Notre Dame Journal of Formal Logic
Volume XXI, Number 2, April 1980
NDJFAM

GENERAL COMPUTABILITY

THOMAS H. PAYNE

1 *Introduction* A theory of computability consists of a domain A of cardinality greater than one and a definition of computability for multivariable partial functions from A into A. The class $\mathcal{F}$ of all partial functions satisfying that definition is called the class of computable functions. In this paper, we consider some properties of $\mathcal{F}$ which have been of importance in various specific theories. In particular, we consider conjunctions of the following properties:

(1) $\mathcal{F}$ is the closure of a family $\mathcal{G}$ under composition.

(2) $\mathcal{F}$ is closed under piecewise composition.

(3) $\mathcal{F}$ is the closure of a family $\mathcal{G}$ under composition and iteration where the iterate f^∞ of a partial function f is $\lambda x\,[f^n(x)]$, where n denotes $\mu m\,[f^m(x) = f^{m+1}(x)]$

(4) $\mathcal{G}$ is closed under piecewise composition.

(5) $\mathcal{G}$ is finite.

(6) $\mathcal{F}$ contains the test function $\lambda w, x, y, z\,[x \text{ if } w = z;\text{ else } y]$.

(7) $\mathcal{F}$ includes a pairing system, i.e., a set $\{\rho, \sigma, p\}$ of partial functions such that $\rho(p(a, b)) = a$ and $\sigma(p(a, b)) = b$ for all a and b in A.

(8) ρ, σ, and the members of $\mathcal{G}$ are total functions on A.

(9) $\mathcal{F}$ has an indexing; i.e., $\mathcal{F}$ is closed under composition and contains a partial function $*$ such that every member of $\mathcal{F}$ is of the form:

$$\lambda x_1, \ldots, x_n\,[(\ldots(a * x_1) * \ldots * x_n)]$$

for some a in A.

(10) $\mathcal{F}$ has a uniform indexing in the sense that $\mathcal{F}$ has an indexing $*$ such that every member of $\mathcal{F}$ is of the form

$$\lambda x_1, \ldots, x_n[g(x_1, \ldots, x_{n-1}) * x_n]$$

for some total g in $\mathcal{F}$.

Received December 11, 1974; Revised January 7, 1980

In this paper, we establish:

a. normal form characterizations for the members of $\mathcal{F}$ when $\mathcal{F}$ satisfies (1); (2); (1) & (7); (1) & (6) & (7) & (8); (3) & (4) & (7); (3) & (5) & (6) & (7)

b. necessary and sufficient conditions for (9) and for (10)

c. the following implications: (3) & (5) & (6) & (7) $\Longleftrightarrow$ (3) & (6) & (10) $\not\Leftarrow$ (6) & (10) $\Longrightarrow$ (7) & (9) $\Longrightarrow$ (10) $\Longrightarrow$ (9) $\Longrightarrow$ (1) & (5)

d. that (2) is implied by (6) & (9), by (1) & (6) & (8), and by (3) & (4) & (7).

Many of our results and methods are extensions of those found in Wagner [5], Strong [3], and Friedman [1], [2]. The theorem on indexability in this paper is an abstract generalization of Kleene's normal-form theorem. It gives a normal-form characterization for the members of the closure, under composition and iteration, of any finite family of partial functions that includes a pairing system and the test function. From this lemma, it follows that such closures are uniformly indexable and that they are exactly the uniformly reflexive structures (URS's) containing semicomputable splinters. It is also shown that there is no first-order axiomatic characterization of such URS's; i.e., they do not constitute a generalized elementary class. The existence of a URS having no semicomputable splinter follows immediately. A more intricate proof of this fact is outlined in [4].

2 *Composition* Henceforth, $\mathcal{F}$ will denote an arbitrary family of multivariable partial functions on an arbitrary set A. $\mathcal{F}^n$ denotes the set consisting of those members of $\mathcal{F}$ having exactly n arguments. We will consider members of A to be 0-ary functions so that $\mathcal{F}^0$ is a subset of A. We let V denote an infinite set disjoint from $\mathcal{F} \cup A$. Its members will be called variables. A function θ from V into A will be called a valuation.

2.1 *Definition* We define the notions of $\mathcal{F}$-term and $\mathcal{F}$-proposition and their values at a valuation θ recursively as follows:

1. t is an $\mathcal{F}$-term iff

   a. t is a variable

   b. t is of the form $ft_1 \ldots t_n$, where f is in $\mathcal{F}^n$ and $t_1, \ldots, t_n$ are $\mathcal{F}$-terms

   c. or t is a conditional; i.e., t is of the form (u) IF P; (v), where u and v are $\mathcal{F}$-terms and P is an $\mathcal{F}$-proposition.

2. P is an $\mathcal{F}$-proposition iff P is an atom (i.e., of the form $u = v$ or of the form $u \neq v$ where u and v are $\mathcal{F}$-terms), or P is a conjunction, disjunction, or negation of $\mathcal{F}$-propositions.

We say that an $\mathcal{F}$-proposition has the value true or false at a given valuation under the usual circumstances provided that the terms of its maximal atoms (those not part of other atoms) are defined at that valuation. The value $t[\theta]$ of an $\mathcal{J}$-term at a valuation θ is

1. $\theta(x)$ if t is the variable x
2. $f(t_1[\theta], \ldots, t_n[\theta])$ if t is the term $ft_1 \ldots t_n$
3. $u[\theta]$ if P is true at θ; $v[\theta]$ if P is false at θ, in the case where t is of the form (u) IF P; (v). (In such a case, t is undefined at θ iff P is undefined, P is true and u is undefined, or P is false and v is undefined.)

$\mathcal{J}$-terms and $\mathcal{J}$-propositions are said to be simple iff they involve no subterms of the form (u) IF P; (v).

2.2 *Definition*

1. If $n > 0$ then we often write

$$f(t_1, \ldots, t_n)$$

instead of the term

$$ft_1 \ldots t_n,$$

and when $n = 2$ we sometimes write

$$ufv$$

for the term

$$fuv\ .$$

2. We will let

$$t_1 \text{ IF } P_1;\ t_2 \text{ IF } P_2;\ \ldots;\ t_n$$

represent the term

$$(t_1) \text{ IF } P_1;\ ((t_2) \text{ IF } P_2;\ (\ldots;\ (t_n) \ldots))$$

3. Also we sometimes write

$$u \text{ IF } P;\ \text{ELSE } v$$

or else

$$u \text{ IF } P;\ v \text{ OTHERWISE}$$

instead of

$$(u) \text{ IF } P;\ (v).$$

2.3 *Definition* Let $x_1, \ldots, x_n$ be variables and t be any $\mathcal{J}$-term. Suppose that t has the same value at θ and θ' whenever θ and θ' assign the same values to $x_1, \ldots, x_n$. Then

$$\lambda x_1, \ldots, x_n[t]$$

denotes that n-ary partial function on A whose value at $(a_1, \ldots, a_n)$ is $t[\theta]$ where θ is any valuation assigning the values $a_1, \ldots, a_n$ to $x_1, \ldots, x_n$, respectively. We assume that this definition includes the case where $n = 0$, so that $\lambda[t] = a$ iff $t[\theta] = a$ for every valuation θ.

