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SEMANTICAL CONSIDERATIONS ON MODAL COUNTERFACTUAL LOGIC WITH COROLLARIES ON DECIDABILITY, COMPLETENESS, AND CONSISTENCY QUESTIONS

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Introduction The title of this paper is almost identical (morphologically) to the title of the famous paper of Kripke [8]. The difference lies in the insertion of the word 'counterfactual' into the title of this paper. Indeed, I am going to argue that this insertion causes all the modifications in the classical modal semantics of Kripke in order to account for the semantic properties of our modal counterfactual logic (MCL). It will be argued that neither nonmodal counterfactual formulas nor modal counterfactual formulas can be given a Kripke-type semantics (K-type semantics).

A nonmodal counterfactual formula (a modal-free counterfactual formula) is a formula which consists of two atomic or molecular sentences with no modal operators such that they are connected by the two-place sentential connective of the counterfactual conditional like $A \mathrel{\Box\!\!\rightarrow} B$ or $A \mathrel{\Diamond\!\!\rightarrow} B$. A modal counterfactual is a nonmodal counterfactual formula in the scope of the classical modal operator of necessity and possibility like $\Box(A \mathrel{\Box\!\!\rightarrow} B)$ or $\Diamond(A \mathrel{\Box\!\!\rightarrow} B)$. It will be shown that in order to treat adequately nonmodal counterfactual formulas one needs a more general model theory than the K-semantics. Such a model theory appears to be *Neighborhood Semantics* (*NS*). In order to account for modal counterfactual formulas one needs an extension of *NS*. Indeed neither K-semantics nor classic *NS* (two-valued *NS*) can serve as a model theory (in the Tarskian sense) for modal counterfactuals. It is only a continuously valued *NS* which can serve as a theory of truth and satisfaction for modal counterfactuals.

1 *Semantics for nonmodal counterfactuals* We shall use Lewis's semantic analysis of counterfactuals as a representative work.[1] Lewis's basic idea is that the antecedent of the counterfactual transforms us from the actual world to a world which is maximally similar to our world except that in it the antecedent holds. Then the counterfactual holds iff the consequent holds in this world.

Consider an assignment \$ to each possible world i of a set $\$_i$ of sets of possible worlds. Then for each i the following holds:

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(A) $\$_i$ is centered on i; that is, the set $\{i\}$, having as its only member i, belongs to $\$_i$.

(B) $\$_i$ is nested; that is, whenever S and T belong to $\$_i$ either S is included in T or T is included in S.

(C) $\$_i$ is closed under unions; that is, whenever G is a subset of $\$_i$ and $\cup G$ is the set of all worlds j, such that j belongs to some member of G, $\cup G$ belongs to $\$_i$.

(D) $\$_i$ is closed under nonempty intersections; whenever G is a nonempty subset of $\$_i$ and $\cap G$ is the set of all worlds j such that j belongs to every member of G, $\cap G$ belongs to $\$_i$.

Given that (A)-(D) are the minimal artifacts of our theory, we set the following truth conditions: $A \mathrel{\Box\!\!\rightarrow} B$ is true at i iff (i) no A-world belongs to any sphere S in $\$_i$ (vacuous case); (ii) some sphere S in $\$_i$ contains at least one A-world and $A \supset B$ holds at every world in S. As for the counterfactual '$A \mathrel{\Diamond\!\!\rightarrow} B$', its conditions are equivalent to those of '$\neg(A \mathrel{\Box\!\!\rightarrow} \neg B)$'.

Conjecture 1: Standard K-semantics is not an adequate nonmodal counterfactual semantics.

The set of possible worlds used by Lewis and the accessibility relation on them is non-Kripkean. In fact the cardinality of the set used by Lewis to evaluate his counterfactuals is much smaller than the cardinality of the set used by Kripke in his normal model structure. More crucial is the different use of the R relation (the accessibility relation). Lewis considered worlds which are accessible in a *certain way* to i. Kripke put no restriction on the relation R except for properties like reflexivity, transitivity, and symmetry which characterize semantically different modal logics. However, Kripke cannot consider only similar worlds to i if he does not modify the nonrestrictive nature of his accessibility relation.

