

## INFINITARY PROPOSITIONAL INTUITIONISTIC LOGIC

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In the first two chapters of his book, Fitting [3] gives an elegant presentation of the semantics and proof theory of propositional intuitionistic logic. The semantics used is based on a notion of intuitionistic models due to Kripke [5], which in the finitary case is shown to be equivalent to the model theory of pseudo-Boolean algebras. The proof theory is essentially the method of tableaux due to Beth [1], with modifications as presented in Smullyan [9]. The purpose of this paper\* is to generalize propositional intuitionistic logic to the infinitary language in which we allow conjunctions and disjunctions over countable collections of formulas.

Section 1 presents our semantics, which is that of complete pseudo-Boolean algebras and homomorphisms. In the infinitary case, it is easy to show that this semantics is not equivalent to the natural generalization of the Kripke models. In 2, we present our proof theory, which we believe combines the best features of the tableau system of Fitting [3] and the system of block tableaux of Smullyan [9]. A proof is an ordered finite branch tree in which a given point may have infinitely many immediate successors. In 3, we first assign countable ordinals to proofs in a straightforward way, and then use induction on the ordinal of a proof to show that the proof theory is correct; i.e., that theorems are valid. Section 4 shows the completeness of the system, using the infinitary version of the Lindenbaum algebra. We show that the collection of theorems is closed under *modus ponens* by proving a tableau version of Gentzen's Hauptsatz. The latter is accomplished by combining elements of the finitary classical proof in [9] with the infinitary proof for classical Gentzen systems due to Feferman [2]. Once the Hauptsatz is obtained, it is easy to generalize the completeness results of Rasiowa and Sikorski [7] to

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the infinitary case. In the concluding remarks, we point out some problems of interest for further investigations.

**1 Semantics** The language  $\mathcal{L}$  that we use consists of the following symbols: (i) An arbitrary collection of atomic formulas, which we denote by  $A, B, C, \dots$ , and (ii) the connectives  $\vee, \wedge, \sim,$  and  $\rightarrow$ .

*Definition 1.1* The formulas of  $\mathcal{L}$  are the members of the least class  $\mathcal{F}$  such that

- (i)  $A \in \mathcal{F}$  for each atomic formula  $A$
- (ii) if  $X \in \mathcal{F}$ , then  $\sim X \in \mathcal{F}$
- (iii) if  $X \in \mathcal{F}$  and  $Y \in \mathcal{F}$ , then  $X \rightarrow Y \in \mathcal{F}$
- (iv) if  $\Phi$  is a nonempty finite or countably infinite subset of  $\mathcal{F}$ , then  $\vee \Phi \in \mathcal{F}$  and  $\wedge \Phi \in \mathcal{F}$ .

From this point on, we use the term “countable” to mean either nonempty finite or countably infinite. We frequently use the symbols  $\forall X_i$  for  $\vee \Phi$  and  $\wedge X_i$  for  $\wedge \Phi$  when no confusion results.

*Definition 1.2* Let  $X \in \mathcal{F}$ . We define  $Sub(X)$ , the collection of subformulas of  $X$ , inductively in the usual manner. We also define  $Subp(X)$ , the collection of proper subformulas of  $x$ , to be  $Sub(X) - \{X\}$ .

We note that to prove all formulas of  $\mathcal{F}$  have a property  $P$ , it suffices to show that the collection of formulas satisfying  $P$  is closed under (i)-(iv) of Definition 1.1. The following propositions are immediate:

*Proposition 1.1* For any  $X \in \mathcal{F}$ ,  $Sub(X)$  is countable.

*Proposition 1.2* Let  $X \in \mathcal{F}$  and let  $X_0, X_1, X_2, \dots$  be a sequence in  $Sub(X)$  such that for each  $i \geq 0$ ,  $X_{i+1} \in Subp(X_i)$ . Then, the given sequence is finite.

The previous result simply says that the relation  $X \in Subp(Y)$  is well-founded.

We now turn to a description of the semantics we will use.

*Definition 1.3* A pseudo-Boolean algebra is an ordered pair  $\langle \mathcal{A}, \leq \rangle$ , where  $\mathcal{A}$  is a nonempty set and  $\leq$  is a partial ordering on  $\mathcal{A}$ , such that for any  $a, b \in \mathcal{A}$ , the following elements of  $\mathcal{A}$  exist:

- (i) the least upper bound  $a \vee b$
- (ii) the greatest lower bound  $a \wedge b$
- (iii) the pseudo-complement of  $a$  relative to  $b$ ,  $a \Rightarrow b$ ; i.e., the largest  $X \in \mathcal{A}$  such that  $a \wedge x \leq b$
- (iv) the zero element  $0$ .

For any  $a \in \mathcal{A}$ , we denote the pseudo-complement of  $a$  by  $a^- = a \Rightarrow 0$ , and the unit element of  $\mathcal{A}$  by  $1 = 0^-$ . With few exceptions, all of the properties of pseudo-Boolean algebras which we shall use are contained in [7], pp. 58-62. Pseudo-Boolean algebras are also referred to in the literature as Heyting algebras.

*Definition 1.4* We call a pseudo-Boolean algebra  $\mathcal{A}$  *complete* if for every countable subset  $\mathcal{C}$  of  $\mathcal{A}$ ,  $\mathcal{C}$  has a least upper bound and a greatest lower bound in  $\mathcal{A}$ .

*Definition 1.5* A *homomorphism*  $h$  is a function  $h: \mathcal{F} \rightarrow \mathcal{A}$ , where  $\mathcal{A}$  is a complete pseudo-Boolean algebra, satisfying the following:

- (i)  $h(\vee \Phi) = \vee \{h(X) \mid X \in \Phi\}$
- (ii)  $h(\wedge \Phi) = \wedge \{h(X) \mid X \in \Phi\}$
- (iii)  $h(\sim X) = h(X)^-$
- (iv)  $h(X \rightarrow Y) = h(X) \Rightarrow h(Y)$ .

It should be clear from (i)-(iv) that a homomorphism  $h$  is completely determined by its values on atomic formulas.