2.4 *Definition* If u and v are $\mathcal{F}$-terms or $\mathcal{F}$-propositions, we say that u is equivalent to v (written $u \sim v$) iff u and v are defined at the same valuations and at such valuations they have the same value. It is easy to show that if $u \sim v$ then $\ldots u \ldots \sim \ldots v \ldots$, where $\ldots u \ldots$ is any $\mathcal{F}$-term or $\mathcal{F}$-proposition involving u, and $\ldots v \ldots$ is the term obtained from it by replacing u by v. If u and v are $\mathcal{F}$-terms then, clearly, $u \sim v$ iff

$$\lambda x_1, \ldots, x_n[u] = \lambda x_1, \ldots, x_n[v]$$

for all variables $x_1, \ldots, x_n$.

2.5 *Definition* An $\mathcal{F}$-term or $\mathcal{F}$-proposition is total iff it is defined at every valuation.

2.6 *Definition* A partial function f is a piecewise composite of $\mathcal{F}$ iff $f = \lambda x_1, \ldots, x_n[t]$ for some $\mathcal{F}$-term t. If t is a simple $\mathcal{F}$-term then we say that f is a composite of F.

The following is an adaptation of a well-known result from recursion theory.

2.7 Lemma *f is a composite of $\mathcal{F}$ iff f is a member of the smallest set $\mathcal{G}$ such that:*

(1) *$\mathcal{F}$ is a subset of $\mathcal{G}$.*
(2) *$\lambda x[a]$ is in $\mathcal{G}^1$ iff a is in $\mathcal{G}^0$.*
(3) *$\lambda x_1, \ldots, x_n[x_m]$ is in $\mathcal{G}$ for every m and n.*
(4) *$\lambda x_1, \ldots, x_n[g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))]$ is in $\mathcal{G}$ whenever g is in $\mathcal{G}^m$ and $f_1, \ldots, f_m$ are in $\mathcal{G}^n$.*

Proof: $\Leftarrow$: Visibly the composites of $\mathcal{F}$ satisfy (1) & (2) & (3) & (4) and hence each member of $\mathcal{G}$ is a composite of $\mathcal{F}$.

$\Rightarrow$: Let f be any composite of $\mathcal{F}$. Then $f = \lambda x_1, \ldots, x_n[t]$ for some variables $x_1, \ldots, x_n$ and some simple $\mathcal{F}$-term t. If t is a variable then f is in $\mathcal{G}$ by (3). Using induction we may suppose that t is a simple $\mathcal{F}$-term of the form $g(t_1, \ldots, t_m)$ where g is in $\mathcal{G}^m$ and $t_1, \ldots, t_m$ are simple $\mathcal{F}$-terms such that $\lambda x_1, \ldots, x_n[t_i]$ is in $\mathcal{G}$ for $i = 1, \ldots, m$. By letting f_i denote $\lambda x_1, \ldots, x_n[t_i]$ for $i = 1, \ldots, m$, we see that f is in $\mathcal{G}$ by (4) since

$$\begin{aligned} f &= \lambda x_1, \ldots, x_n[t] \\ &= \lambda x_1, \ldots, x_n[g(t_1, \ldots, t_m)] \\ &= \lambda x_1, \ldots, x_n[g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))]. \end{aligned}$$

Thus, by induction, every composite of $\mathcal{F}$ is in $\mathcal{G}$. QED

2.8 *Definition* e will denote $\lambda w, x, y, z[x \text{ IF } w = z; \text{ ELSE } y]$.

2.9 Lemma *Every $\mathcal{F}$-proposition is equivalent to an atom.*

Proof: Suppose that P is an $\mathcal{F}$-proposition such that all shorter $\mathcal{F}$-propositions are equivalent to atoms. If P is an atom or a negation then P is obviously equivalent to an atom. If P is a conjunction then we obtain an equivalent atom by replacing its conjuncts by atoms (using our induction hypothesis) and, then, applying one of the following general equivalences:

1. $s = t \; \& \; u = v \sim (u \text{ IF } s = t;\; s \text{ IF } u = u;\; s) = (v \text{ IF } s = t;\; t \text{ IF } v = v;\; t)$
2. $s \neq t \; \& \; Q \sim (s \text{ IF } Q;\; t) \neq (t \text{ IF } s = s;\; t)$.

If P is a disjunction it is equivalent to the negation of a conjunction by De Morgan's law. Thus, in all possible cases P is equivalent to an atom.
QED

2.10 *Definition* An $\mathcal{J}$-term in which every nonconditional subterm is simple and every subproposition is a simple equation is said to be normal.

Note: It is easy to show that the normal $\mathcal{J}$-terms are the smallest class including the simple $\mathcal{J}$-terms and such that the term $(s \text{ IF } t = u;\; v)$ is normal whenever t and u are simple and s and v are normal.

2.11 Theorem *Every $\mathcal{J}$-term can be normalized, i.e., is equivalent to a normal $\mathcal{J}$-term.*

Proof: Let t be any $\mathcal{J}$-term. By the previous lemma we may assume that every $\mathcal{J}$-proposition in t is atomic; and we may assume that these atoms are equations by replacing conditional subterms of the form $s \text{ IF } t \neq u;\; v$ by the equivalent term, $v \text{ IF } t = u;\; s$. If all of the proper subterms of t are normal then we normalize t by induction on the number of misplaced conditionals (those contained in subpropositions or in nonconditional subterms). In such a case, if t is not normal then it is of the form

$$\ldots (u \text{ IF } P;\; v) \ldots$$

where the term $(u \text{ IF } P;\; v)$ is a misplaced conditional of maximal length. But then t is equivalent to the term

$$(\ldots u \ldots) \text{ IF } P;\; (\ldots v \ldots).$$

By induction, we can normalize the subterms $\ldots u \ldots$ and $\ldots v \ldots$. The result can then be normalized by induction since the remaining misplaced conditionals are those in P, fewer by at least one than those in t. If the proper subterms of t are not normal, we can normalize them by induction on length and the procedure just given. QED

2.12 Corollary *Suppose that the members of $\mathcal{J}$ are total. Then the piecewise composites of $\mathcal{J}$ are exactly the composites of $\mathcal{J} \cup \{e\}$.*

Proof: It is clear that e is a piecewise composite of $\mathcal{J}$ and, hence, every composite of $\mathcal{J} \cup \{e\}$ is a piecewise composite of $\mathcal{J}$. Conversely, to obtain an equivalent simple $(\mathcal{J} \cup \{e\})$-term from an $\mathcal{J}$-term, normalize it and replace each conditional part, say $(s \text{ IF } t = u;\; v)$, by the equivalent simple $(\mathcal{J} \cup \{e\})$-term, $e(t, s, v, u)$. QED

Note: If the members of $\mathcal{J}$ are total then for every s, t, u, P, Q

$$(s \text{ IF } P;\; t) \text{ IF } Q;\; u \sim s \text{ IF } P \;\&\; Q;\; t \text{ IF } P;\; u.$$

2.13 Corollary *If the members of $\mathcal{J}$ are total, then every $\mathcal{J}$-term is equivalent to one of the form*

$$t_1 \text{ IF } P_1; \ldots; t_n \text{ IF } P_n; t_{n+1}$$

where $t_1, \ldots, t_{n+1}$ are simple and $P_1, \ldots, P_n$ are conjunctions of simple equations.

