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Automorphisms of ω-Cubes

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1 Preliminaries The word *set* is used for a collection of numbers, *class* for a collection of sets. We write ε for the set of all numbers, o for the empty set of numbers, card Γ for the cardinality of the collection Γ, and $\mathcal{P}_{fin}(\alpha)$ for the class of all finite subsets of α. If f is a function of n variables, i.e., a mapping from a subcollection of ε^n into ε, we denote its domain and range by δf and ρf respectively. A collection of functions is called a *family*. The image under f of the number n is denoted by f_n or $f(n)$, sometimes by both in the same context. We write $\alpha \sim \beta$ for α equivalent to β, $\alpha \simeq \beta$ for α recursively equivalent to β, and $\alpha \oplus \beta$ for the symmetric difference of α and β. The collection of all recursive equivalence types (RETs) is denoted by Ω, that of all isols by Λ. Moreover, $\Omega_0 = \Omega - (0)$, $\Lambda_0 = \Lambda - (0)$, $\varepsilon_0 = \varepsilon - (0)$. The reader is referred to [4] and [8] for the basic properties of RETs and isols. Let $\langle \rho_n \rangle$ be the canonical enumeration of the class $\mathcal{P}_{fin}(\varepsilon)$, i.e., let $\rho_0 = o$ and

$$\rho_{n+1} = \begin{cases} (a_1, \ldots, a_k), \text{ where} \\ n + 1 = 2^{a(1)} + \ldots + 2^{a(k)}, \\ a_1, \ldots, a_k \text{ distinct.} \end{cases}$$

Put $r_n = \text{card } \rho_n$, then r_n is a recursive function. If σ is a finite set, can σ denotes the *canonical index* of σ, i.e., the unique number i such that $\sigma = \rho_i$. For $\alpha \subset \varepsilon$, $i \in \varepsilon$,

$$[\alpha; i] = \{x \mid \rho_x \subset \alpha \ \& \ r_x = i\}, \ 2^\alpha = \{x \mid \rho_x \subset \alpha\} \text{ so that}$$
$$\alpha \simeq \beta \Rightarrow (\forall i)[[\alpha; i] \simeq [\beta; i]], \ \alpha \simeq \beta \Rightarrow 2^\alpha \simeq 2^\beta.$$

If f is a function of one variable, $\delta f^* = 2^{\delta f}$, $f^*(0) = 0$ and

$$f^*(2^{a(1)} + \ldots + 2^{a(k)}) = 2^{fa(1)} + \ldots + 2^{fa(k)},$$

for distinct elements $a_1, \ldots, a_k$ of δf. Equivalently,

$$\delta f^* = 2^{\delta f}, \ \rho_{f^*(x)} = f(\rho_x).$$

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It is readily seen that

(1) $f\ 1-1 \Rightarrow f^*\ 1-1, f \neq g \Rightarrow f^* \neq g^*, (fg)^* = f^*g^*$.

We briefly review the material of [5] which is relevant to the present paper. Note that the vertices of $Q^n = (0, 1)^n$ can be interpreted as the characteristic functions of subsets of $(0, \ldots, n-1)$, $(1, \ldots, n)$ or any other finite set of cardinality n. This suggests the possibility of defining Q^n in terms of $\mathcal{P}_{fin}(\nu)$. With a nonempty set ν we associate the (directed) cube $Q^\nu = \langle 2^\nu, \leqslant\rangle$, where $x \leqslant y \iff \rho_x \subset \rho_y$, for $x, y \in 2^\nu$; we call Q^ν the *ω-cube* on the set ν. An *isomorphism* from Q^μ onto Q^ν is a one-to-one mapping g from 2^μ onto 2^ν such that $x \leqslant y \Rightarrow g(x) \leqslant g(y)$, for $x, y \in 2^\mu$, or equivalently, $\rho_x \subset \rho_y \Rightarrow \rho_{g(x)} \subset \rho_{g(y)}$, for $x, y \in 2^\mu$. An isomorphism is an *ω-isomorphism,* if it has a partial recursive one-to-one extension. The ω-cubes Q^μ and Q^ν are *isomorphic* (*ω-isomorphic*) if there is at least one isomorphism (ω-isomorphism) between them. These equivalence relations are denoted by $\cong$ and $\cong_\omega$. For $N \in \Omega_0$ we define $Q^N = Q^\nu$, for *any* $\nu \in N$. It can be proved that $Q^\mu \cong Q^\nu \iff \mu \sim \nu$, while $Q^\mu \cong_\omega Q^\nu \iff \mu \simeq \nu$. Thus Q^N is uniquely determined by N up to ω-isomorphism, just as Q^n is uniquely determined by n up to isomorphism. We call n the *dimension* of Q^n and Q^ν, for card $\nu = n$; N is the *ω-dimension* of Q^N and Q^ν, for *Req* $\nu = N$. In symbols,

$$\begin{aligned} n &= \dim Q^\nu = \dim Q^n, \quad \text{for card } \nu = n, \ n \in \varepsilon_0, \\ N &= \dim_\omega Q^\nu = \dim_\omega Q^N, \text{for } Req\ \nu = N, N \in \Omega_0. \end{aligned}$$

