \mathbf{c}_{(\Gamma_\kappa)}\rangle_{\kappa < \beta}$$

is a diagonal-free **CA**.

Lemma 1.6 $\mathbf{c}_{(\Gamma)} x \cdot y = 0$ *iff* $\mathbf{c}_{(\Gamma)} y \cdot x = 0$.

Lemma 1.7 *If* $\Gamma \subset \Delta$, *then* $\mathbf{c}_{(\Gamma)}(\mathbf{d}_\Delta \cdot x \cdot y) = \mathbf{c}_{(\Gamma)}(\mathbf{d}_\Delta \cdot x) \cdot \mathbf{c}_{(\Gamma)}(\mathbf{d}_\Delta \cdot y)$.

Lemma 1.8 *If* $\Gamma \cap \Delta \neq 0$, *then* $\mathbf{d}_\Gamma \cdot \mathbf{d}_\Delta = \mathbf{d}_{\Gamma \cup \Delta}$.

Lemma 1.9 $\mathbf{c}_{(\Gamma)} \cdot \mathbf{d}_\Delta = \mathbf{d}_{\Delta \sim \Gamma}$.

Lemma 1.10 *If* $x \in \mathrm{At}\mathfrak{A}$, *then* $x = \prod_{i < \alpha} \mathbf{c}^i x$.

Proof: By 1.5, $x \leqslant \mathbf{c}^i x$ for all $i < \alpha$, hence $\prod_i \mathbf{c}^i x \geqslant x$ and equality follows from (C_9).

Henkin and Tarski have shown ([2], pp. 100-101) that any $\mathbf{CA}_\alpha$ which satisfies 1.10 is representable.

Our goal now is to find a way to uniquely express each atom in terms of the atoms which are less than the generalized diagonal element $\mathbf{d}_\alpha$.

Lemma 1.11 *If $x, y \in \mathrm{At}\mathfrak{A}$ and $x \leqslant \mathbf{c}^i y$, then $\mathbf{c}^i x = \mathbf{c}^i y$.*

Proof: By 1.6 and our hypothesis we see that $y \cdot \mathbf{c}^i x \neq 0$.

Since y is an atom we get $y \leqslant \mathbf{c}^i x$ and, by 1.5,

$$\mathbf{c}^i y \leqslant \mathbf{c}^i \mathbf{c}^i x = \mathbf{c}^i x.$$

The other inequality is obtained similarly using the fact that $x \leqslant \mathbf{c}^i y$.

Theorem 1.12 *If $x, y \in \mathrm{At}\mathfrak{A}$ and $x, y \leqslant \mathbf{d}_\alpha$, then $\mathbf{c}^i x \leqslant \mathbf{c}^i y$ implies $x = y$.*

Proof: By hypotheses 1.5 and 1.7,

$$0 \neq \mathbf{c}^i x = \mathbf{c}^i x \cdot \mathbf{c}^i y = \mathbf{c}^i (x \cdot y),$$

x and y being atoms yields the result.

Theorem 1.13 *If $x \in \mathrm{At}\mathfrak{A}$ and $i < \alpha$, then there is a $y \in \mathrm{At}\mathfrak{A}$ such that $y \leqslant \mathbf{d}_\alpha$ and $\mathbf{c}^i x = \mathbf{c}^i y$.*

Proof: We show this for $i = \alpha - 1$. Construct a sequence y_j as follows

$$\begin{aligned}
y_0 &= \mathbf{c}_0 x \cdot \mathbf{d}_{0\alpha-1} \\
y_1 &= \mathbf{c}_1 y_0 \cdot \mathbf{d}_{1\alpha-1} = \mathbf{c}_1(\mathbf{c}_0 x \cdot \mathbf{d}_{0\alpha-1}) \cdot \mathbf{d}_{1\alpha-1} \\
&= \mathbf{c}_1 \mathbf{c}_0 x \cdot \mathbf{d}_{0\alpha-1} \cdot \mathbf{d}_{1\alpha-1} \\
y_2 &= \mathbf{c}_2 y_1 \cdot \mathbf{d}_{2\alpha-1} = \mathbf{c}_2 \mathbf{c}_1 \mathbf{c}_0 x \cdot \mathbf{d}_{0\alpha-1} \cdot \mathbf{d}_{1\alpha-1} \cdot \mathbf{d}_{2\alpha-1} \\
&\vdots \\
y_{\alpha-2} &= \mathbf{c}^{\alpha-1} x \cdot \mathbf{d}_\alpha.
\end{aligned}$$