Two indirect 'proofs' were given to the incompatibility of Lewis-type semantics with K-semantics. The first was given by an author who was aware of Lewis semantics for counterfactuals. Segerberg [16] showed that counterfactuals semantics can be given in terms of neighborhood semantics. Since *NS* is surely not equivalent with K-semantics one is tempted to conclude that K-semantics is not an adequate model theory for nonmodal counterfactuals. The second indirect 'proof' was given by Montague (1968, 1970) who was not aware and did not intend to refer to Lewis's work. Montague [12] claimed that conditional necessity cannot be treated in terms of K-semantics of relevance relations between two possible worlds. This fits Lewis's formulation of the counterfactual in terms of a variably strict conditional. Montague [11] provided a class of accessibility relations which do not belong to the classical relation. All these nonclassical relations are special because they consider worlds which are accessible *in a certain way* to the base world. One of these nonclassical relations is an accessibility relation between worlds which satisfied a minimal similarity to a degree d. Montague was surely unaware of the work Lewis was going to do in the early seventies and Lewis did not mention Montague's analysis. However, when one compares Lewis's informal discussion with Montague formalization of the properties of this nonclassical accessibility relation one finds an interesting similarity.

Montague's analysis runs as follows: Consider the class M of all sets N such that for some set I of possible worlds: (i) N is a nonempty set of reflexive and symmetric relations having I as their field; (ii) $R \cap S$ is in N whenever R,S are in N; and (iii) for each R in N, there exists an S in N, such that for all $i,j,k \in I$, if iSj and jSk then iRk. If N is in M we regard each member R of N as a relation between worlds of similarity in certain features and to a certain degree. R will not, in general, be transitive. S of which (iii) asserts the existence can be understood as the relation of similarity in the features involved in R but to twice the degree involved in R. This Montaguean characterization fits Lewis's theory (see Lewis [9], 3.2 and 4.2).[2]

Since this Montaguean characterization is given in the general context of accessibility relations which do not enter K-semantics but only NS, it seems that the conclusion concerning the incompatibility of Lewis's analysis with K-semantics follows immediately. However, neither Montague nor Segerberg gave precise truth and satisfaction conditions for counterfactuals in terms of NS. To use the intuitionistic jargon, they gave an existential proof concerning the impossibility of giving a semantics for counterfactuals in terms of K-semantics. But they did not show how effectively we can construct an NS for counterfactuals.

1.1 *NS for nonmodal counterfactuals* In this section I try to provide counterfactuals with NS truth conditions. Recall the basic idea of NS: Let N be a one-place sentential operator (one can generalize to many place operators). N is given as its value a function F_N such that for any $i \in I$, $F_N(i)$ is a set of subsets of I. Since N is not truth functional, the truth value of A is not sufficient if one wants to compute the value of $N(A)$ at i. Hence one must check A's distribution along I. Each distribution of truth values can be represented as a subset H of I. H contains only members of I in which A is true and thus A holds at j iff $j \in H$. Hence $N(A)$ is true at i iff A's distribution (viz H) is one of the sets accepted by $F_N(i)$; i.e., $N(A)$ is true at i iff $H \in F_N(i)$. (The sets in $F_N(i)$ were named by Scott 'the neighborhoods of i'.)[3]

Now let me turn back to Lewis's counterfactuals. In terms of NS his counterfactual can be given the following truth conditions: $A \,\Box\!\!\rightarrow B$ is true at i iff $H \in F_{CF}(i)$ ($CF = \Box\!\!\rightarrow$ counterfactual). Now, $H \in F_{CF}(i)$ iff $S \in H$, when S is the smallest A-permitting sphere around i. Then H is equivalent to the set of all worlds in which '$A \supset B$' holds. Thus $H \in F_{CF}(i)$ iff the set of all worlds in which '$A \supset B$' coincides with the smallest A-permitting sphere.[4] Lewis said that $A \,\Box\!\!\rightarrow B$ is true at i iff some AB-world is closer to i than any $A \neg B$-world. In NS this amounts to the assertion that AB holds at a neighborhood of i whose most distant member is distant to $\leqslant n$ (or similar to $\geqslant n$), while $A \neg B$ holds at a neighborhood of i whose most distant member is distant to $>n$ (or similar to $<n$). Similar formulations can be given in topological terms using the idea of a topological space around i.

2 *Modal counterfactuals* So far we have not compared one counterfactual with another, but have restricted ourselves to the assignment of truth conditions to nonmodal counterfactuals. In this section we consider a possible method of comparing the degree of possibility and necessity of counterfactuals. Since the use of the comparative presupposes that the arguments of the binary predicate satisfy the noncomparative use of the predicate to a degree, it will follow that counterfactuals may be necessary (possible) to a degree. Thus, if A is more necessary than B, A's degree of necessity is higher than B's.