We use the word *graph* in the sense of a simple, connected, countable graph with at least one vertex. Such a graph will be represented by an ordered pair $G = \langle \beta, \eta\rangle$, where $\beta \subset \varepsilon$ and $\eta \subset [\beta; 2]$; a vertex of G is therefore identified with a number, while an edge of G is identified with the canonical index of the set consisting of its endpoints. The relation $\operatorname{can}(p, q) \in \eta$ between the vertices p and q of $G = \langle \beta, \eta\rangle$ is also written: p adj q. With a nonempty set ν we associate the graph $Q_\nu = \langle 2^\nu, \eta\rangle$, where

$$\eta = \{\operatorname{can}(x, y) \in [2^\nu; 2] \mid \operatorname{card}(\rho_x \oplus \rho_y) = 1\}.$$

An *isomorphism* from $Q_\mu = \langle 2^\mu, \theta\rangle$ onto $Q_\nu = \langle 2^\nu, \eta\rangle$ is a one-to-one mapping g from 2^μ onto 2^ν such that $\operatorname{can}(x, y) \in \theta$ implies $\operatorname{can}(g_x, g_y) \in \eta$, for $x, y \in 2^\mu$. An isomorphism is an *ω-isomorphism,* if it has a partial recursive one-to-one extension. The graphs Q_μ and Q_ν are *isomorphic* (*ω-isomorphic*), if there is at least one isomorphism (ω-isomorphism) between them. These equivalence relations between graphs are denoted by $\cong$ and $\cong_\omega$. For $N \in \Omega_0$ we define $Q_N = Q_\nu$, for *any* $\nu \in N$. It can be proved that $Q_\mu \cong Q_\nu \iff \mu \sim \nu$, while $Q_\mu \cong_\omega Q_\nu \iff \mu \simeq \nu$. Thus Q_N is uniquely determined by N up to ω-isomorphism just as Q_n is uniquely determined by n up to isomorphism. We call 2^n the *order* of Q_n and Q_ν, for card $\nu = n$, and 2^N the *order* of Q_N and Q_ν, for *Req* $\nu = N$. In symbols,

$$\begin{aligned} 2^n &= oQ_\nu = oQ_n, \text{ for card } \nu = n, \ n \in \varepsilon_0, \\ 2^N &= oQ_\nu = oQ_N, \text{ for } Req\ \nu = N, N \in \Omega_0. \end{aligned}$$

We shall need two propositions of [5].

Proposition A1.1 ([5], P1.1) *Let g be a one-to-one mapping from 2^μ onto 2^ν. Then*

(a) *g is an isomorphism from Q^μ onto Q^ν iff $g = f^*$, for some one-to-one function f from μ onto ν,*
(b) *g is an ω-isomorphism from Q^μ onto Q^ν iff $g = f^*$, for some one-to-one function f from μ onto ν with a partial recursive one-to-one extension.*

Proposition A1.2 ([5], P3.2) *Let g be an isomorphism (ω-isomorphism) from Q_μ onto Q_ν. Then g is an isomorphism (ω-isomorphism) from Q^μ onto Q^ν iff $g(0) = 0$.*

For a function $f(x)$ we define $\pi f = \{x \in \delta f \mid f(x) \neq x\}$. Let f be a permutation of the set ν. Then f is a *finite* permutation of ν, if πf is finite; f is an *ω-permutation* of ν, if it has a partial recursive one-to-one extension. We write $Per(\nu)$ for the family of all permutations of ν, $Per_\omega(\nu)$ for the family of all ω-permutations of ν, and P_ν for the family of all finite permutations of ν. For the groups under composition formed by these three families we have

$$P_\nu \leqslant Per_\omega(\nu) \leqslant Per(\nu).$$

If ν is finite these three groups are the same. If ν is denumerable we have $Per_\omega(\nu) < Per(\nu)$, since card $Per_\omega(\nu) = \aleph_0$, while card $Per(\nu) = c$. We shall need a characterization of the sets ν for which $P_\nu = Per_\omega(\nu)$. This clearly depends only on $Req\ \nu$. An RET N is *multiple-free,* if every even predecessor of N is finite. Trivially, every finite RET is multiple-free. Let $R = Req\ \varepsilon$. If $A \in \Omega - \Lambda$, we have $R \leqslant A$, where $R = 2R$, hence A is not multiple-free. Thus every multiple-free RET is an isol. There are exactly c infinite isols which are not multiple-free, since every infinite isol which is even or odd is not multiple-free. There also are c infinite isols which are multiple-free, e.g., all infinite, indecomposable isols and every isol which is the sum of two incomparable indecomposable isols ([4], T49).