By an argument similar to 1.12 each y_j is an atom, thus, by 1.5 and 1.9

$$\mathbf{c}^{\alpha-1} y_{\alpha-2} = \mathbf{c}^{\alpha-1}(\mathbf{c}^{\alpha-1} x \cdot \mathbf{d}_\alpha) = \mathbf{c}^{\alpha-1} x$$

and $y_{\alpha-2} \leqslant \mathbf{d}_\alpha$ as desired.

Corollary 1.14 *If $x \in \mathrm{At}\mathfrak{A}$, then there exist $y_0, y_1, \ldots, y_{\alpha-1} \in \mathrm{At}\mathfrak{A}$ such that $y_i \leqslant \mathbf{d}_\alpha$ and $\prod_i \mathbf{c}^i y_i = x$.*

Proof: By 1.13, for each $i < \alpha$ there is a $y_i \in \mathrm{At}\mathfrak{A}$, $y_i \leqslant \mathbf{d}_\alpha$ such that $\mathbf{c}^i y_i = \mathbf{c}^i x$, so by 1.10 $x = \prod_i \mathbf{c}^i x = \prod_i \mathbf{c}^i y_i$.

Lemma 1.15 *If $x_0, x_1, \ldots, x_{\alpha-1} \in \mathrm{At}\mathfrak{A}$ and $j < \alpha$, then*

$$\mathbf{c}^j \Big(\prod_{i<\alpha} \mathbf{c}^i x_i \Big) = \mathbf{c}^j x_j.$$

Proof: We note that $\prod_{i \neq j} \mathbf{c}^i x_i \neq 0$. Now by 1.4 and 1.5

$$\mathbf{c}^j\left(\prod_i \mathbf{c}^i x_i\right) = \mathbf{c}^j\left(\mathbf{c}^j x_j \cdot \prod_{i\neq j} \mathbf{c}^i x_i\right)$$
$$= \mathbf{c}^j\left(\mathbf{c}^j x_j \cdot \mathbf{c}_j\left(\prod_{i\neq j} \mathbf{c}^i x_i\right)\right) = \mathbf{c}^j x_j \cdot \mathbf{c}^j \mathbf{c}_j\left(\prod_{i\neq j} \mathbf{c}^i x_i\right).$$

Now (C_8) implies $\mathbf{c}^j\mathbf{c}_j\left(\prod_{i\neq j} \mathbf{c}^i x_i\right) = 1$.

Theorem 1.16 *If* $x_0, x_1, \ldots, x_{\alpha-1}, y_0, y_1, \ldots, y_{\alpha-1} \in \mathrm{At}\mathfrak{A}$, $x_i, y_i \leq \mathbf{d}_\alpha$ *and* $\prod_{i<\alpha} \mathbf{c}^i x_i = \prod_{i<\alpha} \mathbf{c}^i y_i$, *then* $x_i = y_i$ *for all* $i < \alpha$.

Proof: For each $j < \alpha$, by 1.15,

$$\mathbf{c}^j x_j = \mathbf{c}^j\left(\prod_i \mathbf{c}^i x_i\right) = \mathbf{c}^j\left(\prod_i \mathbf{c}^i y_i\right) = \mathbf{c}^j y_j$$

and so, by 1.12, $x_j = y_j$.

Theorem 1.17 *Let* $\beta = |\mathrm{At}\mathfrak{A}|$, $\gamma = |\mathbf{d}_\alpha \cdot \mathrm{At}\mathfrak{A}|$, *then* $\beta = \gamma^\alpha$.