2.1 *Lewis's comparative connectives* In order to define the notion of 'more closer than' or 'more similar than' Lewis needed a comparative connective. He introduced a family of comparative possibility connectives ([9], pp. 52-56):

a. $A \leqslant B$ means that A is at least as possible as B.

b. $A < B$ means that A is more possible than B.

c. $A \approx B$ means that A and B are equally possible.

Lewis used these connectives in order to show that $A \mathrel{\Box\!\!\rightarrow} B$ means that $\Diamond A$ & $(AB < A \neg B)$. Thus the connective is used to connect atomic sentences. However, interesting results arise when nonatomic expressions are connected by this connective. Let us have formulas of the following form:

$(A \mathrel{\Box\!\!\rightarrow} B) < (A \mathrel{\Box\!\!\rightarrow} C)$
$(A \mathrel{\Diamond\!\!\rightarrow} \neg C) < (A \mathrel{\Diamond\!\!\rightarrow} \neg B)$.

Thus, counterfactuals are themselves compared. Now, given Conjectures 2 and 3 it follows that counterfactuals may be necessary to a degree. Given Conjecture 2 alone, it follows that counterfactuals may be possible to a degree:

Conjecture 2: Given 'A is F-er than B' when F-er is an operation which forms a binary predicate out of an adjective, A and B may satisfy F to a degree.

Generalization: Two compared arguments in a comparative may satisfy to a degree the predicate from which the comparative was derived.

Now back to our specific comparative, that of comparative possibility. We can conjecture that:

Conjecture 3: If the degree of possibility of two arguments may be compared, their degree of necessity may be compared too.

Conjecture 3 rests on the modal equivalence $\Box P \equiv \neg \Diamond \neg P$. Thus if $\ll$ stands for comparative necessity ($A \ll B$ means that A is more necessary than B)

it follows from $\neg A < \neg B$ that $B \ll A$. Indeed it is a biconditional (hence, an equivalence). Thus, given that it makes perfect sense to say that '$(A \boxright B) < (A \boxright C)$' it makes sense also to say that '$(A \boxright \neg C) \ll (A \boxright \neg B)$'.

Now, bearing in mind Lewis's formulation of '$A \boxright B$' in terms of the truth of AB in a closest world than any $A \neg B$-world, we may give similar truth conditions to sentences with the form of $(A \boxright B) \boxright (A \boxright C)$ by equating it with:

$$\Diamond(A \boxright B) \;\&\; ((A \boxright B)\cdot(A \boxright C) < ((A \boxright B)\cdot(A \boxright \neg C))).$$

However this is only a corollary of the use of the comparative connective between counterfactuals. A much more important result follows from Conjecture 2 (and the generalization of it):

Conjecture 2 for MCL: Given that '$(A \boxright B) < (A \boxright C)$' and '$(A \boxright B) \ll (A \boxright C)$' are accepted by our system, it follows that our system contains the following sentences which are true to a degree: '$\Diamond(A \boxright B)$', '$\Diamond(A \boxright C)$', '$\Box(A \boxright B)$', and '$\Box(A \boxright C)$'. Indeed given theorem A it follows that '$\Box(A \boxright B)$ is truer than $\Box(A \boxright C)$' is equivalent to '$(A \boxright B) \ll (A \boxright C)$'.

Thus the classical Kripkean modal operators may have formulas in their scope such that the entire modal sentence is true to a degree iff comparative necessity and possibility are allowed.

3 *Continuously-valued (fuzzy) model theory for modal counterfactuals* Thus it seems that our modal counterfactuals require a nonclassical semantics in the sense of being not two-valued. This result is a direct consequence of our use of the concepts of comparative possibility and necessity. Generally speaking, when two arguments can satisfy a comparative it follows that one can scale these arguments on a scale. Alternatively, if a predicate F can be transformed into a binary predicate of the form 'F-er than', it follows that F may be satisfied to a degree. In our case, our remarks apply to 'necessary' and 'possible' and thus counterfactuals in the scope of these operators may be true to a degree. Thus our semantics must be based on a multivalued model theory. In such a theory, the cardinality of the set of truth values is >2. Roughly speaking, two types of many-valued semantics can be distinguished in the context of truth theories for modal sentences:

1. According to the first conception, the set of truth values consists of a finite number of valuations. Normally the cardinality of this set varies between three to four (consult Łukasiewicz's, Bochvar's, Kleene's, and Herzberger's new four-valued logic). The 'ideology' of this approach is to allow a finite set of truth values and hence a finite set of degrees of truth.