Proof: Let t be any $\mathcal{J}$-term. Normalize it. Then repeatedly replace subterms of the form $(s \text{ IF } P; t) \text{ IF } Q; u$ by the equivalent term $s \text{ IF } P \,\&\, Q; t \text{ IF } P; u$. This process terminates by induction on the number of such subterms and the final result is clearly of the required form. QED

3 *Pairing*

3.1 *Notation*

(1) For any partial functions f and g, fg will denote

$$\lambda x[f(g(x))].$$

(2) ι will denote $\lambda x[x]$.

(3) f^n will denote ι if $n = 0$; $f^{n-1}f$ if n is positive.

(4) p will denote a fixed pairing function on A, i.e., a total one to one function from $A \times A$ into A. $\langle x, y\rangle$ will denote $p(x, y)$.

(5) ρ and σ will denote partial functions such that for all a and b in A, $\rho(\langle a, b\rangle) = a$ and $\sigma(\langle a, b\rangle) = b$.

(6) $\langle x_1, \ldots, x_n\rangle$ will denote x_1 if $n = 1$; $\langle x_1, x_2\rangle$ if $n = 2$; $\langle\langle x_1, \ldots, x_{n-1}\rangle, x_n\rangle$ if $n > 2$. It is undefined if $n = 0$; except that, if f is a constant function, say $f = \lambda x[a]$, then $f(\langle x_1, \ldots, x_n\rangle)$ will be defined to have the value a even when $n = 0$.

(7) For any term t we let $\lambda\langle x_1, \ldots, x_n\rangle [t]$ denote $\lambda x[u]$ where x is any variable not occurring in t and where u is obtained from t by replacing each occurrence of x_1 by the term $\rho^n(x)$, and x_i by the term $\sigma\rho^{n-i}(x)$ for $i = 2, \ldots, n$. Thus we obtain the following equivalence:

$$\lambda\langle x_1, \ldots, x_n\rangle[t](\langle y_1, \ldots, y_n\rangle) \sim \lambda x_1, \ldots, x_n[t](y_1, \ldots, y_n).$$

(8) For every n-ary partial function g, we let $g\hat{}$ denote:

$$\lambda\langle x_1, \ldots, x_n\rangle[g(x_1, \ldots, x_n)]$$

so that we obtain the equivalence:

$$g\hat{}(\langle x_1, \ldots, x_n\rangle) \sim g(x_1, \ldots, x_n).$$

If a is in A, we let $a\hat{}$ denote $\lambda x[a]$.

(9) δ will denote $\lambda x[\langle x, x\rangle]$.

(10) c will denote $\lambda\langle x, y\rangle[\langle y, x\rangle]$.

(11) s will denote $\lambda\langle\langle x, y\rangle, z\rangle[\langle x, \langle y, z\rangle\rangle]$.

(12) $f \times g$ will denote $\lambda\langle x, y\rangle[\langle f(x), g(y)\rangle]$.

Note that $(f_1 \times g_1)(f_2 \times g_2) = (f_1 f_2) \times (g_1 g_2)$.

3.2 Theorem *The composites of $\mathcal{J} \cup \{\rho, \sigma, p\}$ are the partial functions on A of the form $\lambda x_1, \ldots, x_n[g_1 \ldots g_m(\langle x_1, \ldots, x_n\rangle)]$ where $g_1, \ldots, g_m$ are in $\{\rho, s, \delta, c\} \cup \{f^\wedge \times \iota: f \text{ is in } \mathcal{J}\}$.*

Proof: Let $\mathcal{G}$ be the set of all unary composites of $\{\rho, s, \delta, c\} \cup \{f^\wedge \times \iota: f \text{ is in } \mathcal{J}\}$. Note that $g \times \iota$ is in $\mathcal{G}$ whenever g is in $\mathcal{G}$ since:

1. $(g \times \iota) \times \iota = cscsc(g \times \iota)s$ is in $\mathcal{G}$ whenever $g \times \iota$ is in $\mathcal{G}$
2. $(fg) \times \iota = (f \times \iota)(g \times \iota)$ is in $\mathcal{G}$ whenever $f \times \iota$ and $g \times \iota$ are in $\mathcal{G}$
3. $\rho \times \iota = cpcscs$ is in $\mathcal{G}$
4. $\delta \times \iota = cscspcscs\delta$ is in $\mathcal{G}$
5. $c \times \iota = cspcscsscpcss(\delta \times \iota)$ is in $\mathcal{G}$
6. $s \times \iota = (((\rho\rho) \times \iota)c(((\rho c) \times \iota) \times \iota)\delta) \times \iota$ is in $\mathcal{G}$.

Thus, if f and g are in $\mathcal{G}$, so is $\lambda x[\langle f(x), g(x)\rangle] = (f \times \iota)c(g \times \iota)\delta$. And, in general, $\lambda x[\langle g_1(x), \ldots, g_m(x)\rangle]$ is in $\mathcal{G}$ whenever $g_1, \ldots, g_m$ are in $\mathcal{G}$. Let $\mathcal{G}'$ denote $\{g: g^\wedge \text{ is in } \mathcal{G}\}$ which is the set of all functions that can be put into the form given in the theorem. Clearly, $\mathcal{G}'$ is a family of composites of $\mathcal{J} \cup \{\rho, \sigma, p\}$ and includes $\mathcal{J} \cup \{\rho, \sigma, p\}$. Thus we need only show that $\mathcal{J}$ is closed under composition. This follows from Lemma 2.7 and the following facts:

a. a is in $(\mathcal{G}')^0$ iff $\lambda x[a]$ is in $(\mathcal{G}')^1$, since $a^\wedge = \lambda x[a]$ and $\mathcal{G} = (\mathcal{G}')^1$ and, by definition, a is in $(\mathcal{G}')^0$ iff $a^\wedge$ is in $\mathcal{G}$.