*Definition 1.6* A *model* is an ordered triple  $\langle \mathcal{A}, \leq, h \rangle$ , where  $\langle \mathcal{A}, \leq \rangle$  is a complete pseudo-Boolean algebra and  $h$  is a homomorphism from  $\mathcal{F}$  into  $\mathcal{A}$ . For a given  $X \in \mathcal{F}$ , we say  $X$  is *valid* in  $\langle \mathcal{A}, \leq, h \rangle$  if  $h(X) = 1$ . We say  $X$  is *valid* if  $X$  is valid in all models.

In a given model  $\langle \mathcal{A}, \leq, h \rangle$  we will usually denote  $h(X)$  by  $\|X\|$  when no confusion will result. We call  $\|X\|$  the *value* of  $X$  in  $\mathcal{A}$ .

**2 Proof theory** We now present our proof theory, which is a direct generalization of the tableau systems of Fitting [3] and Smullyan [9]. The reduction rules that we use are basically the same as the corresponding finitary rules, with the exception of a special treatment of the infinitary conjunction. Proofs may be infinite trees, allowing for application of reduction rules to infinitary formulas, but we require that the finite branch property holds, i.e., a tree may not contain an infinite sequence of immediate successors.

*Definition 2.1* A *signed formula* is an expression  $TX$  or  $FX$  for  $X \in \mathcal{F}$ . A *block* is a finite set of signed formulas.

We most frequently use the symbols  $\mathbf{S}$ ,  $\mathbf{U}$ ,  $\mathbf{S}_i$ , or  $\mathbf{U}_i$  for blocks. We also find it convenient to use the symbol  $\{\mathbf{S}, H\}$  in place of  $\mathbf{S} \cup \{H\}$ , where  $H$  is a signed formula.

*Definition 2.2* If  $\mathbf{S}$  is a block, let  $\mathbf{S}^T = \{TX \mid TX \in \mathbf{S}\}$  and  $\mathbf{S}^F = \{FX \mid FX \in \mathbf{S}\}$ .

*Definition 2.3* The following eight rules are the *reduction rules*:

$$T\vee: \{\mathbf{S}, T\vee\Phi\} / \{\{\mathbf{S}, TX\} \mid X \in \Phi\}.$$

$$F\vee: \{\mathbf{S}, F\vee\Phi\} / \{\mathbf{S}, FX\} \text{ for any } X \in \Phi.$$

$$T\wedge: \{\mathbf{S}, T\wedge\Phi\} / \{\mathbf{S}, TX\} \text{ for any } X \in \Phi.$$

$$F\wedge: \text{(a) } \{\mathbf{S}, F\wedge\Phi\} / \{\{\mathbf{S}, FX\} \mid X \in \Phi\} \text{ if } \Phi \text{ is finite.}$$

$$\text{(b) } \{\mathbf{S}, F\wedge\Phi\} / \{\{\mathbf{S}^T, FX\} \mid X \in \Phi\} \text{ if } \Phi \text{ is infinite.}$$

$$T\sim: \{\mathbf{S}, T\sim X\} / \{\mathbf{S}, FX\}.$$

$$F\sim: \{\mathbf{S}, F\sim X\} / \{\mathbf{S}^T, TX\}.$$

$T\rightarrow: \{\mathbf{S}, T(X \rightarrow Y)\} / \{\{\mathbf{S}, FX\}, \{\mathbf{S}, TY\}\}.$

$F\rightarrow: \{\mathbf{S}, F(X \rightarrow Y)\} / \{\mathbf{S}^T, TX, FY\}.$

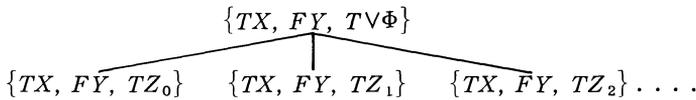
The intuition behind the rules in Definition 2.3 is well-documented, a possible exception being rule  $F\wedge(b)$ . The reason for rule  $F\wedge(b)$  is a direct result of the semantics we have chosen, and an explanation is given in the concluding remarks to this paper.

*Definition 2.4* Let  $\mathbf{U}$  be a block. We say a rule  $R$  *applies* to  $\mathbf{U}$  if by appropriate choice of its members, we can write  $\mathbf{U}$  as the block preceding the line in the statement of rule  $R$ . By an *application* of a rule  $R$  to  $\mathbf{U}$ , we mean the following:  $R$  applies to  $\mathbf{U}$ , and by writing  $\mathbf{U}$  as the block preceding the line, we proceed according to one of the following two cases:

Case 1. If  $R$  is  $F\vee$ ,  $T\wedge$ ,  $T\sim$ ,  $F\sim$ , or  $F\rightarrow$ , we adjoin to  $\mathbf{U}$  as its sole immediate successor the block following the line in the statement of rule  $R$ .

Case 2. If  $R$  is  $T\vee$ ,  $F\wedge$ , or  $T\rightarrow$ , we adjoin to  $\mathbf{U}$  an ordered sequence of immediate successors comprised of all the blocks following the line in the statement of rule  $R$ .

Example 2.1: Suppose  $\mathbf{U} = \{TX, FY, T\vee\Phi\}$ . To apply rule  $T\vee$  to  $\mathbf{U}$ , we may choose any sequence  $\langle Z_i \mid i < \mu \rangle$ , where  $\mu$  is finite or  $\mu = \omega$ , for the members of  $\Phi$ , and then write the diagram



When a rule  $R$  has been applied to a block  $\mathbf{U}$ , we say that  $\mathbf{U}$  has been *reduced* by rule  $R$ . We note that the notation  $\mathbf{U} = \{\mathbf{S}, H\}$  is not meant to exclude consideration of  $H$  as a member of  $\mathbf{S}$ . Thus, applying rule  $T\sim$  to  $\{TX, FY, T\sim Z\}$ , for example, we may adjoin either  $\{TX, FY, FZ\}$  or  $\{TX, FY, T\sim Z, FZ\}$  as the immediate successor. Hence, duplication rules are not necessary.