2. According to the second approach, the set of truth values is the real interval $[0,1]$. Thus each real number in this interval stands for a degree of truth. The first logic which used such a truth values set was not purely

deductive. It was Reichenbach's probability logic in which each probability assignment was taken as a truth value. Thus if $pr(P) = i$ the truth value of P is i. The modern logics in which truth values range over the real interval $[0,1]$ are all tokens of a general *fuzzy logic*. The logical and set theoretical foundations of this logic were set by Zadeh in 1965. In this logic, propositions may be true to any degree in the interval $[0,1]$. Accordingly, elements may belong to a degree to a set. In this way we get the concept of *fuzzy set* which is characterized by a corresponding membership function to the set. Thus if 'x is tall' is true to i it means that x is a member of the set of tall men to the degree i.

Back to our modal counterfactuals, we note that they may be true to a degree. Hence we need a semantics in which the truth set (set of true propositions) is a fuzzy set, at least when modal counterfactuals belong to the set of propositions.

One possible truth theory might have been a fuzzified version (or many-valued version) of Kripke-type semantics. However, Kripke semantics was shown to be inadequate for modal counterfactuals (not only because of its two-valued nature) and thus it cannot serve us. A much more plausible candidate is a fuzzified *NS*. *NS* in its classical form was an adequate semantics for nonmodal counterfactuals. Since the main problem introduced by modal counterfactuals is the possibility of their being true to a degree, a graded *NS* seems to be a plausible semantics.

3.1 *Fuzzy NS* Recall that in classical *NS*, given an operator N and a world i a proposition Np is true at i iff $H \in F_N(i)$, when H is the set of all worlds in which p is true. Now if we let $i,j,k \ldots$ range over worlds, and $\alpha,\beta,\gamma \ldots$ over degrees of truth, we want to say that if our operator is either $\Box$ or $\Diamond$ and our formula in their scope is a nonmodal counterfactual then we may have the case that '$\Box(A \Box\!\!\rightarrow B)$' is true to α at i. In that case we should change the binary (two-valued) truth conditions mentioned above.

Let me advance the following suggestion: Normally we would expect that $\Box(A \Box\!\!\rightarrow B)_i = \alpha$ would be interpreted in the following way: $\Box(A \Box\!\!\rightarrow B)_i$ is true to α iff $A \supset B$ is true in all the alternatives to the closest world to i. Such truth conditions have no intuitive foundations. They just impose a Kripke-style interpretation of necessity which has no significance in our case. Instead I would suggest a more intuitive approach. My basic idea is that if $A \Box\!\!\rightarrow B$ is more necessary than $A \Box\!\!\rightarrow C$ then something in these counterfactuals makes the possibility of $A \Box\!\!\rightarrow \neg B$ much more farfetched than the possibility of $A \Box\!\!\rightarrow \neg C$. What can be the reasons for such a difference? The reason seems to be the type of material implication which is $A \supset B$ and $A \supset C$. If $A \supset B$ is a logically true conditional while $A \supset C$ is a physical truth then the possibility of $A \Box\!\!\rightarrow \neg B$ is lower than the possibility of $A \Box\!\!\rightarrow \neg C$. Accordingly, the necessity of $A \Box\!\!\rightarrow B$ is higher than that of $A \Box\!\!\rightarrow C$. Two special cases exist: when $A \supset B$ is an analytic truth then $\Diamond(A \Box\!\!\rightarrow \neg B)$ is 0 and hence $\Box(A \Box\!\!\rightarrow B)$ is 1. The other case is with a conditional $A \supset B$ where B contradicts A. Then $\Box\neg\Box\neg(A \supset \neg B)$ and thus $\neg\neg\Diamond(A \Box\!\!\rightarrow \neg B)$, and thus $\Box(A \Box\!\!\rightarrow B)$ is false to the degree 1. This explains the intuitive saying that it is always the case that $A \mathrel{\Box\!\!\rightarrow} A$ and never $A \mathrel{\Box\!\!\rightarrow} \neg A$.

