Notre Dame Journal of Formal Logic
Volume XXI, Number 2, April 1980
NDJFAM

TWO REMARKS ON *THE LOGIC OF SIGNIFICANCE AND CONTEXT*

ROSS T. BRADY

I wish to make two remarks on two points that Goddard and Routley made in Chapter 5 of their book, *The Logic of Significance and Context* [6]. The first remark is that the Brady disjunction, 'V',[1] does have applications in ordinary discourse, in answer to the doubts about this expressed by them on pp. 344-5 of [6]. I will defend this position, which I originally made in [1] on p. 30, and also in [2], pp. 172-3, by showing that each of four examples can be interpreted in a way which requires 'V' to be used in its formalization. The second remark is that the need for a functionally complete significance logic such as their system $\mathbf{S}_5$ (or $\mathbf{S}_6$) seems very limited. I show this by examining each of their five arguments for a functionally complete system on pp. 348-351 of [6] and by showing how their weaker significance-complete (for definition, see [6], p. 326) system $\mathbf{S}_3$ (or $\mathbf{S}_4$) will suffice, except in exceptional circumstances, to express what they suppose a functionally complete system is needed for.

1 *The Brady disjunction* 'V'

1.1 *Two predicate disjunction examples* I will consider two examples of predicate disjunctions, of the form '(f or g)x'. I will argue that they can be formalized as a sentential disjunction in the form 'fx V gx' and I will consider the extent to which 'V' can be used in formalizing predicate disjunctions.

The reason I am dealing with predicate disjunctions is that these present clearer cases where 'V' can be used in their formalization than do the sentential disjunctions, of the form 'fx or gx'. I do think, though, that there are many sentential disjunctions which also use 'V' in their formalization, as they have the same interpretation as their corresponding predicate disjunction. Indeed, I want to reject my claim in [1] and [2] that the classical disjunction '$\vee$' should be used for formalizing 'x is a holiday or x likes cheese'.

The first example I consider is the predicate disjunction (scheme) 'x is a holiday or likes cheese', which is my example from [1] and [2]. Note that

Received July 30, 1976

Goddard and Routley misquote this example from [1] as 'x is a holiday or x likes cheese', the corresponding sentential disjunction. Let us use the term '*disjunct*' as applied to a predicate disjunction, '(f or g)x', to mean a disjunct of the corresponding sentential disjunction, 'fx or gx'. That is, 'fx' and 'gx' are the disjuncts of '(f or g)x'. Note that in our example the significance ranges of the disjuncts are disjoint. Hence, there are no assignments to 'x' such that both disjuncts are significant. This means that there are no assignments to 'x' to enable the four significant-significant (s-s) positions of a matrix used to formalize this predicate disjunction to be filled in. This led Goddard and Routley to say, on p. 345 of [6], "it is not even clear that the connective used is matrix-accessible over set $\{t, f, n\}$". However, the object of the present exercise is to show that the disjunction is matrix-accessible by determining such a matrix and, in the process of such a determination, I will assume that there is a satisfactory matrix.

Given the fact that the significance ranges of the two disjuncts are disjoint, that is, that '$\sim(SHx \,\&\, SLx)$' is true, then '$Hx\ D_1\ Lx \simeq Hx\ D_2\ Lx$' is true, where '$Hx$' stands for '$x$ is a holiday', 'Lx' stands for 'x likes cheese', and 'D_1' and 'D_2' are any connectives differing at most in the four s-s positions in their respective matrices. Hence, if one assumes that there is a satisfactory matrix for this disjunction, as long as the positions other than the s-s positions are filled in satisfactorily, it does not matter what values are filled into the s-s positions for this predicate disjunction.

To determine the other positions of matrices for the disjunction, let x be such that 'x is a holiday' is true and 'x likes cheese' is nonsignificant. Being a predicate disjunction, there is a tendency to interpret it in a way which makes it significant for some x, for otherwise it could never be used sensibly. That would imply that the predicate disjunction is not formalized using '$\vee$' and most likely interpreted in such a way as to make the above example true.