Proposition A1.3 ([2], P7, due to B. Cole) *Let $N = Req\ \nu$. Then $P_\nu = Per_\omega(\nu)$ iff N is a multiple-free isol.*

2 Automorphisms of Q^ν and Q_ν

An *automorphism* of Q^ν (of Q_ν) is an isomorphism g from Q^ν (from Q_ν) onto itself; g is an *ω-automorphism* of Q^ν (of Q_ν), if it has a partial recursive one-to-one extension. We define:

$Aut\ Q^\nu$ = the family of all automorphisms of Q^ν,
$Aut_\omega\ Q^\nu$ = the family of all ω-automorphisms of Q^ν,
$Aut\ Q_\nu$ = the family of all automorphisms of Q_ν,
$Aut_\omega\ Q_\nu$ = the family of all ω-automorphisms of Q_ν.

These four families are groups under composition. In case ν is finite we have $Aut_\omega\ Q^\nu = Aut\ Q^\nu$ and $Aut_\omega\ Q_\nu = Aut\ Q_\nu$, since every function with a finite domain is partial recursive. For an elementary discussion of the relationship between the groups $Aut\ Q^\nu$ and $Aut\ Q_\nu$ in the special case $\nu = (1, \ldots, n)$, see [7], Ch. I Section 9. Henceforth the set ν need not be finite, unless this is explicitly stated. If we take $\mu = \nu$ in Propositions A1.1 and A1.2 we obtain:

Proposition A2.1 *Let g be a permutation of 2^ν. Then*

(a) $g \in Aut\, Q^\nu$ *iff* $g = f^*$, *for some* $f \in Per(\nu)$,
(b) $g \in Aut_\omega\, Q^\nu$ *iff* $g = f^*$, *for some* $f \in Per_\omega(\nu)$.

Proposition A2.2 *Let* $g \in Aut\, Q_\nu$ [*or* $\in Aut_\omega\, Q_\nu$]. *Then* $g \in Aut\, Q^\nu$ [*or* $\in Aut_\omega\, Q^\nu$] *iff* $g(0) = 0$.

Remark: Let the mapping ϕ have $Per(\nu)$ as domain and let $\phi(f) = f^*$, $\phi_\omega = \phi | Per_\omega(\nu)$. Then we see by (1) and A2.1 that ϕ is an isomorphism from $Per(\nu)$ onto $Aut\, Q^\nu$, while ϕ_ω is an isomorphism from $Per_\omega(\nu)$ onto $Aut_\omega\, Q^\nu$. The mapping ϕ_ω is effective in the sense that given any $f \in Per_\omega(\nu)$, say by a definition of a partial recursive one-to-one extension $\bar{f}$ of f, we can find a definition of a partial recursive one-to-one extension of f^*, namely $\bar{f}^*$.

We now turn to the question of how $Aut_\omega\, Q_\nu$ can be expressed in terms of $Aut_\omega\, Q^\nu$. The identity function on ε will be denoted by i.

Definition For $a \in \varepsilon$,

$$\delta c_a = \varepsilon,\ c_a(x) = \begin{cases} x + 2^a, & \text{for } a \notin \rho_x. \\ x - 2^a, & \text{for } a \in \rho_x. \end{cases}$$

Note that c_a is a recursive function, $\pi c_a = \varepsilon$, and $c_a c_b = c_b c_a$, for $a, b \in \varepsilon$.

Proposition A2.3 *Let* $a \in \varepsilon$. *Then the function* c_a *is a recursive permutation of* ε, *an involution and a recursive automorphism of the graph* Q_ε.

Proof: Let $a \in \varepsilon$. From now on we keep a fixed and write $f = c_a$. The recursive function f is an involution, since $f^2 = i$ and $f(0) \neq 0$; hence f is a recursive permutation of ε.

Assume x adj y, i.e., card$(\rho_x \oplus \rho_y) = 1$. Then either: (1) $\rho_x \oplus \rho_y = (a)$ or (2) $\rho_x \oplus \rho_y = (b)$, for some $b \neq a$. If (1) holds, $\rho_x = \rho_y \cup (a)$, where $a \notin \rho_y$, or $\rho_y = \rho_x \cup (a)$, where $a \notin \rho_x$. We may assume without loss of generality that $\rho_x = \rho_y \cup (a)$, where $a \notin \rho_y$. Then $x = y + 2^a$, $y = x - 2^a$, hence $f(x) = y$, $f(y) = x$ and $f(x)$ adj $f(y)$. Now assume (2) holds. Since ρ_x and ρ_y only differ in b, where $b \neq a$ we have: either $a \in \rho_x \cap \rho_y$ or $a \notin \rho_x \cup \rho_y$. In the former case $(\rho_x - (a)) \oplus (\rho_y - (a))$ has cardinality 1, hence can$(\rho_x - (a))$ adj can$(\rho_y - (a))$, i.e., $f(x)$ adj $f(y)$. In the latter case, $(\rho_x \cup (a)) \oplus (\rho_y \cup (a))$ has cardinality 1, hence can$(\rho_x \cup (a))$ adj can$(\rho_y \cup (a))$, i.e., $f(x)$ adj $f(y)$.