Thus the more the connection between A and B in $A \mathrel{\Box\!\!\rightarrow} B$ is stronger the more $\Box(A \mathrel{\Box\!\!\rightarrow} B)$ is true. Note that if $A \supset A$ is a logical truth, $A \supset B$ a physical truth (i.e., $A \supset \neg B$ is a physically impossible world), and $A \supset C$ is an historical truth, then the set of worlds in which $A \mathrel{\Box\!\!\rightarrow} A$ holds is larger than the set in which $A \mathrel{\Box\!\!\rightarrow} B$ holds and $A \mathrel{\Box\!\!\rightarrow} C$ holds in the smaller set from all three. Recall that in *NS* we used H for the set of worlds in which the proposition in question holds. Hence, if H_1 stands for the set in which $A \mathrel{\Box\!\!\rightarrow} A$ holds, H_2 for the set in which $A \mathrel{\Box\!\!\rightarrow} B$ holds, and H_3 for the set in which $A \mathrel{\Box\!\!\rightarrow} C$ holds, then $H_1 > H_2 > H_3$. Thus, we no more say just that $H \in F_{CF}(i)$, we are now interested in the cardinality of H. Given that the cardinality of the set denoted by $F_{CF}(i)$ is n and that for the proposition $\Box(A \mathrel{\Box\!\!\rightarrow} B)$ H has the cardinality of m, then the degree of truth of $\Box(A \mathrel{\Box\!\!\rightarrow} B)$ at i is α when α satisfies the equation $\frac{\alpha}{1} = \frac{m}{n}$. In the case $m = n$, $\Box(A \mathrel{\Box\!\!\rightarrow} B)$ will be true to the degree 1. This is the case when H, the truth set of $A \mathrel{\Box\!\!\rightarrow} B$, is the set of all worlds and hence $A \mathrel{\Box\!\!\rightarrow} B$ is completely necessary. Note that when $m = 0$, the truth value of $\Box(A \mathrel{\Box\!\!\rightarrow} B)$ is 0.

Thus, if one recognizes that different true material implication may be necessary to different degrees[5] he must accept one of the two following alternatives:

1. if $A \supset A$, $A \supset B$, and $A \supset C$ are necessary to different degrees then $\Box(A \mathrel{\Box\!\!\rightarrow} A)$, $\Box(A \mathrel{\Box\!\!\rightarrow} B)$, $\Box(A \mathrel{\Box\!\!\rightarrow} C)$ we use different necessity operators. The strength of the operator is a function of the type of sentence in its scope. For example, in $\Box(A \mathrel{\Box\!\!\rightarrow} A)$ the necessity operator is a logical necessity, while if $A \supset \neg B$ describes a physically impossible world, $\Box(A \mathrel{\Box\!\!\rightarrow} B)$ uses an operator for physical necessity.

In such a theory there is no need for degrees of truth and satisfaction as long as one does not compare two different types of necessity. Thus as long as the operators are different, both $\Box(A \mathrel{\Box\!\!\rightarrow} A)$ and $\Box(A \mathrel{\Box\!\!\rightarrow} B)$ are true to the degree 1 because their strength is not compared. (It may be argued that even when there is no comparison between two operators, the need for degrees of truth arises because formulas may be even physical necessities to different degrees. In the epoch of quantum mechanics when laws are interpreted probabilistically, it may be the case that two 'laws' with different probabilities may be said to be necessities to a different degree. Such a theory claims that degrees of truth for modal sentences are needed not only when one compares two different necessity operators, but even within the domain of a single operator. I ignore, intentionally, this claim.)

2. The second approach does not use different operators of necessity. The theory has one operator but it may be satisfied to a degree.

Approaches 1 and 2 may be proved equivalent. One can set a translation function from one theory to the other. One can have a recursive function predicting the degree of truth value of a formula according to (2) given the operator used according to (1). Thus, if L stands for logical necessity, P for physical necessity, and H for historical necessity, and given that $L > P > H$ (in the sense that L quantifies over the largest set of worlds), our translation predicts that if Lq is true and Pr is true, it follows that if in (2) the only operator is N, $Nq > Nr$. If the logician succeeds in fixing an exact (and intuitive) mathematical relation between L, P, and H, then an exact prediction can be made as to the value of the formulas in the scope of N. For example, if being a historical necessity to the degree 1 means being a physical necessity to i, then given that $Pq = 1$ and $Hr = 1$ it follows that $\frac{Nq}{Nr} = \frac{1}{i}$.