*Definition 2.5* Let  $\mathbf{S}$  be a block. By a *tableau* for  $\mathbf{S}$  we mean an ordered finite branch tree  $T$  such that

- (i) the origin of  $T$  is  $\mathbf{S}$
- (ii) the points of  $T$  are blocks
- (iii) for any point  $\mathbf{U}$  of  $T$  which is not an end point, the immediate successors of  $\mathbf{U}$  in  $T$  are the results of applying a reduction rule  $R$  to  $\mathbf{U}$ .

We note that in a tableau  $T$ , a block may have infinitely many immediate successors, as in Example 2.1. The finite branch requirement, however, means that any sequence  $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \dots$  in  $T$ , where each  $\mathbf{U}_{i+1}$  is an immediate successor of  $\mathbf{U}_i$ , must be finite. This property allows us to assign countable ordinals to tableaus in a natural way and thus provides us with the basis for induction proofs.

*Definition 2.6* A block  $\mathbf{S}$  is *closed* if for some  $X \in \mathcal{F}$ , both  $TX \in \mathbf{S}$  and  $FX \in \mathbf{S}$ . A tableau  $T$  is *closed* if each end point of  $T$  is a closed block. A block  $\mathbf{S}$  is *inconsistent* if there exists a closed tableau for  $\mathbf{S}$ . If no closed tableau for  $\mathbf{S}$  exists,  $\mathbf{S}$  is *consistent*.  $X$  is a *theorem* if the block  $\{FX\}$  is inconsistent. If  $X$  is a theorem, we write  $\vdash X$ . A closed tableau for  $\{FX\}$  is called a *proof* of  $X$ .

We now present an example of an infinitary proof, where some obvious simplifications have been made. (See Example 2.2 on facing page.)

**3 Correctness** To show that the tableau system of proof in **2** is correct in the sense that only valid formulas are theorems, we must first establish a basis for induction on tableaux.

*Definition 3.1* Let  $T$  be a tableau for  $\mathbf{S}$ , and let  $\mathbf{U}$  be a block in  $T$ . By the *subtableau* of  $\mathbf{U}$  in  $T$ , we mean the subtree of  $T$  whose origin is  $\mathbf{U}$ .

It is clear from Definition 2.5 that the subtableau of  $\mathbf{U}$  in  $T$  is a tableau for  $\mathbf{U}$ , and if  $T$  is closed, the subtableau of  $\mathbf{U}$  in  $T$  is closed.

The following definition is due essentially to Feferman [2], where he introduces it for infinitary classical Gentzen systems.

*Definition 3.2* Let  $T$  be a tableau for  $\mathbf{S}$ . We define  $Od(T)$ , the *order* of  $T$ , as follows. Suppose  $\mathbf{U}$  is a block in  $T$ . Then,

- (i) if  $\mathbf{U}$  is an end point of  $T$ ,  $\mathbf{U}$  is the subtableau of  $\mathbf{U}$  in  $T$ , so we let  $Od(\mathbf{U}) = 1$ .
- (ii) if  $\mathbf{U}$  is not an end point of  $T$ , let  $T'$  be the subtableau of  $\mathbf{U}$  in  $T$ ,  $\langle \mathbf{U}_i \mid i < \mu \rangle$  the sequence consisting of the immediate successors of  $\mathbf{U}$  in  $T$ , and for each  $i < \mu$ ,  $T^i$  the subtableau of  $\mathbf{U}_i$  in  $T$ . Then, we let

$$Od(T') = \sup_{i < \mu} (Od(T^i) + 1).$$

Clearly, for any tableau  $T$ ,  $Od(T)$  is a countable ordinal, and if  $T'$  is a proper subtableau of  $T$ ,  $Od(T') < Od(T)$ . As examples, if  $T$  is the tableau of Example 2.2,  $Od(T) = 9$ .

We now recall Definition 1.6, where we defined a model  $\langle \mathcal{A}, \leq, h \rangle$ . In what follows, we denote  $h(X)$  by  $\|X\|$  for each  $X \in \mathcal{F}$ .

*Definition 3.3* Let  $\langle \mathcal{A}, \leq, \| \cdot \| \rangle$  be a model. Then, for any block  $\mathbf{S} = \{TX_1, \dots, TX_m, FY_1, \dots, FY_n\}$ , we define

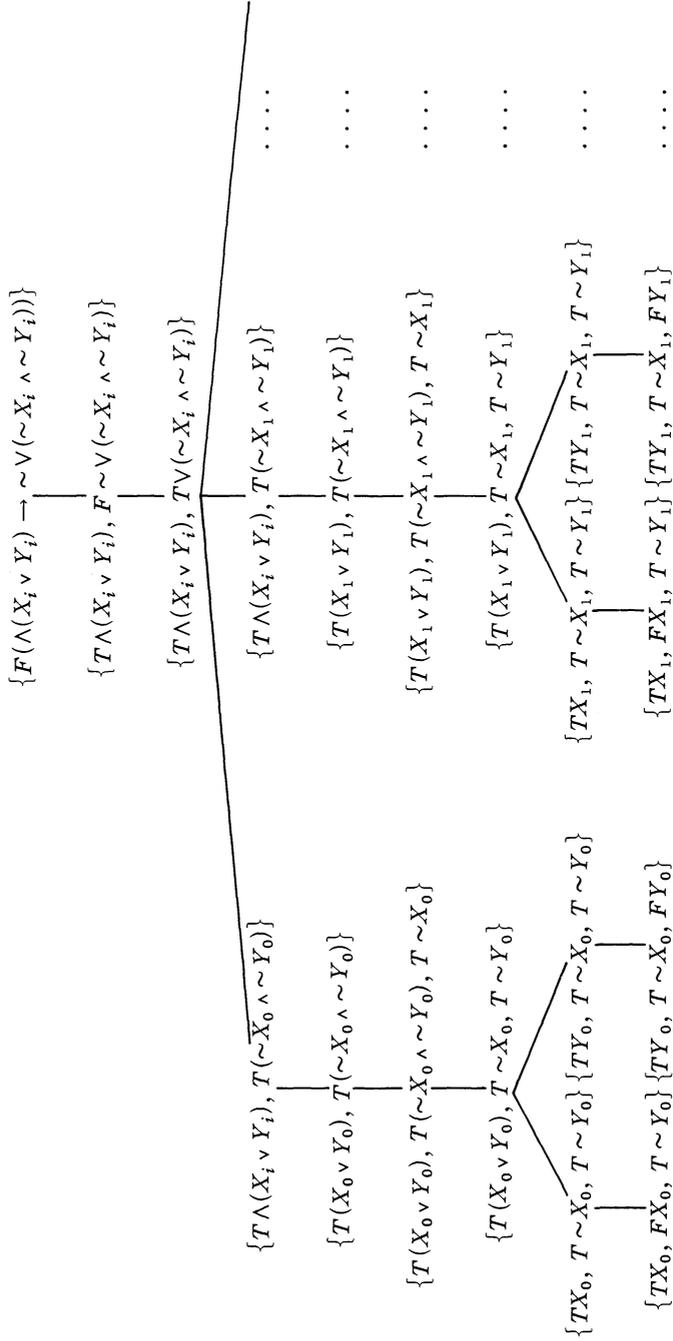
$$\|\mathbf{S}\| = \bigwedge_{i \leq m} \|X_i\| \Rightarrow \bigvee_{j \leq n} \|Y_j\|.$$