b. $\lambda x_1, \ldots, x_n[x_m]$ is in $\mathcal{G}'$ for all m and n since $x_m = \rho c \rho^{n-m}(\langle x_1, \ldots, x_n\rangle)$ if $m > 1$ and $x_1 = \rho^n(\langle x_1, \ldots, x_n\rangle)$.

c. If g is in $(\mathcal{G}')^m$ and $f_1, \ldots, f_m$ are in $(\mathcal{G}')^n$, then for some h in $\mathcal{G}$,

$$\begin{aligned}
&\lambda x_1, \ldots, x_n[g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))] \\
&= \lambda x_1, \ldots, x_n[g^\wedge(\langle f_1{}^\wedge(\langle x_1, \ldots, x_n\rangle), \ldots, f_m{}^\wedge(\langle x_1, \ldots, x_n\rangle)\rangle)] \\
&= \lambda x_1, \ldots, x_n[h(\langle x_1, \ldots, x_n\rangle)] \\
&= h^\wedge
\end{aligned}$$

since $\mathcal{G}$ is closed under unary composition and since whenever $g_1, \ldots, g_m$ are in $\mathcal{G}$ then so is $\lambda x[\langle g_1(x), \ldots, g_m(x)\rangle]$. QED

3.3 Corollary *Let ρ, σ, and the members of $\mathcal{J}$ be total. Then the piecewise composites of $\mathcal{J} \cup \{\rho, \sigma, p\}$ are the functions on A of the form:*

$$\lambda x_1, \ldots, x_n[e^{\wedge k} g_1 \ldots g_m(\langle x_1, \ldots, x_n\rangle)]$$

where $g_1, \ldots, g_m$ are in $\{\rho, s, \delta, c\} \cup \{f^\wedge \times \iota: f \text{ is in } \mathcal{J}\}$.

Proof: Since ρ, σ, p and the members of $\mathcal{J}$ are total, the piecewise composites of $\mathcal{J} \cup \{\rho, \sigma, p\}$ are the composites of $\mathcal{J} \cup \{\rho, \sigma, p\} \cup \{e\}$ which by the previous theorem are of the form:

$$\lambda x_1, \ldots, x_n[g_1 \ldots g_m(\langle x_1, \ldots, x_n\rangle)]$$

where $g_1, \ldots, g_m$ are in $\{\rho, s, \delta, c\} \cup \{f^\wedge \times \iota: f \text{ is in } \mathcal{I}\} \cup \{e^\wedge \times \iota\}$. To see that these can be put into the required form it suffices to note that:

a. $e^\wedge \times \iota = e^\wedge h$ where h denotes

$$\lambda\langle\langle v, w, x, y\rangle, z\rangle[\langle v, \langle w, z\rangle, \langle x, z\rangle, y\rangle]$$

which by the previous theorem is a simple composite of $\{\rho, s, \delta, c\}$.

b. $ge^\wedge = e^\wedge h$ where h denotes $\lambda\langle w, x, y, z\rangle[\langle w, g(x), g(y), z\rangle]$ which is a simple composite of $\{\rho, s, \delta, c, (g^\wedge \times \iota)\}$ and, hence, is a simple composite of $\{\rho, s, \delta, c\} \cup \{f^\wedge \times \iota: f \text{ is in } \mathcal{I}\}$ provided that g is such a composite. QED

4 *Indexing* In most theories of computability one can encode all computable procedures into members of $\mathcal{I}^0$ in such a way that $\lambda x, y[x * y]$ is computable (i.e., in $\mathcal{I}$), where $x * y$ denotes the result of applying to y the procedure encoded into x. In this section, we characterize and study those situations in which such encodings exist. Theorem 5.3 of the next section will show why this includes most theories of computability.

4.1 *Terminology* Henceforth, we let $*$ denote an arbitrary binary partial function on A.

(1) For any $\mathcal{I}$-terms $t_1, \ldots, t_n$, we let $t_1 * \ldots * t_n$ denote

$$(\ldots((t_1 * t_2) * \ldots * t_n) \ldots).$$

(2) A member a of A is a (uniform) $*$-index for f iff

$$f = \lambda x_1, \ldots, x_n[a * x_1 * \ldots * x_n]$$

(and $a * x_1 * \ldots * x_{n-1}$ is total).

(3) For any subset B of A, we let $B*$ denote the family of all partial functions on A having $*$-indices in B. Notice that b is the only $*$-index for the 0-ary function b and, hence, $(B*)^0 = B$. It follows that $B* = C*$ iff $B = C$.

(4) $*$ is a (uniform) indexing of $\mathcal{I}$ iff $(\mathcal{I}^0)*$ is closed under composition and equal to $\mathcal{I}$ (and every member of $\mathcal{I}$ has a uniform $*$-index in $\mathcal{I}^0$). Notice that $*$ is an indexing of $B*$ iff $B*$ is closed under composition.

Note: From the fact that $(a * x_1 * \ldots * x_n \text{ IF } P; b * x_1 * \ldots * x_n) \sim (a \text{ IF } P; b) * x_1 * \ldots * x_n$ it is easy to show that if $\mathcal{I}$ is indexable and contains e then $\mathcal{I}$ is closed under piecewise composition.

The following theorem is an adaptation of some pertinent results from combinatory logic and from Wagner [5].

4.2 Theorem

(I) $*$ *is an indexing of* $B*$ *iff*

(1) *B is closed under $*$, i.e., $a * b$ is in B whenever $a * b$ is defined and a and b are in B*

(2) *$\lambda x, y[x]$ has a $*$-index K in B*

(3) *$\lambda x, y, z[(x * z) * (y * z)]$ has a $*$-index S in B.*

(II) *In such a case the following are equivalent:*

(a) $*$ *is a uniform indexing of* $B*$

(b) $\lambda x, y, z\,[(x * z) * (y * z)]$ *has a uniform* $*$*-index in* B

(c) $\lambda x, y, z\,[x * y * z]$ *has a uniform* $*$*-index* J *in* B

(d) *For every simple* $B*$*-term* t *whose variables are in* $\{z, x_1, \ldots, x_n\}$ *the circular definition*

$$z * x_1 * \ldots * x_n \sim t$$

has a uniform solution in B *in the sense that there exists* b *in* B *such that* $b * x_1 * \ldots * x_{n-1}$ *is total and such that the above equivalence holds when all occurrences of* z *are replaced by* b.

Proof of (I):

$\Rightarrow$: The implication is obvious.

$\Leftarrow$: To prove the converse implication, we first show by induction on the length of t that for every variable x and every simple $(B \cup \{*\})$-term t, the term t is equivalent to a term $t' * x$ where x does not occur in t':

i. $t \sim S * K * K * x$ if t is the variable x

ii. $t \sim K * t * x$ if t is a constant or another variable

iii. $t \sim S * u' * v' * x$ if t is the term $u * v$.

It now follows, by a simple induction, that if the variables of t are among $x_1, \ldots, x_n$ then t is equivalent to a term $t'' * x_1 * \ldots * x_n$ where t'' is a simple $(B \cup \{S, K, *\})$-term containing no variables. Obviously, t'' is in B since S and K are in B and B is closed under $*$.