Such an interpretation is obtained by interpreting 'or' in a way which involves choosing the true disjunct in preference to the nonsignificant one and thus making the disjunction true. This is an extension of the classical two valued disjunction since it involves a choice over three possible values

U	t	f	n
t	t	t	t
f	t	f	n
n	t	n	n .

Goddard and Routley, on p. 345 of [6], mention 'V' and 'U' as minimal operators with respect to the orderings $t < f < n$ and $t < n < f$, respectively. However, this is only as a part of an argument for 'V' being technically useful rather than as a part of an argument, as I am giving, for 'V' having applications in ordinary discourse.

Let us examine some of the differences between these two matrices. Firstly, when assessing the value of the finite disjunction 'p_1 V p_2 V . . . V p_n' the nonsignificant p_i's can be ignored (except when all the p_i's are nonsignificant), and when assessing the value of the finite disjunction 'p_1 U p_2 U . . . U p_n', the false p_i's can be ignored (except when all the p_i's are false). The significance-restricting property of the 'V' case is a more intuitive property of a disjunction than the nonfalsity-restricting property of the 'U' case, since it is rather odd to pick out only the true and nonsignificant p_i's (if any) in determining the value of the finite disjunction. Secondly, 'U' fails the s-n sublogic property (cf., [2], pp. 176-7) whereas 'V' satisfies it. As argued in [2], the s-n sublogic property is a desirable property for a connective to have. Hence, the disjunction 'V' is more intuitive and technically more preferable than the disjunction 'U'.

Thus I have argued that the above two examples of predicate disjunctions can be formalized in the form 'fx V gx' and hence that 'V' does have applications in ordinary discourse. 'V' must be used whenever the intended interpretation involves the appropriate preferential choice of disjuncts of a predicate disjunction. Further, I think 'V' would be used far more often than 'U' or 'v' would, in the formalization of predicate disjunctions, but I think 'v' would be used much more frequently in the formalization of sentential disjunctions than it would be for predicate disjunctions.

1.2 *Two further examples* I will introduce two further examples, one a subject disjunction and one a relation disjunction, where 'V' can be used in their formalization. Let us first consider 'Something is happy', which is best formalized as '(**q**x)Ax' ('Ax' standing for 'x is happy'), as pointed out on p. 172 of [2]. Then consider the example, 'This stone or Tom is happy' where the domain of 'x' in the above example is restricted to two things, and 'Tom' refers to a person.

If 'This stone or Tom is happy' is interpreted as a restricted '**q**'-quantification, then it must be formalized as 'As V At', where 's' stands for 'this stone' and 't' for 'Tom', since '**q**'-quantification over a finite domain can be equivalently formalized by using repeated 'V''s. This provides us with an example of a subject disjunction which can be formalized using 'V'. An argument along the lines of that in **1.1** can also be developed for the formalization of subject disjunctions using 'V'.

Along similar lines, an additional argument can be given for formalizing the first example in **1.1** using 'V'. Consider first 'New Year's Day,

1976, has some property', which, by analogy to the above, is best formalized as '(**q** f)fn' in second-order significance logic, where 'n' stands for 'New Year's Day, 1976'. Next consider 'New Year's Day, 1976, is a holiday or likes cheese', with the domain of predicates restricted to two. With this interpretation as a restricted '**q**'-quantification, 'New Year's Day, 1976, is a holiday or likes cheese' must be formalized as 'Hn V Ln'.

The second example is 'x is spatially or temporally next to y'. Goddard, on p. 156 of [5], gives the example 'is next to' as a heterogeneous relation which can be resolved into the two homogeneous relations: 'is spatially next to' and 'is temporally next to'. As given on p. 350 of [6], such a resolution can most easily be formalized as 'Sxy V Txy', where 'Sxy' stands for 'x is spatially next to y' and 'Txy' stands for 'x is temporally next to y'. It is clear that the relation 'is spatially or temporally next to' normally captures the sense of Goddard's example 'is next to' and hence that 'x is spatially or temporally next to y' would normally be interpreted in a way which would require the formalization 'Sxy V Txy'. I think this provides one of the better examples of the use of 'V' in formalizing ordinary discourse. Also, an argument along the lines of that in **1.1** can be developed for the formalization of relation disjunctions using 'V'.