Now, in the proof of the following theorem, assume that  $\langle \mathcal{A}, \leq, \| \cdot \| \rangle$  is an arbitrary model.

**Theorem 3.1** *If  $\vdash X$ , then  $X$  is valid.*

*Proof:* Suppose that  $\vdash X$ , and let  $T$  be a closed tableau for  $\{FX\}$ . The proof

**Example 2.2:**  $\vdash \bigwedge_{i < \omega} (X_i \vee Y_i) \rightarrow \sim \bigvee_{i < \omega} (\sim X_i \wedge \sim Y_i)$ .



is by induction on  $Od(T)$ . We show that for each block  $\mathbf{U}$  in  $T$ ,  $\|\mathbf{U}\| = 1$ , where 1 is the unit of  $\mathcal{A}$ . Then, in particular, we have

$$\|\{FX\}\| = 1 \Rightarrow \|X\| = 1$$

and hence,  $\|X\| = 1$ .

(i) Suppose  $\mathbf{U} = \{TX_1, \dots, TX_m, FY_1, \dots, FY_n\}$  is an end point of  $T$ . Then,  $\mathbf{U}$  is closed, so for some  $i$ ,  $1 \leq i \leq m$ , and some  $j$ ,  $1 \leq j \leq n$ , we have  $X_i = Y_j = Z$ . Thus,

$$\begin{aligned} \|\mathbf{U}\| &= \bigwedge_{i \leq m} \|X_i\| \Rightarrow \bigvee_{j \leq n} \|Y_j\| \\ &\geq \|Z\| \Rightarrow \bigvee_{j \leq n} \|Y_j\| \\ &\geq \|Z\| \Rightarrow \|Z\| \\ &= 1, \end{aligned}$$

and so,  $\|\mathbf{U}\| = 1$ .

(ii) There are nine cases to consider, one for each of the reduction rules of Definition 2.3. Since each rule has the form

$$\mathbf{U} / \langle \mathbf{U}_i \mid i < \mu \rangle$$

when applied, we must show that if  $\|\mathbf{U}_i\| = 1$  for each  $i < \mu$ , then  $\|\mathbf{U}\| = 1$ . We provide proofs for three of the cases, the others being similar.

Case 1.  $\mathbf{U} = \{\mathbf{S}, T \vee X_i\}$ .

$$\begin{aligned} \text{Then, } &\|\{\mathbf{S}, TX_i\}\| = 1 \text{ for each } i \\ \rightarrow &\|\mathbf{S}^T\| \wedge \|X_i\| \Rightarrow \|\mathbf{S}^F\| = 1 \text{ for each } i \\ \rightarrow &\|\mathbf{S}^T\| \wedge \|X_i\| \leq \|\mathbf{S}^F\| \text{ for each } i \\ \rightarrow &\bigvee_i (\|\mathbf{S}^T\| \wedge \|X_i\|) \leq \|\mathbf{S}^F\| \\ \rightarrow &\|\mathbf{S}^T\| \wedge \bigvee_i \|X_i\| \leq \|\mathbf{S}^F\| \\ \rightarrow &\|\mathbf{S}^T\| \wedge \|\vee X_i\| \leq \|\mathbf{S}^F\| \\ \rightarrow &\|\mathbf{S}^T\| \wedge \|\vee X_i\| \Rightarrow \|\mathbf{S}^F\| = 1 \\ \rightarrow &\|\{\mathbf{S}, T \vee X_i\}\| = 1. \end{aligned}$$

Case 2a.  $\mathbf{U} = \{\mathbf{S}, F(X_1 \wedge X_2)\}$ .

$$\begin{aligned} \text{Then, } &\|\{\mathbf{S}, FX_i\}\| = 1 \text{ for } i = 1, 2 \\ \rightarrow &\|\mathbf{S}^T\| \Rightarrow \|\mathbf{S}^F\| \vee \|X_i\| = 1 \text{ for } i = 1, 2 \\ \rightarrow &\|\mathbf{S}^T\| \leq \|\mathbf{S}^F\| \vee \|X_i\| \text{ for } i = 1, 2 \\ \rightarrow &\|\mathbf{S}^T\| \leq (\|\mathbf{S}^F\| \vee \|X_1\|) \wedge (\|\mathbf{S}^F\| \vee \|X_2\|) \\ \rightarrow &\|\mathbf{S}^T\| \leq \|\mathbf{S}^F\| \vee (\|X_1\| \wedge \|X_2\|) \\ \rightarrow &\|\mathbf{S}^T\| \leq \|\mathbf{S}^F\| \vee \|X_1 \wedge X_2\| \\ \rightarrow &\|\mathbf{S}^T\| \Rightarrow \|\mathbf{S}^F\| \vee \|X_1 \wedge X_2\| = 1 \\ \rightarrow &\|\{\mathbf{S}, F(X_1 \wedge X_2)\}\| = 1. \end{aligned}$$