Proof: Let $D = \mathbf{d}_\alpha \cdot \mathrm{At}\mathfrak{A}$. For each $x_0, x_1, \ldots, x_{\alpha-1} \in D$ we assign the atom $\prod_i \mathbf{c}^i x_i$. By 1.16 this map is one to one and 1.14 shows that it is onto. Hence $\beta = |{}^\alpha D| = \gamma^\alpha$.

Now let $\mathfrak{A}$, $\mathfrak{B}$ be two strong $\mathbf{CA}_\alpha$'s, $D = \mathrm{At}\mathfrak{A} \cdot \mathbf{d}_\alpha$ and $D' = \mathrm{At}\mathfrak{B} \cdot \mathbf{d}_\alpha$.

Theorem 1.18 *If* $|D| = |D'|$, *then* $\mathfrak{A} \cong \mathfrak{B}$.

Proof: Since $|D| = |D'|$, there is a one-to-one map ϕ_1 from D onto D'. Now we extend ϕ_1 to a map from $\mathrm{At}\mathfrak{A}$ onto $\mathrm{At}\mathfrak{B}$. For $x_0, x_1, \ldots, x_{\alpha-1} \in D$, $x'_0, x'_1, \ldots, x'_{\alpha-1} \in D'$ such that $\phi_1(x_i) = x'_i$ we define ϕ_2 as follows:

$$\phi_2\left(\prod \mathbf{c}^i x_i\right) = \prod \mathbf{c}^i x'_i.$$

By 1.14 and 1.16 this map is one to one from $\mathrm{At}\mathfrak{A}$ onto $\mathrm{At}\mathfrak{B}$ and $\phi_2 \upharpoonright D = \phi_1$. Now we extend to a function $\phi: A \to B$ by additivity, that is, for $x \in A$, $x' \in B$

$\phi(x) = x'$ iff (i) if $y \in \mathrm{At}\mathfrak{A}$ and $y \leq x$, then there exists $y' \in \mathrm{At}\mathfrak{B}$ such that $y' \leq x'$ and $\phi_2(y) = y'$.
(ii) if $y' \in \mathrm{At}\mathfrak{B}$ and $y' \leq x'$, then there exists $y \in \mathrm{At}\mathfrak{A}$ such that $y \leq x$ and $\phi_2(y) = y'$.

ϕ is one to one and onto since $\mathfrak{A}$ and $\mathfrak{B}$ are complete and atomic $\mathbf{BA}$'s. Clearly ϕ is a $\mathbf{BA}$ isomorphism. To show ϕ is a $\mathbf{CA}$ isomorphism it is sufficient to show that for any $x \in \mathrm{At}\mathfrak{A}$, $\phi(\mathbf{c}_i x) = \mathbf{c}_i(\phi(x)) = \mathbf{c}_i x'$. The result then follows by the complete additivity of $\mathbf{c}_i$.

By 1.14 there exists $x_0, x_1, \ldots, x_{\alpha-1} \in D$, $x'_0, x'_1, \ldots, x'_{\alpha-1} \in D'$ such that $\phi(x_i) = x'_i$, $x = \prod_i \mathbf{c}^i x_i$, $x' = \prod_i \mathbf{c}^i x'_i$ and $\phi(x) = x'$. By 1.4, 1.5 and (C_8),

$$\mathbf{c}_j x = \mathbf{c}_j\left(\prod_i \mathbf{c}^i x_i\right) = \prod_{i\neq j} \mathbf{c}^i x_i.$$

Let $y \leq \mathbf{d}_\alpha$, then

$$z = \prod_{i\neq j} \mathbf{c}^i x_i \cdot \mathbf{c}^j y \leq \mathbf{c}_j x$$

and $z \in \mathrm{At}\mathfrak{A}$ by (C_9). $\phi(z) = z' = \prod_{i\neq j} \mathbf{c}^i x'_i \cdot \mathbf{c}^j y' \leq \mathbf{c}_j x'$, where $y' = \phi(y)$.