3.1.1 *Corollaries* In our (fuzzy) modal counterfactual logic some truths of the classical propositional calculus fail at the moment our propositional variables are propositions of the form '$\Box(A \mathrel{\Box\!\!\rightarrow} B)$', '$\Box(A \mathrel{\Box\!\!\rightarrow} C)$'. In the classical propositional calculus (CPC) P,Q,R . . . range over sentences. Let $P = \Box(A \mathrel{\Box\!\!\rightarrow} B)$, $Q = \Box(A \mathrel{\Box\!\!\rightarrow} C)$, and $R = \Box(A \mathrel{\Box\!\!\rightarrow} D)$. Then as corollaries of our truth conditions in a fuzzy NS, the following classical tautologies fail in our modal counterfactual propositional logic:

Failure 1: $\Box(A \mathrel{\Box\!\!\rightarrow} B) \vee \neg\Box(A \mathrel{\Box\!\!\rightarrow} B)$ (in CPC: $P \vee \neg P$)

Failure 2: $\Box(A \mathrel{\Box\!\!\rightarrow} B) \rightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} C)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} B)))$ (in CPC: $P \rightarrow (Q \rightarrow P)$

Failure 3: $\neg(\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} C)))$ (in CPC: $\neg P \rightarrow (P \rightarrow Q)$)

Failure 4: $(((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \wedge (\Box(A \mathrel{\Box\!\!\rightarrow} C))) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} D))) \leftrightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} C)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} D))))$ (in CPC: $((P \wedge Q) \rightarrow R) \leftrightarrow (P \rightarrow ((Q \rightarrow R)))$.)

Failure 5: $((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} C)) \wedge \neg(\Box(A \mathrel{\Box\!\!\rightarrow} C))) \rightarrow \neg(\Box(A \mathrel{\Box\!\!\rightarrow} B)))$ (in CPC: $(P \rightarrow (Q \wedge \neg Q)) \rightarrow \neg P$.)

Failure 6: $((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \wedge \neg(\Box(A \mathrel{\Box\!\!\rightarrow} B))) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} C))$. (in CPC: $(P \wedge \neg P) \rightarrow Q$).

Failure 7: $(\Box(A \mathrel{\Box\!\!\rightarrow} C)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} B) \vee \neg(\Box(A \mathrel{\Box\!\!\rightarrow} B)))$ (in CPC: $(Q \rightarrow (P \vee \neg P)))$.

It is to be noted that these seven failures disappear if the sentences of our modal counterfactual logic can be either true or false, i.e., if propositions cannot be true to a degree. That these seven failures are a function of the possibility to be true to a degree is obvious from the fact that the same failures occur in fuzzy propositional logic in which the sentential variables are nonmodal atomic sentences. It is an established fact that in a fuzzified classical propositional calculus these seven tautologies fail.

On the other hand the following principles of CPC hold in our modal counterfactual propositional logic (MCPC):

(a) De Morgan laws
(b) Associative laws
(c) Distributive laws

(d) Commutative laws

(e) $\Box(A \mathrel{\Box\!\!\rightarrow} B) \rightarrow \Box(A \mathrel{\Box\!\!\rightarrow} B)$ (in *CPC*: $P \rightarrow P$)

(f) $((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} C)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} D)))) \rightarrow (((\Box(A \rightarrow B)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} C))) \rightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} D))))$ (in *CPC*: $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R)))$

(g) $(\neg(\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow \neg(\Box(A \mathrel{\Box\!\!\rightarrow} C))) \rightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} C)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} B)))$ (in *CPC*: $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P))$

(h) $(((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} C))) \wedge \neg(\Box(A \mathrel{\Box\!\!\rightarrow} C))) \rightarrow \neg(\Box(A \mathrel{\Box\!\!\rightarrow} B))$ (in *CPC*: $((P \rightarrow Q) \wedge \neg Q) \rightarrow \neg P)$

(i) $((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} C))) \rightarrow (((\Box(A \mathrel{\Box\!\!\rightarrow} C)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} D))) \rightarrow ((\Box(A \mathrel{\Box\!\!\rightarrow} B)) \rightarrow (\Box(A \mathrel{\Box\!\!\rightarrow} D))))$ (in *CPC*: $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R)))$.

It is not surprising to find out that the same principles hold in the propositional calculus of fuzzy logic. This similarity permits us to set the following conjecture:

Conjecture 4: In any system of logic (in the sense of a Tarskian model theoretic treatment of the notions of truth and satisfaction in a model) in which the set of truth values is the real interval $[0,1]$ and in which propositions can be mapped into each point on this continuum, the division between rejected principles and accepted ones will be the same as in the propositional calculus of fuzzy logic.

Then our modal counterfactual logic (*MCL*) is simply a special case obeying this conjecture through the rejection made by its propositional calculus.