Proof of (II):

(d) $\Rightarrow$ (c): by taking t to be the term $x_1 * x_2 * x_3$ in (d).

(b) $\Rightarrow$ (a): by adding the hypothesis that t' is total in the proof of (I) above.

(c) $\Rightarrow$ (b): since any $*$-index for $\lambda x, y\,[J * (J * S * x) * y]$ is a uniform $*$-index for $\lambda x, y, z\,[(x * z) * (y * z)]$.

(a) $\Rightarrow$ (d): Let t be any simple $B*$-term whose variables are among $z, x_1, \ldots, x_n$. Choose a in B to be a uniform $*$-index for $\lambda z, x_1, \ldots, x_n[t']$ where t' is obtained from t by replacing all occurrences of z by $z * z$. Then $a * a$ is in B and is a uniform solution to the circular definition $z * x_1 * \ldots * x_n \sim t$. QED

As a consequence of the above theorem one can show that if $\mathcal{I}$ is a subset of $\mathcal{G}$ and $*$ is a (uniform) indexing of $\mathcal{I}$, then $*$ is a (uniform) indexing of $\mathcal{G}$ iff $\mathcal{G}$ is the closure of $\mathcal{G}^0 \cup \mathcal{I}$ under composition.

4.3 Theorem *Suppose that* $*$ *is a uniform indexing of* $B*$ *and that* $B*$ *includes* $\{\rho, \sigma, p\}$. *Let* $t_1, \ldots, t_m$ *be simple* $B*$*-terms whose variables are among* $x_1, \ldots, x_n, z_1, \ldots, z_m$. *Then the following system of circular definitions has a uniform solution:*

$$\begin{array}{c} z_1 * x_1 * \ldots * x_n \sim t_1 \\ \vdots \\ z_m * x_1 * \ldots * x_n \sim t_m \end{array}$$

in the sense that there exist $b_1, \ldots, b_m$ *in* B *such that* $b_i * x_1 * \ldots * x_{n-1}$ *is total and* $b_i * x_1 * \ldots * x_n \sim t_i$ *for* $i = 1, \ldots, m$.

Proof: For $j = 1, \ldots, m$ let c_j be a $*$-index for

$$\lambda y, x_1, \ldots, x_n[\sigma\rho^{m-j}(y * x_1 * \ldots * x_n)].$$

For $i = 1, \ldots, m$ let t_i' be obtained from t_i by replacing all occurrences of z_j by $c_j * z$ for $j = 1, \ldots, m$. Let a be a uniform solution to the circular definition

$$z * x_1 * \ldots * x_n \sim \langle z, t_1', \ldots, t_m'\rangle.$$

Then $c_i * a * x_1 * \ldots * x_n$ is equivalent to t_i with all occurrences of z_i replaced by $c_i * a$. Thus, it suffices to let $b_i = c_i * a$ for $i = 1, \ldots, m$. QED

4.4 Theorem *Suppose that* $\mathcal{F}$ *is closed under composition and includes* $\{\rho, \sigma, p\}$ *and that there exists a partial functions* $*$ *and* $\#$ *in* $\mathcal{F}^2$ *such that* $(\mathcal{F}^0)*$ *includes* $\{\lambda x, y[f(\langle x, y\rangle)]: f$ *is in* $\mathcal{F}^1\}$ *and*

$$\langle x, y\rangle \# z \sim x * \langle y, z\rangle.$$

Then $\#$ *is a uniform indexing of* $\mathcal{F}$.

Proof: It suffices to show, by induction on n, that if f is in $\mathcal{F}^{n+1}$ then there exists a total g in $\mathcal{F}^n$ such that

$$f = \lambda x_1, \ldots, x_{n+1}[g(x_1, \ldots, x_n) \# x_{n+1}].$$

We do this by letting $g = \lambda x_1, \ldots, x_n[g'(\langle x_1, x_2\rangle, x_3, \ldots, x_n)]$ where g' is chosen, by induction, such that g' is total and

$$g'(x_1, \ldots, x_{n-1}) \# x_n \sim f(\rho(x_1), \sigma(x_1), x_2, \ldots, x_n).$$

In case $n = 1$, we let $g = \lambda x[\langle a, x\rangle]$ where a is a $*$-index for $\lambda\langle x, y\rangle[f(\langle x, y\rangle)]$ and, hence, $g(x) \# y \sim \langle a, x\rangle \# y \sim a * \langle x, y\rangle \sim f(x, y)$. QED

4.5 Corollary *Suppose that* $\mathcal{F}$ *includes* $\{\rho, \sigma, p\}$ *and that* $*$ *is an indexing of* $\mathcal{F}$. *Then* $\lambda x, y[\rho(x) * \langle\sigma(x), y\rangle]$ *is a uniform indexing of* $\mathcal{F}$.

5 *Iteration*

5.1 *Definition* For any function f, f^∞ denotes $\lambda x[f^n(x)]$, where n denotes $\mu m[f^m(x) = f^{m+1}(x)]$. f^∞ is called the iterate of f. $\mathcal{F}$ is said to be closed under iteration iff f^∞ is in $\mathcal{F}$ whenever f is in $\mathcal{F}$. The closure of $\mathcal{F}$ under composition and iteration is called the iterative closure of $\mathcal{F}$ and its members are called iterative composites of $\mathcal{F}$.

5.2 Theorem *Suppose that* $\mathcal{F}$ *includes* $\{\rho, \sigma, p\}$ *and is closed under* *piecewise composition. Then the iterative composites of $\mathfrak{I}$ are closed under piecewise composition and are of the form:*

$$\lambda x_1, \ldots, x_n[\sigma h^{\infty}\gamma(\langle x_1, \ldots, x_n\rangle)]$$

where h is in $\mathfrak{I}^1$ and γ denotes $\lambda x[\langle x, x, x, x, x, x\rangle]$.