Now let z be any atom such that $z \leqslant \mathbf{c}_{(j)}x$, and we show that z is obtained in the above manner. We know that $z = \prod_i \mathbf{c}^i y_i$ for some $y_0, y_1, \ldots, y_{\alpha-1} \in D$, hence

$$z = \prod_i \mathbf{c}^i y_i \leqslant \mathbf{c}_{(j)}x = \prod_{i \neq j} \mathbf{c}^i x.$$

So, for any $m \neq j < \alpha$, by 1.15,

$$\mathbf{c}^m z = \mathbf{c}^m \left(\prod_i \mathbf{c}^i y_i\right) = \mathbf{c}^m y_m \leqslant \mathbf{c}^m \left(\prod_{i \neq j} \mathbf{c}^i x_i\right) = \mathbf{c}^m x_m.$$

By 1.12, $x_m = y_m$ and $z = \prod_{i \neq j} \mathbf{c}^i x_i \cdot \mathbf{c}^j y$ as desired. So we have shown that for every atom $z \leqslant \mathbf{c}_{(j)}x$, $\phi(z) \leqslant \mathbf{c}_{(j)}(\phi(x))$. By an analogous argument we obtain that if $\phi(y)$ is an atom, $\phi(y) \leqslant \mathbf{c}_{(j)}(\phi(x))$, then $y \leqslant \mathbf{c}_{(j)}x$, which completes the proof.

Theorem 1.19 *Every strong* $\mathbf{CA}_\alpha$ *is isomorphic to a full* $\mathbf{CSA}_\alpha$.

Proof: Let $\mathfrak{A}$ be a strong $\mathbf{CA}_\alpha$, $\beta = |D|$. Let $\mathfrak{A}'$ be a full $\mathbf{CSA}_\alpha$ generated by a set of cardinality β, hence $\beta = |\mathrm{At}\mathfrak{A}' \cdot \mathbf{d}_\alpha|$ so, by 1.18, $\mathfrak{A} \cong \mathfrak{A}'$.

Let β and γ be cardinal numbers, from set theory we know that, with the assumption of the generalized continuum hypothesis, $2^\beta = 2^\gamma$ implies $\beta = \gamma$.

Theorem 1.20 (GCH) *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are two strong* $\mathbf{CA}_\alpha$'s *such that* $|A| = |B|$, *then* $\mathfrak{A} \cong \mathfrak{B}$.

Proof: Since $\mathfrak{A}$ and $\mathfrak{B}$ are complete and atomic $\mathbf{BA}$'s, $|A| = 2^\beta$ and $|B| = 2^\gamma$ for some cardinals β and γ, where $\beta = |\mathrm{At}\mathfrak{A}|$ and $\gamma = |\mathrm{At}\mathfrak{B}|$. By the GCH we see that $\beta = \gamma$. By 1.17, $\beta = \beta_1^\alpha$ and $\gamma = \gamma_1^\alpha$ where $\gamma_1 = |D'|$ and $\beta_1 = |D|$. Hence $\beta_1 = \gamma_1$, and so 1.18 yields $\mathfrak{A} \cong \mathfrak{B}$.

The independence of these additional two axioms can be exhibited by considering the following two $\mathbf{CSA}_2$'s. Let $\mathfrak{A}$ represent the cylindric set algebra of all subsets of $\mathcal{R}^2$, the real plane. The Cartesian product $\mathfrak{A} \times \mathfrak{A}$ satisfies all the axioms except (C_8), since, for any non-empty set x in $\mathcal{R}^2$, $\mathbf{c}_{(2)}(\langle x, 0\rangle) = \langle \mathcal{R}^2, 0\rangle$. Now let $\mathfrak{B}$ be the complete atomic subalgebra of $\mathfrak{A}$ generated by lines of slope 1. $\mathfrak{B}$ satisfies $\mathbf{c}_i x = 1$ for all x, hence (C_8) holds but (C_9) is falsified for any atom.