2 *Functional completeness* Certain connectives of Goddard and Routley's functionally complete system $\mathbf{S}_5$ (or $\mathbf{S}_6$) have no clear intuitive interpretations in terms of a natural language, e.g., 'Q', 'D', '1' and 'T_n', which are as follows:

Q	
t	t
f	n
n	f

D	
t	n
f	f
n	t

1	t	f	n
t	t	f	n
f	n	n	n
n	n	n	n

T_n	
t	t
f	n
n	n .

Note that there is at least one 'n' in the s or s-s positions of these matrices. There is a prima facie case for omitting the above connectives from significance logic, since such a logic should ideally just formalize those sentences and arguments that can be expressed in a natural language. However, Goddard and Routley, on pp. 347-352 of [6], present five arguments in favour of a functionally complete system. I will counter each of their arguments in turn and give my general conclusions at the end.

(i) In their first argument, Goddard and Routley show that some connections between homogeneous and heterogenous relations can be expressed using 'Q', but go on to show that the connections they wish to express can be made using 'V' instead of 'Q'. However, as they admit, these connections can be then expressed within a quantified $\mathbf{S}_3$ (or $\mathbf{S}_4$), the connectives of which are significance-complete but not functionally complete.

(ii) In their second argument, Goddard and Routley use 'Q' in their formalization of a theory of descriptions. The description operator '($\mathbf{1}x$)(fx, gx)' is defined as '($\mathbf{E}x$)(fx & $(y)(Tfx \supset T(x = y))$ & gx) & $Q((\mathbf{E}x)fx)$'. If this definition is expressed as an equivalence, using '$\simeq$', as Goddard and Routley do, then the wff obtained is itself '$\simeq$'-equivalent to:

(*) $(T(\mathbf{E}x)fx \rightarrow ((\iota x)(fx, gx) \simeq (\mathbf{E}x)(fx \;\&\; (y)(Tfx \supset T(x = y)) \;\&\; gx))) \;\&$ $(\sim T(\mathbf{E}x)fx \rightarrow \sim S(\iota x)(fx, gx))$,

which does not contain 'Q' and can be formalized using a quantified $\mathbf{S}_3$. Hence, instead of *defining* '$(\iota x)(fx, gx)$' in a quantified functionally complete logic, one can introduce '$(\iota x)(fx, gx)$' as a *primitive*, given 'f', 'g', and 'x', and add (*) as an axiom to a quantified $\mathbf{S}_3$. So, a functionally complete logic can be avoided here, with the small price of losing an explicit definition and direct substitution of '$(\iota x)(fx, gx)$' in favour of an equivalent contextual definition which amounts to a definition by cases.

(iii) In their third argument, Goddard and Routley use 'Q' in their formalization of an interpretation of traditional logic. They approximate qu(All f are g) by '$(x)(fx \supset gx) \;\&\; Q(T(\mathbf{E}x)fx \;\&\; T(\mathbf{E}x)gx)$'. Let qu(All f are g) be symbolized as '$f\forall g$' and let the approximation be expressed as an equivalence, using '$\simeq$'. Then the wff so obtained is '$\simeq$'-equivalent to:

(**) $((T(\mathbf{E}x)fx \;\&\; T(\mathbf{E}x)gx) \rightarrow (f\forall g \simeq (x)(fx \supset gx))) \;\&\; ((\sim T(\mathbf{E}x)fx \vee \sim T(\mathbf{E}x)gx) \rightarrow$ $\sim Sf\forall g)$,

which does not contain 'Q' and, similarly to the corresponding expression in (ii), can be formalized using a quantified $\mathbf{S}_3$. Hence, instead of defining '$f\forall g$' or using a wff to represent qu(All f are g) in a quantified functionally complete logic, one can introduce '$f\forall g$' as a primitive and add (**) as an axiom to a quantified $\mathbf{S}_3$, yielding, as in (ii), a contextual definition amounting to a definition by cases. So, again, a functionally complete logic can be avoided.