Proof: To show that the iterative composites are of the required form, we use the Theorem 3.2 and the following facts:

1. $f = \sigma h^{\infty}\gamma$ where h denotes

$$\begin{array}{lll} \lambda\langle u, v, w, x, y, z\rangle[& \langle u, v, w, x, y, f(y)\rangle & \text{IF } z \neq f(y) \\ & \langle u, v, w, x, y, z\rangle & \text{OTHERWISE }] \end{array}$$

2. $(\sigma f^{\infty}\gamma)(\sigma g^{\infty}\gamma) = \sigma h^{\infty}\gamma$ where h denotes

$$\begin{array}{lll} \lambda\langle u, v, w, x, y, z\rangle[& \langle u, \gamma(u), \gamma(u), x, y, z\rangle & \text{IF } v \neq \gamma(u) \\ & \langle u, v, g(w), x, y, z\rangle & \text{IF } w \neq g(w) \\ & \langle u, v, w, \gamma\sigma(w), \gamma\sigma(w), z\rangle & \text{IF } x \neq \gamma\sigma(w) \\ & \langle u, v, w, x, f(y), z\rangle & \text{IF } y \neq f(y) \\ & \langle u, v, w, x, y, \sigma(y)\rangle & \text{IF } z \neq \sigma(y) \\ & \langle u, v, w, x, y, z\rangle & \text{OTHERWISE }] \end{array}$$

3. $(\sigma f^{\infty}\gamma)^{\infty} = \sigma h^{\infty}\gamma$ where h denotes

$$\begin{array}{lll} \lambda\langle u, v, w, x, y, z\rangle[& \langle u, v, w, \gamma(w), \gamma(w), z\rangle & \text{IF } x \neq \gamma(w) \\ & \langle u, v, w, x, f(y), z\rangle & \text{IF } y \neq f(y) \\ & \langle u, v, w, x, y, \sigma(y)\rangle & \text{IF } z \neq \sigma(y) \\ & \langle u, v, z, x, y, z\rangle & \text{OTHERWISE }]. \end{array}$$

To show that the iterative composites are closed under piecewise composition, consider a conditional term t over the iterative closure of $\mathfrak{I}$. We may assume that it is of the form:

$$t_1 \text{ IF } t_2 = t_3;\ t_4$$

where $t_i \sim \sigma f_i^{\infty}\gamma(\langle x_1, \ldots, x_n\rangle)$ and f_i is in $\mathfrak{I}^1$ for $i = 1, 2, 3, 4$. Then t is equivalent to $\sigma h^{\infty}\gamma(\langle x_1, \ldots, x_n\rangle)$ where h denotes

$$\begin{array}{lll} \lambda\langle u, v, w, x, y, z\rangle\, [& \langle u, \gamma(u), \gamma(u), \gamma(u), \gamma(u), z\rangle & \text{IF } v \neq \gamma(u) \\ & \langle u, v, f_1(w), f_4(x), y, z\rangle & \text{IF } w \neq f_1(w) \text{ OR } x \neq f_4(x) \\ & \langle u, v, w, x, f_2(y), z\rangle & \text{IF } \sigma(w) = \sigma(x) \text{ AND } y \neq f_2(y) \\ & \langle u, v, w, x, f_3(y), z\rangle & \text{IF } \sigma(w) \neq \sigma(x) \text{ AND } y \neq f_3(y) \\ & \langle u, v, w, x, y, \sigma(y)\rangle & \text{IF } z \neq \sigma(y) \\ & \langle u, v, w, x, y, z\rangle & \text{OTHERWISE }]. \end{array}$$

QED

5.3 Theorem on indexability *There is a composite q of $\{e, \rho, \sigma, p\}$ such that for any subset B of A and any partial functions $f_1, \ldots, f_k$ on A*

$$\lambda x, y[\sigma\rho((\iota \times (f_1\hat{} \times \ldots \times f_k\hat{}))\,q)^{\infty}(\langle x, y\rangle)]$$

is a uniform indexing of the iterative closure $\mathfrak{I}$ of $B \cup \{e, \rho, \sigma, p\} \cup \{f_1, \ldots, f_k\}$ provided that $\mathfrak{I}^0$ contains more than one member and intersects the domain of $f_1\hat{} \times \ldots \times f_k\hat{}$.

Proof: Since $\mathcal{J}^0$ has more than one element and δ is in $\mathcal{J}$ then $\mathcal{J}^0$ has infinitely many members, e.g., $\langle a, b\rangle$, $\delta(\langle a, b\rangle)$, $\delta\delta(\langle a, b\rangle)$, . . . where a and b are distinct members of $\mathcal{J}^0$. By way of notation, we let ρ', c', s', δ', e', f', d, r, b, Δ, ∇, $\sigma(\nabla)$ be distinct members of $\mathcal{J}^0$. We let f denote $f_1{}^\wedge \times \ldots \times f_k{}^\wedge$ and let a be any member of $\mathcal{J}^0 \cap \text{Dom } f$. We let # denote $\lambda x, y[\sigma\rho((\iota \times f)q)^\infty(\langle x, y\rangle)]$ and let $*$ denote $\lambda x, y[\langle x, \rho(y)\rangle \# \sigma(y)]$ so that $\langle x, y\rangle \# z \sim x * \langle y, z\rangle$. We will construct q in such a way that

a. $\langle\langle\nabla, g', \Delta\rangle, \nabla\rangle * \langle x, y\rangle \sim g(\langle x, y\rangle)$ if g is in $\{\rho, c, s, \delta, e\}$

b. $\langle\langle\nabla, b, d, \Delta\rangle, \nabla\rangle * \langle x, y\rangle \sim b$ if b is in $\mathcal{J}^0$

c. $\langle\langle\nabla, \Delta, f', \Delta\rangle, \nabla\rangle * \langle x, y\rangle \sim \iota \times f(\langle x, y\rangle)$

d. $\langle\langle\nabla, r, a_1, \ldots, a_n, b, \Delta\rangle, \nabla\rangle * \langle x, y\rangle \sim g^\infty(\langle x, y\rangle)$ if
$\langle\langle\nabla, a_1, \ldots, a_n, \Delta\rangle, \nabla\rangle * \langle x, y\rangle \sim g(\langle x, y\rangle)$

e. $\langle\langle\nabla, a_1, \ldots, a_n, b_1, \ldots, b_m, \Delta\rangle, \nabla\rangle * \langle x, y\rangle \sim gh(\langle x, y\rangle)$ if
$\langle\langle\nabla, a_1, \ldots, a_n, \Delta\rangle, \nabla\rangle * \langle x, y\rangle \sim g(\langle x, y\rangle)$ and
$\langle\langle\nabla, b_1, \ldots, b_m, \Delta\rangle, \nabla\rangle * \langle x, y\rangle \sim h(\langle x, y\rangle)$.

Then # is a uniform indexing of $\mathcal{J}$ by Theorem 4.4, Theorem 3.2, and the fact that $\mathcal{J}$ is the closure of $B \cup \{f, e, \rho, \sigma, p\}$ under composition and iteration.