2 *Strong* $\mathbf{CA}_2$ *and relation algebras* In [3] McKinsey gave an axiomatization of a subclass of relation algebras which we will denote by **MRA**. We show that the theory of **MRA** is definitionally equivalent to the theory of strong $\mathbf{CA}_2$.

Definition 2.1 By an **MRA** we mean an algebraic structure

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, |\rangle$$

such that 0 and 1 are distinguished elements of A, $+$, $\cdot$ and $|$ are binary operations on A, $-$ is an unary operation on A, and such that for any $x, y, u, v \in A$, the following postulates are satisfied:

(M_0) $\langle A, +, \cdot, -, 0, 1\rangle$ *is a complete and atomic* **BA**
(M_1) $x|(y|z) = (x|y)|z$
(M_2) *If* $x \leqslant u$ *and* $y \leqslant v$, *then* $x|y \leqslant u|v$
(M_3) *If* $x \neq 0$, *then* $1|(x|1) = 1$
(M_4) *If* $z \in \mathrm{At}\mathfrak{A}$, *and* $z \leqslant x|y$, *then there exist* $x', y' \in \mathrm{At}\mathfrak{A}$ *such that* $x' \leqslant x$, $y' \leqslant y$ *and such that* $z = x'|y'$
(M_5) *If* $x, y, z \in \mathrm{At}\mathfrak{A}$, $x|y \neq 0$, $y|x \neq 0$, $x|z \neq 0$ *and* $z|x \neq 0$, *then* $y = z$.

The relational operation converse and the constant $1'$ can be defined in this system and need not be taken as primitives. McKinsey has shown ([3], Thm. B, p. 94) that each **MRA** is isomorphic to a system where A is the full power set of $U \times U$, for some non-empty set U, and $|$ is the standard relative product on A.

We know that given any relation algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, |, 1'\rangle$$

the system

$$\mathbf{c}\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathbf{c}_\kappa, \mathbf{d}_{\kappa\lambda}\rangle_{\kappa, \lambda < 2}$$

where the non-Boolean operations are defined as follows:

$$\mathbf{c}_0 x = 1|x,\ \mathbf{c}_1 x = x|1,\ \mathbf{d}_{\kappa\kappa} = 1 \text{ and } \mathbf{d}_{\kappa\lambda} = 1' \text{ for } \kappa \neq \lambda$$

is a $\mathbf{CA}_2$. By McKinsey's result it follows that if $\mathfrak{A}$ is an **MRA**, then $\mathbf{c}\mathfrak{A}$ as defined above is a strong $\mathbf{CA}_2$.

Now let $\mathfrak{A}$ be an arbitrary, but fixed, strong $\mathbf{CA}_2$.

Theorem 2.2 *If* $x \in \mathrm{At}\mathfrak{A}$, *then there exists a unique* $y \in \mathrm{At}\mathfrak{A}$ *such that* $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \leqslant \mathbf{d}_{01}$ *and* $\mathbf{c}_0 y \cdot \mathbf{c}_1 x \leqslant \mathbf{d}_{01}$.

Proof: First we show the existence of such an atom. By 1.14, there exists $x_0, x_1 \in \mathrm{At}\mathfrak{A}$ such that $x_0, x_1 \leqslant \mathbf{d}_{01}$ and $\mathbf{c}_0 x_0 \cdot \mathbf{c}_1 x_1 = x$. Let $y = \mathbf{c}_0 x_1 \cdot \mathbf{c}_1 x_0$. By 1.5, ($C_8$) and 1.10, $\mathbf{c}_0 x \cdot \mathbf{c}_1 y = \mathbf{c}_0(\mathbf{c}_0 x_0 \cdot \mathbf{c}_1 x_1) \cdot \mathbf{c}_1(\mathbf{c}_0 x_1 \cdot \mathbf{c}_1 x_0) = \mathbf{c}_0 x_0 \cdot \mathbf{c}_1 x_0 = x_0 \leqslant \mathbf{d}_{01}$. Similarly, $\mathbf{c}_0 y \cdot \mathbf{c}_1 x = x_1 \leqslant \mathbf{d}_{01}$.