Remark: Let $a \in \nu$, $f = c_a | 2^\nu$, then $f \in Aut_\omega\, Q_\nu$. However, $f(0) = 2^a$, hence $f \notin Aut_\omega\, Q^\nu$ by A2.2. Thus $Aut_\omega\, Q^\nu < Aut_\omega\, Q_\nu$, whenever ν is nonempty.

Definition For $\alpha \in \mathcal{P}_{fin}(\varepsilon)$.

$$\delta c_\alpha = \varepsilon,\ c_\alpha = \begin{cases} i, & \text{if } \alpha = o, \\ c_{a(1)} \cdot \ldots \cdot c_{a(k)}, & \text{if } \alpha \neq o,\ \text{card}\, \alpha = k,\ \alpha = (a_1, \ldots, a_k). \end{cases}$$

Proposition A2.4 *For every finite set* α, c_α *is a recursive permutation of* ε. *Moreover,* $c_\alpha c_\beta = c_{\alpha \oplus \beta}$, *for* $\alpha, \beta \in \mathcal{P}_{fin}(\varepsilon)$. *Also,* c_α *is an involution for* $\alpha \neq o$.

Proof: Let $\alpha \in \mathcal{P}_{fin}(\varepsilon)$. The first statement follows immediately from the definition of c_α. Now assume $\alpha, \beta \in \mathcal{P}_{fin}(\varepsilon)$, $\gamma = \alpha \cap \beta$. Then γ is finite and

$c_\alpha c_\beta = c_{\alpha-(p)} c_{\beta-(p)}$, for each $p \in \gamma$. We conclude that $c_\alpha c_\beta = c_{\alpha-\gamma} c_{\beta-\gamma}$, where $\alpha - \gamma$, $\beta - \gamma$ are disjoint; then $c_\alpha c_\beta = c_{(\alpha-\gamma)\cup(\beta-\gamma)} = c_{\alpha\oplus\beta}$. Let $\alpha \neq o$; then $c_\alpha \neq i$ and $c_\alpha^2 = c_{\alpha\oplus\alpha} = c_o = i$. Thus c_α is an involution.

Notations: If ν is known from the context,

$$c_a^\# = c_a | 2^\nu,\ c_\alpha^\# = c_\alpha | 2^\nu, \text{ for } a \in \nu,\ \alpha \in \mathcal{P}_{fin}(\nu),$$
$$C_\nu = \{c_\alpha^\# | \alpha \in \mathcal{P}_{fin}(\nu)\}.$$

Proposition A2.5 *The mapping $\phi(\alpha) = c_\alpha$ from $\mathcal{P}_{fin}(\varepsilon)$ onto C_ε is an isomorphism from the group $\langle \mathcal{P}_{fin}(\varepsilon), \oplus\rangle$ onto the group formed by C_ε under composition. Similarly, the mapping $\phi(\alpha) = c_\alpha^\#$ is an isomorphism from the group $\langle \mathcal{P}_{fin}(\nu), \oplus\rangle$ onto the group formed by C_ν under composition.*

Proof: Since $\phi(\alpha \oplus \beta) = c_\alpha c_\beta$ it suffices to show that ϕ is one-to-one. For $\alpha, \beta \in \mathcal{P}_{fin}(\varepsilon)$,

$$\alpha \neq \beta \Rightarrow \alpha \oplus \beta \neq o \Rightarrow c_{\alpha\oplus\beta} \neq i \Rightarrow c_\alpha c_\beta \neq i \Rightarrow c_\alpha \neq c_\beta^{-1} \Rightarrow c_\alpha \neq c_\beta .$$

Remark: If ν is infinite, the Abelian group $\langle \mathcal{P}_{fin}(\nu), \oplus\rangle$ is isomorphic to $\mathbf{Z}_2^{\aleph_0}$, i.e., the direct sum of $\aleph_0$ copies of $\mathbf{Z}_2$.

If H and K are subgroups of a group G with unit element i, we say that G is the *semidirect* product of H by K (written: $G = H \times K$), if $HK = G$, $H \cap K = (i)$, $H \lhd G$. We call G the *direct* product of H and K, if we also have $K \lhd G$, i.e., if both H and K are normal subgroups of G.

Proposition A2.6 *For $\nu \subset \varepsilon$,*

(a) $Aut\ Q_\nu = C_\nu \times Aut\ Q^\nu$,
(b) $Aut_\omega\ Q_\nu = C_\nu \times Aut_\omega\ Q^\nu$.