Case 2b.  $\mathbf{U} = \{\mathbf{S}, F \wedge X_i\}$ , where  $F \wedge X_i$  is infinitary.

$$\begin{aligned} \text{Then, } &\|\{\mathbf{S}^T, FX_i\}\| = 1 \text{ for each } i \\ \rightarrow &\|\mathbf{S}^T\| \Rightarrow \|X_i\| = 1 \text{ for each } i \end{aligned}$$

- $\|\mathbf{S}^T\| \leq \|X_i\|$  for each  $i$
- $\|\mathbf{S}^T\| \leq \bigwedge_i \|X_i\|$
- $\|\mathbf{S}^T\| \leq \|\bigwedge X_i\|$
- $\|\mathbf{S}^T\| \leq \|\mathbf{S}^F\|_{\vee} \|\bigwedge X_i\|$
- $\|\mathbf{S}^T\| \Rightarrow \|\mathbf{S}^F\|_{\vee} \|\bigwedge X_i\| = 1$
- $\|\{\mathbf{S}, F \bigwedge X_i\}\| = 1$ .

Case 3.  $\mathbf{U} = \{\mathbf{S}, F \sim X\}$ .

- Then,  $\|\{\mathbf{S}^T, TX\}\| = 1$
- $\|\mathbf{S}^T\| \wedge \|X\| \Rightarrow 0 = 1$
  - $\|\mathbf{S}^T\| \wedge \|X\| = 0$
  - $\|\mathbf{S}^T\| \leq \|X\|^- = \|\sim X\|$
  - $\|\mathbf{S}^T\| \leq \|\mathbf{S}^F\|_{\vee} \|\sim X\|$
  - $\|\mathbf{S}^T\| \Rightarrow \|\mathbf{S}^F\|_{\vee} \|\sim X\| = 1$
  - $\|\{\mathbf{S}, F \sim X\}\| = 1$ .

We may thus show that  $\|\mathbf{U}\| = 1$  for each block  $\mathbf{U}$  in  $T$ , and so  $\|X\| = 1$ . QED

**4 Completeness** To prove the completeness of our proof theory, we first establish that the class of theorems is closed under *modus ponens*: i.e., if  $\vdash X$  and  $\vdash X \rightarrow Y$ , then  $\vdash Y$ . We accomplish this by proving a tableau version of Gentzen’s Hauptsatz. More specifically, we show that for any block  $\mathbf{S}$  and any  $X \in \mathcal{F}$ , if both  $\{\mathbf{S}, TX\}$  and  $\{\mathbf{S}, FX\}$  are inconsistent, then  $\mathbf{S}$  is inconsistent. The proof is constructive in nature and does not use any semantical notions.

For any block  $\mathbf{U}$ , we say  $\mathbf{U}$  closes if  $\mathbf{U}$  is inconsistent; i.e., if there is a closed tableau for  $\mathbf{U}$ . We say  $\mathbf{U}$  closes via  $H$  if there exists a closed tableau  $T$  for  $\mathbf{U}$  in which  $H \in \mathbf{U}$  is the first member to which a reduction rule  $R$  is applied. We say  $\mathbf{U}$  closes with  $Od = \gamma$  if there exists a closed tableau  $T$  for  $\mathbf{U}$  with  $Od(T) = \gamma$ .

Now, it is well-known that for any countable ordinal  $\gamma$ , we may write uniquely

$$\gamma = \omega^{e_1} n_1 + \omega^{e_2} n_2 + \dots + \omega^{e_k} n_k \quad ,$$

where  $e_1 > e_2 > \dots > e_k \geq 0$  and each  $n_i$  is finite. This expression is referred to as the *normal expansion* of  $\gamma$ . The following definition is due to Feferman [2].

*Definition 4.1* Let  $\gamma$  and  $\delta$  be countable ordinals. Then, using the normal expansions, we define the *linear sum* of  $\gamma$  and  $\delta$  to be

$$\gamma \oplus \delta = \omega^{e_1} (n_1 + m_1) + \omega^{e_2} (n_2 + m_2) + \dots + \omega^{e_k} (n_k + m_k) \quad ,$$

where either  $n_j \neq 0$  or  $m_j \neq 0$  for  $1 \leq j \leq k$ .

The linear sum is easily seen to be commutative and an increasing function of either argument. These facts are important for the proof of Theorem 4.1.

*Definition 4.2* Let  $\mathbf{U}_1$  and  $\mathbf{U}_2$  be blocks. Suppose that  $\mathbf{U}_1$  closes with

$Od = \gamma$  and  $\mathbf{U}_2$  closes with  $Od = \delta$ , and let  $\alpha = \gamma \oplus \delta$ . Then, we say that  $\mathbf{U}_1$  and  $\mathbf{U}_2$  close with *combined order*  $\alpha$ . Let  $X \in \mathcal{F}$  and let  $\alpha$  be a countable ordinal. We say that  $X$  is  $\alpha$ -*eliminable* if for any block  $\mathbf{S}$ , if  $\{\mathbf{S}, TX\}$  and  $\{\mathbf{S}, FX\}$  close with combined order  $\alpha$ , then  $\mathbf{S}$  closes. We say that  $X$  is *eliminable* if  $X$  is  $\alpha$ -eliminable for all  $\alpha$ .

We now prove the Hauptsatz in the following form:

**Theorem 4.1** *For every  $X \in \mathcal{F}$ ,  $X$  is eliminable.*

*Proof:* Let  $X \in \mathcal{F}$  and let  $\alpha$  be a countable ordinal. We prove that  $X$  is  $\alpha$ -eliminable by a double induction argument on formulas and on  $\alpha$ . The induction hypothesis is as follows:

- (A) For each  $Y \in \text{Subp}(X)$ ,  $Y$  is eliminable
- (B) For each  $\beta < \alpha$ ,  $X$  is  $\beta$ -eliminable.