Now assume y and y' have the desired property. By (C_9) there are atoms z and z' such that $z = \mathbf{c}_0 x \cdot \mathbf{c}_1 y$, $z' = \mathbf{c}_0 x \cdot \mathbf{c}_1 y'$ and $z, z' \leqslant \mathbf{d}_{01}$. 1.4 and ($C_8$) imply $\mathbf{c}_0 z = \mathbf{c}_0 z'$ and we conclude, by 1.12, $z = z'$. Now by (C_3) and (C_8)

$$\mathbf{c}_1 y = \mathbf{c}_1 z = \mathbf{c}_1 z' = \mathbf{c}_1 y'.$$

Similarly $\mathbf{c}_0 y = \mathbf{c}_0 y'$ and, by 1.10, $y = y'$.

Remark. If we replace (C_9) by 2.2 in the axiom system for strong $\mathbf{CA}_2$'s we obtain an equivalent theory.

Now we define a binary operation $|$ on A as follows:

Definition 2.3 For $x, y \in \mathrm{At}\mathfrak{A}$,

$$x|y = \begin{cases} 0, & \text{if } \mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01} = 0 \\ \mathbf{c}_0 y \cdot \mathbf{c}_1 x, & \text{otherwise.} \end{cases}$$

For $x, y \in A$, let $X = \{u: u \leqslant x \text{ and } u \in \mathsf{At}\mathfrak{A}\}$, and $Y = \{v: v \leqslant y \text{ and } v \in \mathsf{At}\mathfrak{A}\}$. Then

$$x|y = \sum_{v \in Y} \sum_{u \in X} (u|v).$$

Lemma 2.4 *If $x, y \in \mathsf{At}\mathfrak{A}$, then $x|y = 0$ iff $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01} = 0$.*

Proof: Sufficiency follows from 2.3. If $x|y = 0$, then $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01} = 0$ or $\mathbf{c}_0 y \cdot \mathbf{c}_1 x = 0$. But $\mathbf{c}_0 y \cdot \mathbf{c}_1 x = 0$ implies $\mathbf{c}_0 x \cdot \mathbf{c}_1 y = 0$ and hence $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01} = 0$.

We show that the resulting system

$$\mathbf{m}\mathfrak{A} = \langle A, +, \cdot, 0, 1, |\rangle$$

is an **MRA**. (M_0) follows from (C_0), (M_2) and (M_4) follow from the additive definition of $|$.

Lemma 2.5 *If $x, y \in \mathsf{At}\mathfrak{A}$, $x|y \neq 0$ and $y|x \neq 0$, then $x|y \leqslant \mathbf{d}_{01}$ and $y|x \leqslant \mathbf{d}_{01}$.*

Proof: By 2.4 and (C_9), $x|y \neq 0$ implies $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01} \neq 0$. Hence, since $y|x \neq 0$, $y|x = \mathbf{c}_0 x \cdot \mathbf{c}_1 y \leqslant \mathbf{d}_{01}$. Similarly, $x|y \leqslant \mathbf{d}_{01}$.

Theorem 2.6 *If $x, y, z \in \mathsf{At}\mathfrak{A}$, $x|y \neq 0$, $y|x \neq 0$, $x|z \neq 0$ and $z|x \neq 0$, then $y = z$.*

Proof: By 2.5, $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \leqslant \mathbf{d}_{01}$, $\mathbf{c}_0 y \cdot \mathbf{c}_1 x \leqslant \mathbf{d}_{01}$, $\mathbf{c}_0 x \cdot \mathbf{c}_1 z \leqslant \mathbf{d}_{01}$ and $\mathbf{c}_0 z \cdot \mathbf{c}_1 x \leqslant \mathbf{d}_{01}$. Hence, by 2.2, $y = z$.