(iv) In their fourth argument, Goddard and Routley, argue that ". . . given a significance-complete logic it is imperative to advance to a functionally complete logic. For in no merely significance-complete logic is it possible to define a constant non-significant sentence." They point out that the Słupecki '$\dot{\mathbf{T}}$' can be defined in terms of a nonsignificant constant 'n_0' as follows:

$$\dot{\mathbf{T}}\Delta =_{df} T\Delta \;\&\; n_0.$$

They have previously shown that " '$\dot{\mathbf{T}}$' together with significance-complete connectives provides a functionally complete connective set." Also, on p. 346 of [6], there are two examples of sentence-schemes, viz., 'the lazy number thinks that p' and 'p is thinking',[2] which they claim, privately if not specifically in the book, can be formalized as '$\dot{\mathbf{T}}p$', thus yielding ordinary discourse applications for '$\dot{\mathbf{T}}$'. This would independently justify the introduction of '$\dot{\mathbf{T}}$' as a connective so as to obtain a functionally complete system.

(a) Firstly, I will deal with the argument from the nonsignificant constant 'n_0'. As with the addition of the false constant 'f_0' to classical two-valued sentential logic, the addition of 'n_0' to a sentential significance logic is usually taken to yield an alternative formulation of that logic. However, by comparison with the nonlogical constants of first-order

theories, such a constant would be an axiomatically characterized nonlogical constant in an axiomatization of the significance logic, $\mathbf{S}_5$ (or $\mathbf{S}_6$). So, the addition of 'n_0' does not yield an alternative formulation of the *pure* sentential significance logic, but it yields an *applied* logic with nonlogical constant 'n_0', adopting the pure and applied logic distinction from Church, [4], p. 173-4. Further connectives, such as 'Q' and 'D', can be defined in terms of 'n_0' and connectives from $\mathbf{S}_3$, in a similar way to the definition of the further quantifier '$\mathbf{E}$' in terms of the nonlogical predicate constant 'E' and the connectives and quantifiers of $Q\mathbf{S}_6$, i.e., $(\mathbf{E}x)A =_{df} (\mathbf{q}x)(\mathbf{E}(x)\ \&\ A)$ (cf., p. 531 of [6]). But there is no need to add to the pure logic any connectives or quantifiers which are definable in terms of a nonlogical constant of the applied logic. Further, it is clear that it is the pure sentential logic that is being formulated in Chapter 5 of [6] as no sentential constants are added in any of the respective morphologies. Also, Goddard and Routley are quite wrong in suggesting that such sentential constants can be defined in terms of connectives as such a definition would define a constant in terms of free sentential variables. Hence, there is no need to advance their pure sentential logic from a significance-preserving one to a functionally complete one because of the addition of 'n_0' and the connectives 'Q', 'D', and the like, need only be defined in the applied logic.

(b) Secondly, I will deal with the argument from the Słupecki '$\dot{\mathbf{T}}$'. The two examples, where Goddard and Routley suggest that '$\dot{\mathbf{T}}$' can be employed in their formalization, that is, 'The lazy number thinks that p' and 'p is thinking', can be formalized as 'T^2n_lp' and 'T^1p', respectively, where 'T^2xp' stands for 'x thinks that p', 'T^1p' stands for 'p is thinking' and 'n_l' with the adjectival subscript 'l' stands for 'the lazy number'. These formalizations capture the respective sentence-schemes much more precisely than '$\dot{\mathbf{T}}p$' does, as they use predicate, operator, and individual constants and show more of the sentential structure than '$\dot{\mathbf{T}}p$' does. So, '$\dot{\mathbf{T}}p$' is a poor formalization for these sentence-schemes because it uses a logical connective (or operator) instead of the more specific nonlogical constants, 'T^2', 'n_l', and 'T^1', used above. These nonlogical constants can be included in an applied significance logic with some characterizing axioms providing additional properties for them over and above the Słupecki '$\dot{\mathbf{T}}$' property given by '$\vdash\sim ST^2n_lp$' and '$\vdash\sim ST^1p$'.

Hence, there is no need to introduce '$\dot{\mathbf{T}}$' into the pure sentential logic in order to formalize the two sentence-schemes, as they are much better formalized without it. The same can be said for other such sentence-schemes which either have a significance clash within their subject or atomic predicate expressions or have a predicate applying to sentences when the significance range of the predicate is disjoint with the class of all sentences.