Specifically, we define q to be: (an explanation follows)

$$
\lambda\langle v, w, x, y\rangle\left[\begin{array}{ll}
\langle \rho(v), w, e^\wedge(x), a\rangle & \text{IF } \sigma(v) = e' \\
\langle \rho(v), w, \rho(x), a\rangle & \text{IF } \sigma(v) = \rho' \\
\langle \rho(v), w, s(x), a\rangle & \text{IF } \sigma(v) = s' \\
\langle \rho(v), w, \delta(x), a\rangle & \text{IF } \sigma(v) = \delta' \\
\langle \rho(v), w, c(x), a\rangle & \text{IF } \sigma(v) = c' \\
\langle \rho(v), w, \rho(x), \sigma(x)\rangle & \text{IF } \sigma(v) = f' \\
\langle \rho(v), w, \langle x, y\rangle, a\rangle & \text{IF } \sigma(v) = \Delta \\
\langle \rho\rho(v), w, \sigma\rho(v), a\rangle & \text{IF } \sigma(v) = d \\
\langle \rho(v), \langle w, \rho(v), x\rangle, x, a\rangle & \text{IF } \sigma(v) = b \\
\langle \rho(w), \langle \rho(w), x\rangle, x, a\rangle & \text{IF } \sigma(v) = r \text{ AND } x \neq \sigma(w) \\
\langle \rho(v), \rho\rho(v), x, a\rangle & \text{IF } \sigma(v) = r \text{ AND } x = \sigma(w) \\
\langle v, w, x, a\rangle & \text{OTHERWISE}
\end{array}\right],
$$

so that

$$
q(\langle\langle a_1, \ldots, a_n\rangle, \langle u, \langle b_1, \ldots, b_m\rangle, t\rangle, x, y\rangle) \sim
$$

$$
\begin{array}{ll}
\langle\langle a_1, \ldots, a_{n-1}\rangle, \langle u, \langle b_1, \ldots, b_m\rangle, t\rangle, g(x), a\rangle & \text{if } a_n = g' \text{ is in } \{e', \rho', s', \delta', c'\} \\
\langle\langle a_1, \ldots, a_{n-1}\rangle, \langle u, \langle b_1, \ldots, b_m\rangle, t\rangle, \rho(x), \sigma(x)\rangle & \text{if } a_n = f' \\
\langle\langle a_1, \ldots, a_{n-1}\rangle, \langle u, \langle b_1, \ldots, b_m\rangle, t\rangle, \langle x, y\rangle, a\rangle & \text{if } a_n = \Delta \\
\langle\langle a_1, \ldots, a_{n-2}\rangle, \langle u, \langle b_1, \ldots, b_m\rangle, t\rangle, a_{n-1}, a\rangle & \text{if } a_n = d \\
\langle\langle a_1, \ldots, a_{n-1}\rangle, \langle u, \langle b_1, \ldots, b_m\rangle, t, \langle a_1, \ldots, a_{n-1}\rangle, x\rangle, x, a\rangle & \text{if } a_n = b \\
\langle\langle b_1, \ldots, b_m\rangle, \langle u, \langle b_1, \ldots, b_m\rangle, x\rangle, x, a\rangle & \text{if } a_n = r \text{ and } x \neq t \\
\langle\langle a_1, \ldots, a_{n-1}\rangle, \langle u\rangle, x, a\rangle & \text{if } a_n = r \text{ and } x = t \\
\langle\langle a_1, \ldots, a_n\rangle, \langle u, \langle b_1, \ldots, b_m\rangle, t\rangle, x, a\rangle & \text{otherwise.}
\end{array}
$$

From the above definition it is obvious that q is a composite of $\{e, \rho, \sigma, p, \rho', c', s', \delta', e', f', d, b, r, \Delta, \nabla, \sigma(\nabla), a\}$. However, we have made our choices of $\rho', c', s', \delta', e', f', d, b, r, \Delta, \nabla, \sigma(\nabla), a$ fixed only to simplify the construction. We may consider them to be additional parameters of q in which case q is in the closure of $\{e, \rho, \sigma, p\}$ under composition.

In considering the construction of q, it helps to imagine a machine $\langle v, w, x, y\rangle$ consisting of an instruction stack v, a loop-control stack w, a main register x, and a special register y, and having $(\iota \times f)q$ as its next-state function. The stack u has $\sigma(u)$ as its top item, $\sigma\rho(u)$ as its second, etc.; the same holds for w. ∇ serves as a null value; and $\sigma(\nabla)$ serves as a halt instruction. $\rho', s', \delta', c', e'$ serve as instructions to perform the corresponding operation (function) on the content of the main register. d is an instruction to delete the next item (any constant) from the instruction stack and load it into the main register. b initializes for a loop by pushing the content of the instruction stack and then the main register onto the loop-control stack. r is a conditional-repeat instruction that tests whether the content of main register is the same at the end of the current iteration as at the end of the last (top item of loop-control stack). If so, it deletes the top two items from the loop-control stack. If not, it replaces the top item of the loop-control stack by the content of the main register and copies the second item of the loop-control stack (the content of the instruction stack upon entering the loop) into the instruction stack. Δ combines the contents of the main and special registers into a pair and loads it into the main register. Because the next-state function is $(\iota \times f)q$, f is applied to the special register on each step. f' loads the first and second components of the content of the main register into the main and special registers, respectively. Other instructions load the value a into the special register so that $(\iota \times f)q(\langle v, w, x, y\rangle)$ will be defined. QED

6 *Axiomatic characterization* Wagner [5] has shown that $*$ is a uniform indexing of $A*$ and $A*$ contains e iff $(A, *)$ satisfies the following first-order axiom:

There exist S and E such that

(1) $S \neq E$

(2) and for all x, y, and z there exists w such that $S * x * y = w$ and for all v, $w * x = v$ iff $v = (x * z) * (y * z)$ (i.e., S is a uniform $*$-index for $\lambda x, y, z[(x * z) * (y * z)]$)

(3) and for all w, x, y, and z, $E * w * x * y * z = x$ if $w = z$; and $E * w * x * y * z = y$ if $w \neq z$ (i.e., E is a $*$-index for e).

This follows, as in Strong [3], from the characterization of uniform indexings by noting that in such a case $\lambda x, y[x]$ has a $*$-index, namely $K = S * (E * E) * I$ where I denotes $S * (E * S * E) * (S * S * S * S)$ which is a $*$-index for ι. In such a case, $(A, *)$ is said to be a uniformly reflexive structure. For the sake of generality and notational convenience, we extend this notion to include families $\mathfrak{J}$ where $\mathfrak{J}^0$ is a proper subset of A, by means of the following definition.

6.1 *Definition* $\mathcal{J}$ is a uniformly reflexive structure (URS) iff $\mathcal{J}$ is uniformly indexable and contains e.

Wagner [5] has shown that every URS includes a pairing system, e.g., $\{\lambda x, y[E * a * x * y], \lambda x[x * a], \lambda x[x * b]\}$ where a and b are distinct members of A and E is a $*$-index for e.

6.2 *Definition* If g is in $\mathcal{J}^1$ and a is in $\mathcal{J}^0$ and $T = \{a, g(a), gg(a), \ldots\}$ is infinite then T is said to be an $\mathcal{J}$-splinter.

For example, if $\mathcal{J}$ contains δ and a constant $a = \langle b, c\rangle$ where $b \neq c$ then $\{a, \delta(a), \delta\delta(a), \ldots\}$ is an $\mathcal{J}$-splinter.