Proof: To prove (a) it suffices to show:

(1) $f \in Aut\ Q_\nu \Rightarrow (\exists g)(\exists h)[f = gh\ \&\ g \in C_\nu\ \&\ h \in Aut\ Q^\nu]$,
(2) $C_\nu \cap Aut\ Q^\nu = (i)$,
(3) $C_\nu \lhd Aut\ Q_\nu$.

Re (1). Let $f \in Aut\ Q_\nu$, $f(0) = b$, $\beta = \rho_b$, then $\beta \in \mathcal{P}_{fin}(\nu)$ and $c_\beta^\# \in C_\nu$. Hence $c_\beta^{\#^{-1}} f(0) = 0$, $c_\beta^{\#^{-1}} f \in Aut\ Q^\nu$ and $c_\beta^\# \cdot c_\beta^{\#^{-1}} f$ is an expression of f in the desired form.

Re (2). $f \in C_\nu \cap Aut\ Q^\nu \Rightarrow f(0) = 0\ \&\ f \in C_\nu \Rightarrow f = c_o^\# = i$.

Re (3). We only need to show

$$(c_\beta^\# h)^{-1} C_\nu (c_\beta^\# h) \subset C_\nu, \text{ for } \beta \in \mathcal{P}_{fin}(\nu),\ h \in Aut\ Q^\nu.$$

Since $c_\beta^{\#^{-1}} C_\nu c_\beta^\# = C_\nu$, it suffices to prove that $h^{-1} C_\nu h \subset C_\nu$, for $h \in Aut\ Q^\nu$. Note that $c_\beta^\# \text{can}(\xi) = \text{can}(\beta \oplus \xi)$, for $\xi \in \mathcal{P}_{fin}(\nu)$. Assume $h \in Aut\ Q^\nu$ and $g \in C_\nu$, say $h = f^*$, for $f \in Per(\nu)$ and $g = c_\beta^\#$, for $\beta \in \mathcal{P}_{fin}(\nu)$. Put $\gamma = f^{-1}(\beta)$, then $\gamma \in \mathcal{P}_{fin}(\nu)$, and for $\sigma \in \mathcal{P}_{fin}(\nu)$,

$$h^{-1} g h(\text{can}\ \sigma) = (f^*)^{-1} c_\beta^\# f^*(\text{can}\ \sigma) = (f^{-1})^* c_\beta^\# f^*(\text{can}\ \sigma) =$$
$$(f^{-1})^* c_\beta^\# [\text{can}\ f(\sigma)] = (f^{-1})^* \text{can}[\beta \oplus f(\sigma)] =$$
$$\text{can}\ f^{-1}[\beta \oplus f(\sigma)] = \text{can}[f^{-1}(\beta) \oplus \sigma] = \text{can}(\gamma \oplus \sigma) = c_\gamma^\#(\sigma).$$

Thus $h^{-1}gh \in C_\nu$. We have proved that $h^{-1}C_\nu h \subset C_\nu$. We now consider (b). First of all, C_ν consists of ω-automorphisms of Q_ν, hence $C_\nu \leqslant Aut_\omega\, Q_\nu$, while $Aut_\omega\, Q^\nu \leqslant Aut_\omega\, Q_\nu$. To finish the proof of (b) it suffices to show that

$$(1') \qquad f \in Aut_\omega\, Q_\nu \Rightarrow (\exists g)(\exists h)[f = gh \;\&\; g \in C_\nu \;\&\; h \in Aut_\omega\, Q^\nu],$$

since the ω-analogues (2′) and (3′) of (2) and (3) follow immediately from (2) and (3). Let $f \in Aut_\omega\, Q_\nu$, $f(0) = b$, $\beta = \rho_b$. Then $f = c^\#_\beta \cdot c^\#_\beta f$, where $c^\#_\beta \in C_\nu$, $c^\#_\beta f \in Aut\, Q^\nu$. Both $c^\#_\beta$ and f have partial recursive one-to-one extensions, hence so has $c^\#_\beta f$. It follows that $c^\#_\beta f \in Aut_\omega\, Q^\nu$.

Remark: If card $\nu \geqslant 2$ the two semidirect products are not direct. First consider the product in (a). Let $p,\, q \in \nu$, $p \neq q$, f the permutation of ν which interchanges p and q, and $h = f^*$. Put $g = c^\#_p$, then $g \in C_\nu$, hence $g \in Aut\, Q_\nu$. Since $g = g^{-1}$ we have $ghg \in g\, Aut\, Q^\nu g^{-1}$. However, $ghg(0) = gh(2^p) = g(2^{f(p)}) = g(2^q) = 2^p + 2^q$, so that $ghg(0) \neq 0$ and $ghg \notin Aut\, Q^\nu$; thus $Aut\, Q^\nu \lhd Aut\, Q_\nu$ is false. The functions g and h can also be used to show that $Aut_\omega\, Q^\nu \lhd Aut_\omega\, Q_\nu$ is false.