(v) In their fifth argument, Goddard and Routley argue that '$\upharpoonright$' is required for the full theory of restricted quantification since '$(\mathbf{q}X)(G(X) \upharpoonright F(X))$' takes the same value as '$(\mathbf{q}X)F(X)$' with the range of 'X' in '$(\mathbf{q}X)F(X)$' restricted to those X such that '$G(X)$' is true, given that '$(\mathbf{q}X)G(X)$' is true.

Also, the matrix for '$\uparrow$' is uniquely determined by this property, so that none other will suffice. This argument is also given in [2], p. 181.

However, it can be shown that $(\mathbf{q}X)_{G(X)}F(X) \simeq (\mathbf{P}X)_{G(X)\,\&\,SF(X)}F(X)$, given that '$(\mathbf{q}X)(TG(X)\ \&\ SF(X))$' is true, where the subscripts are the predicates to whose truth-ranges the quantification is restricted.

In order to determine the equivalent unrestricted form for the restricted '**P**'-quantifier, consider the definition:

$$(\mathbf{P}X)F(X) =_{df} \sim(\mathbf{U}X)\sim F(X) \quad \text{(cf. [6], p. 494)},$$

and then restrict both sides using the restricting predicate '$G(X)$' to obtain:

$$(\mathbf{P}X)_{G(X)}F(X) =_{df} \sim(\mathbf{U}X)_{G(X)}\sim F(X).$$

This is subject, of course, to '$(\mathbf{q}X)G(X)$' being true. Using the equivalent unrestricted form for the restricted '**U**'-quantifier on p. 330 of [6], i.e.,

$$(\mathbf{U}X)_{G(X)}F(X) \simeq (\mathbf{U}X)(G(X) \rightarrow F(X)),$$

we obtain:

$$(\mathbf{P}X)_{G(X)}F(X) \simeq \sim(\mathbf{U}X)(G(X) \rightarrow \sim F(X)).$$

By the definition of '**P**', we obtain:

$$(\mathbf{P}X)_{G(X)}F(X) \simeq (\mathbf{P}X) \sim (G(X) \rightarrow \sim F(X)), \quad \text{___ (1)}$$

which gives the equivalent unrestricted form for the restricted '**P**'-quantifier.

Subject to the condition that '$(\mathbf{q}X)(G(X)\ \&\ SF(X))$' is true,

$$\begin{aligned}(\mathbf{P}X)_{G(X)\,\&\,SF(X)}F(X) &\simeq (\mathbf{P}X) \sim ((G(X)\ \&\ SF(X)) \rightarrow \sim F(X)), \text{ using (1)},\\ &\simeq (\mathbf{P}X)\,T(T_nG(X)\ \&\ F(X)), \text{ since } \sim((p\ \&\ Sq) \rightarrow \sim q)\\ &\simeq T(T_np\ \&\ q),\\ &\simeq T(\mathbf{q}X)(T_nG(X)\ \&\ F(X)), \text{ since } (\mathbf{P}X)TA \simeq T(\mathbf{q}X)A.\end{aligned}$$

By using the condition above,

$T(\mathbf{q}X)(G(X)\ \&\ SF(X))$, and hence
$(\mathbf{q}X)\,T(G(X)\ \&\ SF(X))$, since $T(\mathbf{q}X)A \simeq (\mathbf{q}X)TA$,
$(\mathbf{q}X)(TG(X)\ \&\ TSF(X))$, since $T(p\ \&\ q) \simeq Tp\ \&\ Tq$,
$(\mathbf{q}X)(ST_nG(X)\ \&\ SF(X))$, since $Tp \simeq ST_np$ and $TSp \simeq Sp$,
$(\mathbf{q}X)S(T_nG(X)\ \&\ F(X))$, since $Sp\ \&\ Sq \simeq S(p\ \&\ q)$,
$S(\mathbf{q}X)(T_nG(X)\ \&\ F(X))$, since $(\mathbf{q}X)SA \simeq S(\mathbf{q}X)A$.