Now suppose that for some block  $\mathbf{S}$ ,  $\{\mathbf{S}, TX\}$  closes with  $Od = \gamma$ ,  $\{\mathbf{S}, FX\}$  closes with  $Od = \delta$ , and  $\gamma \oplus \delta = \alpha$ . We must show that  $\mathbf{S}$  closes.

Suppose either  $\{\mathbf{S}, TX\}$  or  $\{\mathbf{S}, FX\}$  is closed (i.e.,  $\gamma = 1$  or  $\delta = 1$ .) If  $\{\mathbf{S}, TX\}$  is closed, then either  $\mathbf{S}$  is closed or  $FX \in \mathbf{S}$ , in which case  $\{\mathbf{S}, FX\} = \mathbf{S}$  and  $\mathbf{S}$  closes. The argument is the same for  $\{\mathbf{S}, FX\}$ .

The body of the proof consists of exhausting the cases in which either  $\{\mathbf{S}, TX\}$  or  $\{\mathbf{S}, FX\}$  closes via  $H \in \mathbf{S}$ , and, then, those in which  $\{\mathbf{S}, TX\}$  closes via  $TX$  and  $\{\mathbf{S}, FX\}$  closes via  $FX$ . Since the arguments in many of these cases are virtually identical, we include only a few representative cases here.

**Case 1.**  $\{\mathbf{S}, TX\}$  closes via  $H = T \vee X_i$ .

Suppose that  $\{\mathbf{S}, TX\}$  closes via  $H$ . Then, for each  $i$ ,  $\{\mathbf{S}, TX_i, TX\}$  closes with  $Od < \gamma$ . But  $\{\mathbf{S}, TX_i, FX\}$  closes with  $Od \leq \delta$  for each  $i$ , so by (B),  $\{\mathbf{S}, TX_i\}$  closes for each  $i$ . Thus,  $\mathbf{S}$  closes via  $H$ .

**Case 2.**  $\{\mathbf{S}, TX\}$  closes via  $H = F \sim Y$  and  $\{\mathbf{S}, FX\}$  closes via  $FX$ .

**Subcase 2.1.**  $X = \vee X_i$ . Suppose that  $\{\mathbf{S}, T \vee X_i\}$  closes via  $H \in \mathbf{S}$ , and  $\{\mathbf{S}, F \vee X_i\}$  closes via  $F \vee X_i$ . Then,  $\{\mathbf{S}^T, TY, T \vee X_i\}$  closes with  $Od < \gamma$  and for some  $i$ ,  $\{\mathbf{S}, F \vee X_i, FX_i\}$  closes with  $Od < \delta$ .

Now, suppose first that  $\delta > 2$ . Then,  $\{\mathbf{S}, F \vee X_i, TX_i\}$  closes with  $Od = 2 < \delta$ . But  $\{\mathbf{S}, T \vee X_i, TX_i\}$  closes with  $Od \leq \gamma$  so by (B),  $\{\mathbf{S}, TX_i\}$  closes. Also,  $\{\mathbf{S}, T \vee X_i, FX_i\}$  closes with  $Od \leq \gamma$ , so again by (B),  $\{\mathbf{S}, FX_i\}$  closes. Since we have that both  $\{\mathbf{S}, TX_i\}$  and  $\{\mathbf{S}, FX_i\}$  close, it follows from (A) that  $\mathbf{S}$  closes.

Finally, suppose that  $\delta = 2$ . Then,  $\{\mathbf{S}, F \vee X_i, FX_i\}$  is closed. We must have  $TX_i \in \mathbf{S}$ . But then,  $TX_i \in \mathbf{S}^T$ , so  $\{\mathbf{S}^T, TY, F \vee X_i\}$  closes with  $Od = \delta$ . Thus, by (B),  $\{\mathbf{S}^T, TY\}$  closes, and so  $\mathbf{S}$  closes via  $H$ .

**Subcase 2.2**  $X = \wedge X_i$ , finitary. It suffices to assume  $X = X_1 \wedge X_2$ . Suppose that  $\{\mathbf{S}, T(X_1 \wedge X_2)\}$  closes via  $H \in \mathbf{S}$ , and  $\{\mathbf{S}, F(X_1 \wedge X_2)\}$  closes via  $F(X_1 \wedge X_2)$ . Then,  $\{\mathbf{S}^T, TY, T(X_1 \wedge X_2)\}$  closes with  $Od < \gamma$ , and  $\{\mathbf{S}, F(X_1 \wedge X_2), FX_i\}$  closes with  $Od < \delta$  for  $i = 1, 2$ .

Now, suppose first that  $\delta > 2$ . Then,  $\{\mathbf{S}, F(X_1 \wedge X_2), TX_1, TX_2\}$  closes with  $Od = 2 < \delta$ . Since  $\{\mathbf{S}, T(X_1 \wedge X_2), TX_1, TX_2\}$  closes with  $Od \leq \gamma$ , we have by (B) that  $\{\mathbf{S}, TX_1, TX_2\}$  closes. Also,  $\{\mathbf{S}, T(X_1 \wedge X_2), FX_i\}$  closes with  $Od \leq \gamma$  for  $i = 1, 2$ , so  $\{\mathbf{S}, FX_i\}$  closes for  $i = 1, 2$  by (B). Now, since  $\{\mathbf{S}, FX_2\}$  closes,  $\{\mathbf{S}, FX_2, TX_1\}$  closes, so by (A),  $\{\mathbf{S}, TX_1\}$  closes. But  $\{\mathbf{S}, FX_1\}$  closes, so again by (A), it follows that  $\mathbf{S}$  closes.