Theorem 2.6 shows us that the system $\mathbf{m}\mathfrak{A}$ satisfies (M_5). If we wish to define the converse in this system for $x \in \mathsf{At}\mathfrak{A}$ we define $\breve{x}$ to be the unique atom y such that $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \leqslant \mathbf{d}_{01}$ and $\mathbf{c}_0 y \cdot \mathbf{c}_1 x \leqslant \mathbf{d}_{01}$.

Lemma 2.7 *If $x \in \mathsf{At}\mathfrak{A}$, then $\mathbf{c}_\kappa x \cdot \mathbf{d}_{01} \in \mathsf{At}\mathfrak{A}$.*

Proof: Follows from 1.12.

Lemma 2.8 *If $x \in \mathsf{At}\mathfrak{A}$, then $x|1 = \mathbf{c}_1 x$.*

Proof: $x|1 = x|\left(\sum_{y \in \mathsf{At}\mathfrak{A}} y\right) = \sum_{\substack{y \in \mathsf{At}\mathfrak{A} \\ x|y \neq 0}} (x|y) = \sum_{\substack{y \in \mathsf{At}\mathfrak{A} \\ x|y \neq 0}} \mathbf{c}_1 x \cdot \mathbf{c}_0 y \leqslant \mathbf{c}_1 x.$

Now let $z \leqslant \mathbf{c}_1 x$, $z \in \mathsf{At}\mathfrak{A}$, and let $y = \mathbf{c}_0 x \cdot \mathbf{d}_{01} \in \mathsf{At}\mathfrak{A}$. By (C_3) and 1.9, $\mathbf{c}_0 y = \mathbf{c}_0 x$. Let $w = \mathbf{c}_1(\mathbf{c}_0 y \cdot \mathbf{d}_{01}) \cdot \mathbf{c}_0 z$. $w \in \mathsf{At}\mathfrak{A}$ by (C_9). So, by 1.3.9 [1],

$$\mathbf{c}_0 x \cdot \mathbf{c}_1 w \cdot \mathbf{d}_{01} = \mathbf{c}_0 x \cdot \mathbf{c}_1(\mathbf{c}_0 y \cdot \mathbf{d}_{01}) \cdot \mathbf{d}_{01} = \mathbf{c}_0 x \cdot \mathbf{c}_0 y \cdot \mathbf{d}_{01} = y.$$

Hence $x|w \neq 0$, so $x|w = \mathbf{c}_0 w \cdot \mathbf{c}_1 x = \mathbf{c}_0 z \cdot \mathbf{c}_1 x = z$, since $z \leqslant \mathbf{c}_0 z$ and $z \leqslant \mathbf{c}_1 x$. Hence $z \leqslant x|1$ and the proof is complete.

Theorem 2.9 $x|1 = \mathbf{c}_1 x$.

Proof: By 2.8 and the additivity of $|$ and $\mathbf{c}_1$.

Theorem 2.10 $1|x = \mathbf{c}_0 x$.

Proof: Similar to 2.9.

Corollary 2.11 *If $x \neq 0$, then $1|(x|1) = 1$.*

Proof: If $x \neq 0$, then, by 2.9 and 2.10, $1|(x|1) = \mathbf{c}_0\mathbf{c}_1 x = 1$.

Now we show that $|$ is associative, and therefore that $\mathbf{m}\mathfrak{A}$ satisfies (M_0)-(M_5), and hence is an **MRA**.

Lemma 2.12 *If* $x, y, z \in \mathrm{At}\mathfrak{A}$, *then* $x|(y|z) = (x|y)|z$.

Proof: Case 1. $y|z = 0$. Then $x|(y|z) = 0$ and, by 2.4, $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01} = 0$. If $x|y = 0$, then $(x|y)|z = 0$ and we have equality. Assume $x|y \neq 0$, then $x|y = \mathbf{c}_0 y \cdot \mathbf{c}_1 x$. But

$$\mathbf{c}_0(\mathbf{c}_0 y \cdot \mathbf{c}_1 x) \cdot \mathbf{c}_1 z \cdot \mathbf{d}_{01} = \mathbf{c}_0 y \cdot \mathbf{c}_1 z \cdot \mathbf{d}_{01} = 0$$

so, by 2.4, $(x|y)|z = 0$.