3 ω-Groups Consider countable groups $G = \langle \nu, g\rangle$, where $\nu \subset \varepsilon$, g is the group operation and $h(x) = x^{-1}$, for $x \in \nu$. If such a group G is finite, i.e., if the set ν is finite, the functions g and h are partial recursive, but if G is denumerable, this need not be the case. The group $G = \langle \nu, g\rangle$ is *r.e.*, if ν is *r.e.* and g is partial recursive (hence so is h). We call G an *ω-group*, if both g and h have partial recursive extensions. Thus every r.e. group is an ω-group and so is each of its subgroups. ω-groups were introduced by Hassett [6] and also studied by Applebaum [1]-[3]. The *order* oG of the ω-group $G = \langle \nu, g\rangle$ is defined as $Req\ \nu$; thus oG has the usual meaning iff G is finite. An *ω-isomorphism* from the ω-group $G_1 = \langle \nu_1, g_1\rangle$ onto the ω-group $G_2 = \langle \nu_2, g_2\rangle$ is an isomorphism from G_1 onto G_2 with a partial recursive one-to-one extension. G_1 is *ω-isomorphic* to G_2 (written: $G_1 \cong_\omega G_2$), if there is at least one ω-isomorphism from G_1 onto G_2. Two finite groups are therefore ω-isomorphic iff they are isomorphic. Let $N \in \Omega_0$, $\nu \in N$. In this section we shall show that the group P_ν of all finite permutations of ν can be represented by (i.e., is isomorphic to) an ω-group $\mathbf{P}_\nu$ of order $N!$, while the group $C_\nu = \{c^\#_\alpha \mid \alpha \in \mathcal{P}_{fin}(\nu)\}$ can be represented by an ω-group $\mathbf{Z}_2(\nu)$ of order 2^N. We first define a Gödel-numbering for the family P_ε of all finite permutations of ε. Let i again denote the identity mapping on ε and let q_{n-1} stand for the n^{th} odd prime number, for $n \geqslant 1$.

Notations: For $f \in P_\varepsilon$, $\nu \subset \varepsilon$,

$$\tilde{f} = \begin{cases} 1, & \text{if } f = i, \\ 2^{n+1} \prod\limits_{i=0}^{n} [q(x_i)]^{fx(i)+1}, & \text{if } f \neq i,\ \pi f = (x_0, \ldots, x_n), \end{cases}$$

$\mathbf{P}_\varepsilon = \langle \eta, p\rangle$, where $\eta = \{\tilde{f} \mid f \in P_\varepsilon\}$, $p(\tilde{f}, \tilde{g}) = \widetilde{fg}$,

$\mathbf{P}_\nu = \langle \tilde{\nu}, p_\nu\rangle$, where $\tilde{\nu} = \{\tilde{f} \in \eta \mid \pi f \subset \nu\}$, $p_\nu = p \mid \tilde{\nu} \times \tilde{\nu}$.

Thus η is an infinite, recursive set, p a partial recursive function and $\mathbf{P}_\varepsilon$ a *r.e.* group isomorphic to P_ε. Moreover, for every choice of the set ν, $\mathbf{P}_\nu$ is an ω-group isomorphic to P_ν.

In order to represent the group C_ε by a *r.e.* group it suffices by A2.5 to do this for the group $\langle \mathcal{P}_{fin}(\varepsilon), \oplus \rangle$.

Notations: For $\nu \subset \varepsilon$,

$$\mathbf{Z}_2(\varepsilon) = \langle 2^\varepsilon, g \rangle, \text{ where } g(x, y) = \text{can } (\rho_x \oplus \rho_y),$$
$$\mathbf{Z}_2(\nu) = \langle 2^\nu, g_\nu \rangle, \text{ where } g_\nu = g \mid 2^\nu \times 2^\nu.$$

Clearly, $2^\varepsilon = \varepsilon$ and $\mathbf{Z}_2(\varepsilon)$ is a *r.e.* group, while $\mathbf{Z}_2(\nu) \leqslant \mathbf{Z}_2(\varepsilon)$, for $\nu \subset \varepsilon$. Moreover, the group C_ν can be represented by the ω-group $\mathbf{Z}_2(\nu)$.

Proposition A3.1 *For $\mu, \nu \subset \varepsilon$,*

(a) $\mu \simeq \nu \Longleftrightarrow \mathbf{P}_\mu \cong_\omega \mathbf{P}_\nu$,
(b) $\mu \simeq \nu \Rightarrow \mathbf{Z}_2(\mu) \cong_\omega \mathbf{Z}_2(\nu)$.