Finally, suppose  $\delta = 2$ . Then,  $\{\mathbf{S}, F(X_1 \wedge X_2), FX_i\}$  is closed for  $i = 1, 2$ . We must have  $TX_1 \in \mathbf{S}$  and  $TX_2 \in \mathbf{S}$ , so  $TX_1 \in \mathbf{S}^T$  and  $TX_2 \in \mathbf{S}^T$ . Then,  $\{\mathbf{S}^T, TY, F(X_1 \wedge X_2)\}$  closes with  $Od = \delta$ . Recalling that  $\{\mathbf{S}^T, TY, T(X_1 \wedge X_2)\}$  closes with  $Od < \gamma$ , we have by (B) that  $\{\mathbf{S}^T, TY\}$  closes. Hence,  $\mathbf{S}$  closes via  $H$ .

Subcase 2.3  $X = \wedge X_i$ , infinitary. Suppose that  $\{\mathbf{S}, T \wedge X_i\}$  closes via  $H \in \mathbf{S}$ , and  $\{\mathbf{S}, F \wedge X_i\}$  closes via  $F \wedge X_i$ . We have then that  $\{\mathbf{S}^T, TY, T \wedge X_i\}$  closes with  $Od < \gamma$ , and  $\{\mathbf{S}^T, FX_i\}$  closes with  $Od < \delta$  for each  $i$ . But clearly,  $\{\mathbf{S}^T, TY, F \wedge X_i\}$  closes with  $Od \leq \delta$ , so by (B),  $\{\mathbf{S}^T, TY\}$  closes. Thus,  $\mathbf{S}$  closes via  $H$ .

Case 3.  $\{\mathbf{S}, TX\}$  closes via  $TX$  and  $\{\mathbf{S}, FX\}$  closes via  $FX$ .

Subcase 3.1  $X = Y \rightarrow Z$ . Suppose that  $\{\mathbf{S}, T(Y \rightarrow Z)\}$  closes via  $T(Y \rightarrow Z)$ , and  $\{\mathbf{S}, F(Y \rightarrow Z)\}$  closes via  $F(Y \rightarrow Z)$ . Then, both  $\{\mathbf{S}, T(Y \rightarrow Z), FY\}$  and  $\{\mathbf{S}, T(Y \rightarrow Z), TZ\}$  close with  $Od < \gamma$ , and  $\{\mathbf{S}^T, TY, FZ\}$  closes with  $Od < \delta$ .

First, we note that  $\{\mathbf{S}, F(Y \rightarrow Z), FY\}$  closes with  $Od \leq \delta$ , so by (B),  $\{\mathbf{S}, FY\}$  closes. But also,  $\{\mathbf{S}, F(Y \rightarrow Z), TZ\}$  closes with  $Od \leq \delta$ , so again by (B),  $\{\mathbf{S}, TZ\}$  closes. Now, since  $\{\mathbf{S}^T, TY, FZ\}$  closes,  $\{\mathbf{S}, TY, FZ\}$  closes, and since  $\{\mathbf{S}, TZ\}$  closes,  $\{\mathbf{S}, TY, TZ\}$  closes. Hence, by (A),  $\{\mathbf{S}, TY\}$  closes. But we already know  $\{\mathbf{S}, FY\}$  closes, so again by (A),  $\mathbf{S}$  must close.

Subcase 3.2  $X = \wedge X_i$ , infinitary. Suppose that  $\{\mathbf{S}, T \wedge X_i\}$  closes via  $T \wedge X_i$ , and  $\{\mathbf{S}, F \wedge X_i\}$  closes via  $F \wedge X_i$ . Then,  $\{\mathbf{S}, T \wedge X_i, TX_j\}$  closes with  $Od < \gamma$  for some  $j$ , and  $\{\mathbf{S}^T, FX_i\}$  closes with  $Od < \delta$  for each  $i$ . Now,  $\{\mathbf{S}, F \wedge X_i, TX_j\}$  closes with  $Od \leq \gamma$ , so by (B),  $\{\mathbf{S}, TX_j\}$  closes. But also,  $\{\mathbf{S}, FX_j\}$  closes, since  $\{\mathbf{S}^T, FX_j\}$  does. Thus, by (A),  $\mathbf{S}$  closes.

We may thus show that, in any case,  $\mathbf{S}$  closes. QED

The next result follows from Theorem 4.1 by exactly the same argument as [9], p. 115, for the finitary classical version.

Corollary 4.1 (*Modus Ponens*) *If  $\vdash X$  and  $\vdash X \rightarrow Y$ , then  $\vdash Y$ .*

It is now a straightforward matter to show that our tableau system is complete. It is well-known that the Lindenbaum algebra  $\mathcal{A}_0$  of finitary intuitionistic logic is a pseudo-Boolean algebra, and that  $\vdash X$  if and only if  $|X| = 1$ , the unit element of  $\mathcal{A}_0$  (see [7], IX, 2.2). We may define  $\mathcal{A}_0$  for the collection of theorems in  $\mathcal{F}$  in the same manner as in [7] and use the argument there to show that  $\mathcal{A}_0$  is a pseudo-Boolean algebra. Since by Corollary 4.1, the class of theorems is closed under *modus ponens*, we have the following exactly as in the finitary case:

Theorem 4.2  $\vdash X$  if and only if  $|X| = 1$  in  $\mathcal{A}_0$ .

It remains to show that: (i)  $\mathcal{A}_0$  is nondegenerate, and (ii)  $\mathcal{A}_0$  is complete. The former is trivial.

Theorem 4.3  $\mathcal{A}_0$  is nondegenerate.