This conjecture, if true, has very interesting results as it is shown in Almog [4]. The results are concerned with the notion of decidability via the notions of finite model property and finite frame property. My point is that if all the logics in which degrees of truth are allowed, retain (and reject, respectively) the same truths from the classical propositional calculus, then they must have a special and unique tool which does this rejection (or acceptance). Since normally this tool is the finite model property it follows that the finite models of these logics have a special structure. The *finite model property* (*FMP*) can be characterized in the following way: the logic L has the *FMP* iff it is characterized by a class of finite models iff each nontheorem is rejected by some finite model of the logic.

Now since all the logics which allow degrees of truth reject (retain) the same principles of the propositional calculus of classical logic, it follows that all these logics would behave in the same way with respect to decidability problems. Recall that in the fuzzy logic propositional calculus (*FPC*) we have a connective '$\Vdash$' of entailment such that when A and B are atomic sentences, '$A \Vdash B$' may be true to the degree i, $0 \leqslant i \leqslant 1$. Such a fuzzy implication connective has been proposed in the literature by Łukasiewicz, Gödel or some versions of sequence logics (especially the C-standard version). A modern counterpart of such a fuzzy connective has been suggested by Scott [13]. It follows that in *FPC* the following holds:

$A \Vdash_i B$ iff $\Vdash_i A \supset B$ (when $\Vdash$ stands for validity and $\supset$ for the two-valued Principia Mathematica implication).

In such a system, we have degrees of theoremhood exemplified by the fuzzification of the notion of validity. Thus our question is: Does our modal counterfactual logic exemplify the concept of graded validity? The answer is definitively 'yes'. In light of our conjecture it follows that all the features of *FPC* apply to the *MCPC* of our *MCL*. From this fact, some conclusions concerning completeness, consistency, and decidability of *MCL* follow:

Given that S is a formula of *MCL* ($S = A \,\Box\!\!\rightarrow B$), in light of (4) S and hence $\neg S$ may be theorems of *MCL* to a degree. Then it follows:

Completeness: *MCL* is fuzzily complete to the degree i iff the sentence 'either S or $\neg S$ is in *MCL*' is true to i. i will vary from 0.5 to 1. It will be complete to the degree 1 iff $S \epsilon MCL$ (or $\neg S \epsilon MCL$) is true to 1 and $\neg S \epsilon MCL$ (or $S \epsilon MCL$) is true to 0. (This is completeness in the classical sense.) *MCL* will be complete to 0.5 iff S is a theorem of *MCL* to 0.5 and thus $\neg S$ is also a theorem of *MCL* to 0.5. Since disjunction is equivalent to the maximum of the disjuncts, *MCL* will be at least complete to 0.5 (if the stated above holds for all S and their negations).

Consistency: *MCL* is fuzzily consistent to i iff the sentence 'S and $\neg S$ are theorems of *MCL*' is true to $1 - i$. Since $|S| = 1 - |\neg S|$ it follows that *MCL* will be at least consistent to 0.5 (if it holds for all S).

Decidability: *MCL* is decidable iff there is an effective procedure to determine to which degree S is a theorem of *MCL*.

We see that we need fuzzified notions of completeness, consistency, and decidability for logics which allow formulas to be theorems to a degree. There is no sense in saying either that *MCL* is undecidable or that it is decidable. For if we take a logic to be equivalent to the set of its sound principles (accepted theorems) then *MCL* is neither a decidable set nor an undecidable. *MCL* would be decidable if there were an effective procedure to determine whether a particular S is a member of *MCL*. Since the relation 'member of' is not two-valued in *MCL*, i.e., S may be a member of *MCL* to a degree, it follows that the extension of the relation 'is member of' in *MCL* is a fuzzy set, i.e., certain arguments (certain formulas) satisfy partly the relation with regard to *MCL*. Hence *MCL* needs a fuzzified concept of decidability. The need for a fuzzy notion of decidability remains even if we arithmetize *MCL*. Then decidability questions will take the form of: Is it true that the Gödel numbers of the formulas of *MCL* form a decidable class? The answer will be that since *MCL* contains formulas which are partly theorems of it or, more sharply, *MCL* partly contains some formulas, it will follow that the Gödel numbers of the formulas of *MCL* form a fuzzily decidable set, i.e., a set whose members do not allow a two-valued mapping into 'accepted members' and 'rejected members' but rather demand a sequence of sets 'accepted to i', 'accepted to j', 'accepted to k', when i,j,k range over real numbers in the interval $[0,1]$.