Proof: (a) The $\Rightarrow$ part follows immediately from the definitions of the concepts involved and of $\tilde{f}$. The $\Leftarrow$ part is due to Applebaum ([3], Section 3). (b) Let $\mu \simeq \nu$, say $\mu \subset \delta q$, $q(\mu) = \nu$, where q is a partial recursive one-to-one function. Put $f = q^*$, then $2^\mu \subset \delta f$, $f(2^\mu) = 2^\nu$, where f is also a partial recursive one-to-one function. Moreover, for $x, y \in \delta f$,

$$\begin{aligned} g[f(x), f(y)] &= \text{can}[\rho_{f(x)} \oplus \rho_{f(y)}] = \text{can}[\rho_{q^*(x)} \oplus \rho_{q^*(y)}] \\ &= \text{can}[q(\rho_x) \oplus q(\rho_y)] = \text{can } q[\rho_x \oplus \rho_y] = \text{can } q\rho_{g(x,y)} \\ &= \text{can } \rho_{q^*g(x,y)} = q^*g(x, y) = fg(x, y). \end{aligned}$$

Thus f is an isomorphism from $\mathbf{Z}_2(\delta f)$ onto $\mathbf{Z}_2(\rho f)$, while $f \mid 2^\mu$ is an ω-isomorphism from $\mathbf{Z}_2(\mu)$ onto $\mathbf{Z}_2(\nu)$.

Definition $\mathbf{Z}_2^N = \mathbf{Z}_2(\nu)$, $\mathbf{P}_N = \mathbf{P}_\nu$, for $\nu \in N$, $N \in \Omega_0$.

In view of A3.1 the ω-groups $\mathbf{Z}_2^N$ and $\mathbf{P}_N$ are unique up to ω-isomorphism.

Proposition A3.2 $o\mathbf{Z}_2^N = 2^N$ *and* $o\mathbf{P}_N = \mathbf{N}!$, *for $N \in \Omega_0$.*

Proof: Let for $\nu \in \mathcal{P}_{fin}(\varepsilon)$, $\Phi(\nu) = \{x \mid \rho_x \subset \nu\}$, $\Psi(\nu) = \{\tilde{f} \in \eta \mid \pi f \subset \nu\}$, then Φ and Ψ are recursive, combinatorial operators inducing the functions 2^n and $n!$ respectively. Hence for $N = Req\ \nu$, we have $o\mathbf{Z}_2^N = Req\ \Phi(\nu) = 2^N$ and $o\mathbf{P}_N = Req\ \Psi(\nu) = N!$.

4 The main result

Theorem *Let $\nu \in N$ and $N \in \Omega_0$. Then*

(a) *$Aut_\omega\, Q_\nu = C_\nu \times Aut_\omega\, Q^\nu$, i.e., $Aut_\omega\, Q_\nu$ is the semidirect product of C_ν by $Aut_\omega\, Q^\nu$,*
(b) *the group C_ν can be represented by the ω-group $\mathbf{Z}_2^N$ of order 2^N,*
(c) *if N is a multiple-free isol, the group $Aut_\omega\, Q^\nu$ can be represented by the ω-group $\mathbf{P}_N$ of order $N!$,*
(d) *if N is a multiple-free isol, the group $Aut_\omega\, Q_\nu$ can be represented by an ω-group of order $2^N \cdot N!$*

Proof: Parts (a), (b), and (c) follow from A1.3, A2.5, A2.6, A3.2 and the Remark following A2.2. To prove (d) assume that N is a multiple-free isol. We shall use the recursive function $j(x, y) = x + (x + y)(x + y + 1)/2$. Define a set β_ν and a function h_ν by:

$$\beta_\nu = \{j(a, \tilde{f}) | a \in 2^\nu \;\&\; \tilde{f} \in \mathbf{P}_\nu\},$$
$$\delta h_\nu = \beta_\nu,\; h_\nu j(a, \tilde{f}) = c_\alpha^\# f^*, \text{ where } \alpha = \rho_a.$$

We claim: (i) h_ν maps β_ν one-to-one onto $Aut_\omega(Q_\nu)$, and (ii) there is a group operation t_ν on β_ν such that $G_\nu = \langle \beta_\nu, t_\nu\rangle$ is an ω-group which is isomorphic to $Aut_\omega\, Q_\nu$.

Re (i). Let $j(a, \tilde{f}) \in \beta_\nu$. Then $a \in 2^\nu$, $\alpha \in \mathcal{P}_{fin}(\nu)$, $c_\alpha^\# \in C_\nu$ and $\tilde{f} \in \mathbf{P}_\nu$, $f \in P_\nu$, $f^* \in Aut_\omega\, Q^\nu$. Thus $c_\alpha^\# f^* \in Aut_\omega\, Q_\nu$ by A2.6. If a ranges over 2^ν, then $c_\alpha^\#$ ranges over C_ν. Also, if $\tilde{f}$ ranges over $\mathbf{P}_\nu$, then f ranges over P_ν and since $P_\nu = P_\omega(\nu)$ (N being multiple-free), f^* ranges over $Aut_\omega\, Q^\nu$. Thus h_ν maps β_ν onto $Aut_\omega\, Q_\nu$. The fact that $C_\nu \cap Aut_\omega\, Q^\nu = (i)$ implies that each member of $Aut_\omega\, Q_\nu$ can be expressed in exactly one way as $c_\alpha^\# f^*$, with $c_\alpha^\# \in C_\nu$ and $f \in P_\nu$; thus the function h_ν is one-to-one.