6.3 *Definition* A set B included in A is said to be semicomputable in $\mathcal{J}$ iff it is the domain of a member of $\mathcal{J}^1$.

6.4 Theorem *Let $T = \{a, g(a), gg(a), \ldots\}$ be any $\mathcal{J}$-splinter. Then the following are equivalent:*

(1) *There is a pairing system $\{\rho, \sigma, p\}$ and a partial function h on A whose domain intersects $\mathcal{J}^0$ such that $\mathcal{J}$ is the iterative closure of $\mathcal{J}^0 \cup \{h, e, \rho, \sigma, p\}$.*

(2) *$\mathcal{J}$ is a URS closed under iteration.*

(3) *$\mathcal{J}$ is a URS closed under minimization in the sense that if f is in $\mathcal{J}^1$ then $\mathcal{J}^1$ contains*

$$\lambda x[g^n(a) \text{ where } n = \mu m[f(g^m(a), x) = a]].$$

(4) *$\mathcal{J}$ is a URS and T is semicomputable.*

(5) *There is a pairing system $\{\rho, \sigma, p\}$ and a partial function h on A whose domain intersects $\mathcal{J}^0$ such that $\mathcal{J}$ is the closure of $\mathcal{J}^0 \cup \{h, e, \rho, \sigma, p\}$ under composition and* DO-UNTIL *constructions, in the sense that if P is an $\mathcal{J}$-proposition and f is in $\mathcal{J}^1$ then $\mathcal{J}^1$ also contains the partial function* DO f UNTIL P *which is defined to be:* $\cap\{h: h = \lambda x[x \text{ IF } P; hf(x) \text{ OTHERWISE}]\}$.

Proof:

(1) $\Rightarrow$ (2): Note that $\mathcal{J}^0$ includes T which is infinite. Hence, (2) follows from (1) by the theorem on indexability.

(2) $\Rightarrow$ (3): Let f be any member of $\mathcal{J}^1$. Let h denote $\lambda\langle w, x\rangle[\langle w, x\rangle \text{ IF } f(w, x) = a; \langle g(w), x\rangle \text{ OTHERWISE}]$. Let h' denote $\lambda y[\langle a, y\rangle]$. Then $h^\infty h'(y) = g^n(a)$ where $n = \mu m[f(g^m(a), y) = a]$.

(3) $\Rightarrow$ (4): $\lambda x[g^n(a)$ *where* $n = \mu m[e(g^m(a), a, x, x) = a]]$ is the identity function on T and is clearly in $\mathcal{J}$ if (3) holds.

(4) $\Rightarrow$ (5): Suppose that (4) holds and that $T = \text{Dom } h$ where h is in $\mathcal{J}$. Then it is obvious that $\mathcal{J}$ is generated under composition alone by $\mathcal{J}^0 \cup \{*\}$ where $*$ is any indexing of $\mathcal{J}$. So it remains only to show that $\mathcal{J}$ is closed under

DO-UNTIL constructions. We do so by noting that for every f in $\mathcal{F}^1$, DO f UNTIL P is equal to

$$\lambda x [\sigma f'(\langle a, x\rangle)]$$

where f' is defined circularly as follows:

$$f'(\langle w, x\rangle) \sim \langle w, x\rangle \text{ IF } (h(w) = h(w)) \ \&\ P; \text{ ELSE } f'(\langle g(w), f(x)\rangle).$$

(5) $\Rightarrow$ (1): Suppose that (5) holds and that f is in $\mathcal{F}^1$. Then f^∞ is in $\{h\colon h = \lambda x [x \text{ IF } x = f(x); hf(x) \text{ OTHERWISE}]\}$. To show that it is the smallest member of this set, note that for any such h, if $f^n(x) = f^{n+1}(x)$ then $h(x) = f^n(x)$ by induction on n. QED

Friedman has shown that there exists a URS having no semicomputable splinter (see Strong [4]). The following theorem extends his result by showing that any first-order extension of the axioms for URS's will have a model containing no semicomputable splinter.

6.5 Theorem *Uniformly reflexive structures are finitely axiomatizable while URS's containing a semicomputable splinter are not determined by any set of first-order axioms.*

Proof: Let B be any subset of A and let $*$ be a binary partial function on A. By our characterization of uniform indexings, it is clear that $B*$ is a URS iff $(A, B, *)$ satisfies the following first-order statements:

1. There exist x and y such that $x \neq y$.
2. There exists S in B such that for all x, y, and z there exists w such that $S * x * y = w$ and for all v, $v = w * z$ iff $v = (x * z) * (y * z)$.
3. There exists E in B such that for all w, x, y, and z, either $E * w * x * y * z = x$ and $w = z$ or $E * w * x * y * z = y$ and $w \neq z$.
4. For all x, y, and z, if x and y are in B and $x * y = z$ then z is in B.

Let Γ be any set of first-order statements including the four above and having a model M. Let $(A, B, *)$ be a δ-incomplete ultrapower of M. Then $(A, B, *)$ is a model for Γ and is $\aleph_0$-saturated, in the sense that if $\{\phi_1(x), \phi_2(x), \ldots\}$ is a finitely satisfiable family of formulas in the language of Γ then it is simultaneously satisfiable; i.e., there exists a in A such that $(A, B, *)$ satisfies $\phi_i(a)$ for $i = 1, 2, \ldots$. But if $T = \{b * d, b * b * d, \ldots\}$ is a $B*$-splinter included in the domain of some member of $B*^1$, say $\lambda x[c * x]$, then the set of all formulas of the form

$$x \neq b * \ldots * b * d \ \&\ (\text{for some } y,\ y = c * x)$$

is countable and finitely satisfiable. Thus, there exists a in the domain of $\lambda x[c * x]$ such that a is not in $\{b * d, b * b * d, \ldots\}$ and hence T is not the domain of $\lambda x[c * x]$. QED

REFERENCES

[1] Friedman, H., "Axiomatic recursive function theory," in *Logic Colloquium '69*, ed. by R. O. Gandy and C. M. E. Yates, North-Holland Publishing Co., Amsterdam, 1971, pp. 113-137.

[2] Friedman, H., "Algorithmic procedures, generalized Turing algorithms and elementary recursion theory," in *Logic Colloquium '69*, ed. by R. O. Gandy and C. M. E. Yates, North-Holland Publishing Co., Amsterdam, 1971, pp. 361-389.

[3] Strong, H. R., "Algebraically generalized recursive function theory," *IBM Journal of Research and Development*, vol. 12 (1968), pp. 465-475.

[4] Strong, H. R., "Models for algebraically generalized recursive function theory," *The Journal of Symbolic Logic*, vol. 24 (1970), pp. 401-409.

[5] Wagner, E. G., "Uniformly reflexive structures: On the nature of Gödelization and relative computability," *Transactions of the American Mathematical Society*, vol. 144 (1969), pp. 1-41.

University of California, Riverside
Riverside, California