Such a transition from a binary concept of decidability to a many-valued concept requires some modifications in some classical mathematical concepts which are strongly connected with the concept of decidability. Such notions are the notions of recursive functions and recursive relations. Since these notions are connected via Church's thesis (C) to the concepts of calculable relations and functions it follows that the latter are also in the scope of the needed modifications:

(C) All (effectively) calculable functions and relations are recursive.

I do not want to go into the question of how such a modification is to be done. A general discussion of the problem of decision procedures in fuzzified modal logics appears in Almog.[6] I will only remark that the notion of recursive function might be found to be inadequate for a theory (in the sense of Tarski and Robinson of the term 'Theory') in which formulas are theorems to a degree. Instead we might need a notion of partial recursiveness according to which a mechanical decision procedure is *almost* available; i.e., the 'Turing machine' gives you an algorithm which covers almost the whole field, in the sense of calculating almost all effectively calculable functions and relations. My remarks are not purely theoretic. It is a fact of computer science that such special partial recursive functions are possible. (I hope to report in the near future on my results in setting a program in which the program includes fuzzy algorithms. Some interesting work had been done in this field by Zadeh [17].[7]

Recall the classical approach to decidability: for any formula S of a theory T, if S is a theorem of T then with some finite length of time the machine M will show that S is a theorem of T. Suppose now that the set of formulas of T which are not theorems of T is also recursively enumerable, then this set could be generated by another machine, call it M'. Thus in a finite time M will show that S is a theorem of T and M' will show that $\neg S$ is not.

But now recall that in a fuzzy logic like our MCL, S (hence, $\neg S$) may be theorems to a degree. Hence the whole concept of a mechanical procedure via Turing machines will have to be modified. According to the new approach where the sentence '*S is a theorem of T*' may take values in the continuum $[0,1]$, M and M' become only special cases, i.e., cases in which the above-mentioned sentence is either completely true (this is the case for M) or completely false (this is the case for M'). All other cases are not accounted for by the classical procedure of the classical Turing machines.

Let me conclude by returning to our *FMP* (*finite model property*). We said that since all the logics which allow degrees of truth reject (retain) the same principles of the *CPC* it follows that they must have special models (finite ones) which allow a special *FMP*. However, this is not the whole story. Segerberg [15] and Hansson and Gärdenfors [7] have argued that one can form a stronger variant of the *FMP* where for each nontheorem there exists a computable upper limitation to the size of the model that falsifies the formula in question. Thus one is not bound to know whether a logic is finitely axiomatizable in order to know whether it is decidable. One must only check whether a certain formula is true in all models smaller than the given limitation in order to know whether it is a theorem. This notion is called the *finite frame property*. (The concepts of frame and model are used in the sense assigned to them by the model theories of modal logics.)

The finite model property (*FMP*) is trivially entailed by the finite frame property (*FFP*). The converse is said to hold too, *pace* Segerberg [15]. Thus our remarks concerning the connection between the *FMP* and logics with degrees of truth apply again. Since for a logic L to have the *FFP* means that each nontheorem is rejected by some finite frame of the logic, it follows that logics with degrees of truth do not have only special models (finite ones) but also special finite frames which permit all the logics allowing degrees of truth (to their formulas) to accept and retain the same principles of the classical two-valued logic. Since *MCL* via its *MCPC* is a special case of these logics it follows that it has such special finite models and frames.

NOTES

1. Other semantical analyses were suggested by Stalnaker, Thomason, Nute, and Pollock; however, I regard Lewis's suggestion as the most comprehensive account both from a philosophical point of view (see [10]) and a formal point of view.

2. In an unpublished note of mine, "Montague-semantics for Lewis' conditionals," I show that a modal logic with the accessibility relation which Montague considers at the end of his paper [11] is practically equivalent to Lewis's counterfactual logic.

3. This analysis is due to M. J. Cresswell.

4. One may disregard the informal meaning of 'coincide'. What is important is the existence of a one-to-one mapping between the two sets.

5. Fuzzification of implications had been suggested by Gödel, Łukasiewicz, Belnap, Scott, and the sequence logics.

6. In [1] and [2] I touch the questions of fuzzy recursiveness and decidability in several *relevance* logics.

7. Bellacicco had recently reported on new results in the domain of fuzzy algorithms [6]. Moreover, if my fuzzified Quantum logic is reasonable, it can be shown on the basis of my [3], [4], and [5] that degrees of theoremhood and a graded concept of validity are needed in quantum logics based on Scott's approximate lattices [13].

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