*Proof:* Let  $A$  be any atomic formula of  $\mathcal{F}$ . Then, no closed tableau for  $\{FA\}$  exists, so  $A$  is not a theorem. Hence  $|A| \neq 1$  in  $\mathcal{A}_0$ . QED

Theorem 4.4  $\mathcal{A}_0$  is a complete pseudo-Boolean algebra. For any countable subset  $\Phi$  of  $\mathcal{F}$ ,

$$\begin{aligned} \vee\{|X| \mid X \in \Phi\} &= |\vee\Phi| \\ \wedge\{|X| \mid X \in \Phi\} &= |\wedge\Phi|. \end{aligned}$$

*Proof:* For convenience, we write  $\vee\Phi = \vee X_i$ . For any  $i$ , we have that  $\vdash X_i \rightarrow \vee X_i$  by the following closed tableau:

$$\begin{array}{c} \{F(X_i \rightarrow \vee X_i)\} \\ \mid \\ \{TX_i, F\vee X_i\} \\ \mid \\ \{TX_i, F\vee X_i, FX_i\}. \end{array}$$

Thus,  $|X_i| \leq |\vee X_i|$  for each  $i$ . Suppose now that  $|X_i| \leq |Y|$  for each  $i$ . Then,  $\vdash X_i \rightarrow Y$  for each  $i$ , so there exists a closed tableau for  $\{F(X_i \rightarrow Y)\}$  for each  $i$ , and hence, for  $\{TX_i, FY\}$ . But consider  $\vee X_i \rightarrow Y$ . If we begin a tableau as follows:

$$\begin{array}{c} \{F(\vee X_i \rightarrow Y)\} \\ \mid \\ \{T\vee X_i, FY\} \quad \dots \\ \swarrow \quad \downarrow \quad \searrow \\ \{TX_0, FY\} \quad \{TX_1, FY\} \quad \{TX_2, FY\} \quad \dots, \end{array}$$

we see that  $\{F(\vee X_i \rightarrow Y)\}$  closes. Hence,  $\vdash \vee X_i \rightarrow Y$ , and so  $|\vee X_i| \leq |Y|$ . But this gives us that

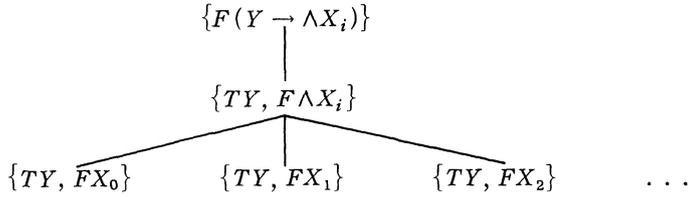
$$\vee |X_i| = |\vee X_i|$$

so countable least upper bounds exist in  $\mathcal{A}_0$ .

Now, consider  $\wedge\Phi = \wedge X_i$ . For any  $i$ , we have  $\vdash \wedge X_i \rightarrow X_i$  by the following closed tableau:

$$\begin{array}{c} \{F(\wedge X_i \rightarrow X_i)\} \\ \mid \\ \{T\wedge X_i, FX_i\} \\ \mid \\ \{T\wedge X_i, TX_i, FX_i\}. \end{array}$$

Thus,  $|\wedge X_i| \leq |X_i|$  for each  $i$ . Suppose now that  $|Y| \leq |X_i|$  for each  $i$ . Then,  $\vdash Y \rightarrow X_i$  for each  $i$ , so there exists a closed tableau for  $\{F(Y \rightarrow X_i)\}$  for each  $i$ , and hence, for  $\{TY, FX_i\}$ . But consider  $Y \rightarrow \wedge X_i$ . If we begin a tableau as follows:



we see that  $\{F(Y \rightarrow \wedge X_i)\}$  closes. Hence,  $\vdash Y \rightarrow \wedge X_i$ , and so  $|Y| \leq |\wedge X_i|$ . But this gives us that

$$|\wedge X_i| = |\wedge X_i|,$$

so countable greatest lower bounds exist in  $\mathcal{A}_0$ . Hence,  $\mathcal{A}_0$  is complete. QED

It is now easy to see that we have a canonical homomorphism

$$h_0: \mathcal{F} \rightarrow \mathcal{A}_0$$

given by  $h_0(X) = |X|$  for each  $X \in \mathcal{F}$ , and so  $\langle \mathcal{A}_0, \leq, h_0 \rangle$  is a model in the sense of Definition 1.6. This gives us our final result:

**Theorem 4.5** *If  $X$  is valid, then  $\vdash X$ .*

*Proof:* Suppose  $X$  is not a theorem. Then,  $|X| \neq 1$  in  $\mathcal{A}_0$ . But this means  $X$  is not valid in  $\mathcal{A}_0$ , so  $X$  is not valid. QED

Hence, we see that, as in the finitary case, the Lindenbaum algebra  $\mathcal{A}_0$  for our proof theory provides a countermodel for all nontheorems.

**5 Concluding remarks** There appear to be some directions in which further investigation would prove interesting.

Fitting shows in [3] that in the finitary case, the algebraic semantics and the Kripke semantics are equivalent. Hence, his tableau system is shown to be complete for both semantics. In the infinitary case, however, the reduction rules corresponding to the algebraic semantics we have used are not the natural generalization of the reduction rules in Fitting. Specifically, if we replaced the rule  $F\wedge(b)$  in Definition 2.3 by the following:

$$F\wedge^*: \{S, F\wedge\Phi\} / \{\{S, FX\} | X \in \Phi\},$$

which is the infinitary version of the  $F\wedge$  rule in [3], we would get a stronger proof system. It is not hard to construct a tableau proof of the formula

$$Y = \bigwedge_i (X \vee X_i) \rightarrow X \vee \bigwedge_i X_i$$

in this stronger system, but  $Y$  is not valid in the algebraic semantics since the inequality

$$\bigwedge_i (a \vee a_i) \leq a \vee \bigwedge_i a_i$$

does not, in general, hold in all pseudo-Boolean algebras. For a counter-example, see [7], p. 135.

It is easy to show that the stronger system described above yields theorems which are valid in the Kripke semantics for infinitary propositional logic. Hence, it would be interesting to generalize the construction of a Hintikka collection, as in [3], to the infinitary case, and thus show the stronger system is complete.

For the proof theory we have presented, it would likewise be interesting to prove a stronger completeness theorem; i.e., to show that a countable collection  $\Phi$  of formulas is consistent if and only if it has a model. This would necessitate, it appears, construction of an appropriate topological space whose open sets would provide the model.

Since the submission of this paper, the above directions have been taken by Nadel [6], and the reader is referred to his results.

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