Hence,

$$\begin{aligned}(\mathbf{P}X)_{G(X)\,\&\,SF(X)}F(X) &\simeq (\mathbf{q}X)(T_nG(X)\ \&\ F(X)), \text{ since } Sp \rightarrow (Tp \simeq p),\\ &\simeq (\mathbf{q}X)(G(X) \uparrow F(X)), \text{ since } T_np\ \&\ q \simeq p \uparrow q,\\ &\simeq (\mathbf{q}X)_{G(X)}F(X), \text{ by using the equivalent unrestricted}\\ &\quad\ \text{form for the restricted '}\mathbf{q}\text{'-quantifier.}\end{aligned}$$

Since '$(\mathbf{q}X)(G(X)\ \&\ SF(X))$' is true iff '$(\mathbf{q}X)(TG(X)\ \&\ SF(X))$' is true, the required equivalence is shown, subject to the given condition.

Hence, the connective '$\uparrow$' is not required for a restricted quantification

theory dealing with the restricted '**q**'-quantifier since the equivalent unrestricted form of '$(\mathbf{q}X)_{G(X)}F(X)$', namely '$(\mathbf{q}X)(G(X) \uparrow F(X))$', can be replaced by the equivalent unrestricted form of '$(\mathbf{P}X)_{G(X)\,\&\,SF(X)}F(X)$', namely '$(\mathbf{P}X) \sim ((G(X) \,\&\, SF(X)) \rightarrow \sim F(X))$', all of whose connectives are contained in the significance-complete system $\mathbf{S}_3$. This, however, is subject to the condition that '$(\mathbf{q}X)(TG(X) \,\&\, SF(X))$' is true, that is, that the truth-range of '$G(X)$' intersects with the significance-range of '$F(X)$'. This condition is normally adhered to in practice and is a three-valued generalization of the condition that '$(\mathbf{q}X)G(X)$' is true in two-valued logic, since the assumption that '$F(X)$' is significant for all X is required to keep the logic two-valued. In fact, the condition that '$(\mathbf{q}X)(TG(X) \,\&\, SF(X))$' is true could well be introduced as a normal requirement on restricted quantification generally.

(vi) Thus, I have countered each of the five arguments that Goddard and Routley use to support a functionally complete system. I have shown in each case that, except in exceptional circumstances in cases (ii), (iii), and (v), a significance-complete logic will suffice where Goddard and Routley supposed that a functionally complete logic was needed.

My points in (ii) and (iii) can be made for all examples where expressions are to be nonsignificant under certain specifiable conditions, which differ from the normal nonsignificance conditions of these expressions. The need for a functionally complete logic here is very rare since not only do these examples requiring it not conform to the usual intuitions but the need for explicit definitions and direct substitution in such examples is rare. Also, in point (v), if the condition that '$(\mathbf{q}X)(TG(X) \,\&\, SF(X))$' is true is not satisfied then one would be restricting '**q**'-quantification to a class which is disjoint with the significance range of '$F(X)$', which is a rather pointless restriction of quantification and for which there would be a very rare need.

However, there is no conclusive argument to say that there are no examples which would require a functionally complete logic and so I would suggest retaining functionally complete systems in case the need for them arises, but concentrating on the development of significance-complete systems as these are the ones which will be of general use and the connectives of which do have ordinary discourse applications (cf. [6], pp. 327-9).

NOTES

1. The symbols and terminology for this paper are taken from Goddard and Routley [6].
2. I will adopt the convention of leaving out the quotation function, '*qu* (. . .)', inside quotation marks.

REFERENCES

[1] Brady, R. T., "A 4-valued theory of classes and individuals," Ph.D. thesis, 1971, deposited in the University Library, St. Andrews.

[2] Brady, R. T., "Significance logics," *Notre Dame Journal of Formal Logic*, vol. XVII (1976), pp. 161-183.

[3] Brady, R. T., "Significance range theory," *Notre Dame Journal of Formal Logic*, vol. XXI (1980), pp. 319-345.

[4] Church, A., *Introduction to Mathematical Logic,* Princeton University Press, Princeton, New Jersey, 1956.

[5] Goddard, L., "Predicates, relations and categories," *Australasian Journal of Philosophy*, vol. 44 (1966), pp. 139-171.

[6] Goddard, L. and R. Routley, *The Logic of Significance and Context*, Vol. 1, Scottish Academic Press, Edinburgh, 1973.

La Trobe University
Bundoora, Victoria, Australia