Case 2. $x|y = 0$. Then $(x|y)|z = 0$ and, by an argument similar to Case 1, $x|(y|z) = 0$.

Case 3. $x|(y|z) = 0$ and $y|z \neq 0$. Then $y|z = \mathbf{c}_0 z \cdot \mathbf{c}_1 y$ and, by 2.4,

$$0 = \mathbf{c}_1 x \cdot \mathbf{c}_0(\mathbf{c}_0 z \cdot \mathbf{c}_1 y) \cdot \mathbf{d}_{01} = \mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01}.$$

So $x|y = 0$ and $(x|y)|z = 0$.

Case 4. $(x|y)|z = 0$ and $x|y \neq 0$. Similar to Case 3.

Case 5. $(x|y)|z \neq 0$ and $x|(y|z) \neq 0$. So $x|y \neq 0$ and $y|z \neq 0$, hence

$$(x|y)|z = \mathbf{c}_1(\mathbf{c}_1 x \cdot \mathbf{c}_0 y) \cdot \mathbf{c}_0 z = \mathbf{c}_1 x \cdot \mathbf{c}_0 z$$

and

$$x|(y|z) = \mathbf{c}_1 x \cdot \mathbf{c}_0(\mathbf{c}_1 y \cdot \mathbf{c}_0 z) = \mathbf{c}_1 x \cdot \mathbf{c}_0 z.$$

Theorem 2.13 $x|(y|z) = (x|y)|z$.

Proof: By 2.12 and the additivity of $|$.

Let $\mathfrak{A}$ be a strong $\mathbf{CA}_2$, then $\mathbf{m}\mathfrak{A}$ is an **MRA** and $\mathbf{cm}\mathfrak{A}$ is a strong $\mathbf{CA}_2$. Theorems 2.9 and 2.10 imply that $\mathfrak{A} = \mathbf{cm}\mathfrak{A}$. Now let $\mathfrak{A}$ be an **MRA**. We wish to show that $\mathfrak{A} = \mathbf{mc}\mathfrak{A}$. By McKinsey's result we know that $\mathfrak{A} \cong \mathfrak{B}$, where

$$\mathfrak{B} = \langle B, \cup, \cap, -, 0, U \times U, |' \rangle$$

in which U is a non-empty set, B the power set of $U \times U$, and $|$ is the natural relative product. If we show that $\mathfrak{B} = \mathbf{mc}\mathfrak{B}$, then $\mathfrak{A} = \mathbf{mc}\mathfrak{A}$. It is sufficient to show $x|y = x|'y$, for $x, y \in \mathrm{At}\mathfrak{A}$. Any atom $x \in B$ is a set which consists of a single ordered pair. Let $x = \{\langle s, t\rangle\}$ and $y = \{\langle u, v\rangle\}$ be atoms of $\mathfrak{B}$. In $\mathbf{c}\mathfrak{B}$

$$\mathbf{c}_0 x = \{\langle z, t\rangle : z \in U\} \qquad \mathbf{c}_1 x = \{\langle s, z\rangle : z \in U\}$$
$$\mathbf{c}_0 y = \{\langle z, v\rangle : v \in U\} \qquad \mathbf{c}_1 y = \{\langle u, z\rangle : z \in U\}.$$

If $x|y = 0$, then $t \neq u$, which implies $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01} = 0$. Hence $x|'y = 0$. If $x|y \neq 0$, then $t = u$ and $x|y = \{\langle s, v\rangle\}$. Then $\mathbf{c}_0 x \cdot \mathbf{c}_1 y \cdot \mathbf{d}_{01} = \{\langle t, t\rangle\} \neq 0$, so $x|'y = \mathbf{c}_1 x \cdot \mathbf{c}_0 y = \{\langle s, v\rangle\}$.

We have now established a one-to-one correspondence between the class of **MRA** and strong $\mathbf{CA}_2$ and, since they are interdefinably related, we conclude that the two theories are definitionally equivalent.

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