Re (ii). Let for $x, y \in \beta_\nu$ the unique element $z \in \beta_\nu$ such that $h_\nu(z) = s_x s_y$, where $s_x = h_\nu(x)$, $s_y = h_\nu(y)$, be denoted by $t_\nu(x, y)$. Put $G_\nu = \langle \beta_\nu, t_\nu\rangle$, then $G_\nu \cong Aut_\omega\, Q_\nu$. In order to show that G_ν is an ω-group we define $\beta_\varepsilon, h_\varepsilon, t_\varepsilon$ in terms of ε as we defined β_ν, h_ν, t_ν in terms of ν. Put $G_\varepsilon = \langle \beta_\varepsilon, t_\varepsilon\rangle$, then $G_\nu \leqslant G_\varepsilon$ and it can be proved that G_ε is a *r.e.* group. Hence G_ν is an ω-group. We note in passing that h_ε maps G_ε onto a *proper* subgroup of $Aut_\omega\, Q_\varepsilon$, since $Req\; \varepsilon$ is not multiple-free, hence $P_\varepsilon \subset_+ P_\omega(\varepsilon)$. Clearly,

$$oG_\nu = Req\; \beta_\nu = Req\; 2^\nu \cdot Req\; P_\nu = 2^N \cdot N!$$

5 ***Concluding remarks*** (A) Uniformity. Let us call an ω-group *uniform,* if it is a subgroup of a *r.e.* group. Remmel [9] proved that an ω-group need not be uniform. Let ν be a nonempty set. Then $\mathbf{Z}_2(\nu) \leqslant \mathbf{Z}_2(\varepsilon)$ and $\mathbf{P}_\nu \leqslant \mathbf{P}_\varepsilon$, where $\mathbf{Z}_2(\varepsilon)$ and $\mathbf{P}_\varepsilon$ are *r.e.* groups, hence $\mathbf{Z}_2(\nu)$ and P_ν are uniform ω-groups. In view of the proof of the theorem of Section 4 we conclude that the groups $Aut_\omega\, Q^\nu$ and $Aut_\omega\, Q^\nu$ can be represented by uniform ω-groups, for every nonzero, multiple-free isol N.

(B) The simplex. The graph $G = \langle \beta, \eta\rangle$ is called an *ω-graph*, if it has a minimal path algorithm, i.e., if there is an effective procedure which enables us, given any two distinct vertices of G, to find a shortest path between them. It was proved in [5] that Q_ν is an ω-graph for every nonempty set ν. We briefly indicate how one can associate with every nonempty set ν an ω-graph S_ν which is related to a simplex as Q_ν is related to a cube. Put $\nu^* = \{2x \in \varepsilon | x \in \nu\} \cup (1)$. Define $S_\nu = \langle \nu^*, \eta\rangle$, where $\eta = [\nu^*; 2]$, i.e., let S_ν be the complete graph on ν^*. Clearly, $\mu \simeq \nu$ implies $\mu^* \simeq \nu^*$ and $S_\mu \cong_\omega S_\nu$. There is only one minimal path between two distinct vertices of S_ν, namely the edge joining them; thus S_ν is an ω-graph. Define $S_N = S_\nu$, for $\nu \in N$, $N \in \Omega_0$, then the ω-graph S_N is unique up to ω-isomorphism. We call N the *ω-dimension* of S_ν and S_N. If $S_\nu = \langle \nu^*, \eta\rangle$ we have $Req\; \nu^* = N + 1$ and $Req\; \eta = [N + 1; 2]$, the canonical extension of the recursive, combinatorial function $n(n + 1)/2$. Since $\eta = [\nu^*; 2]$ we see that every permutation of ν^* preserves adjacency, i.e., is an automorphism of S_ν. An automorphism of S_ν is called an *ω-automorphism,* if it has a partial recursive one-to-one extension. Thus $Aut\, S_\nu = Per(\nu^*)$ and $Aut_\omega\, S_\nu = Per_\omega(\nu^*)$. We conclude that for $\nu \in N$, $N \in \Lambda_0$ and N multiple-free, the group $Aut_\omega\, S_\nu$ can be represented by the uniform ω-group $\mathsf{P}_{\nu*}$ of order $(N + 1)!$.

(C) Opposite vertices. Call the RET $N = Req\ \nu$ *finite*, if the set ν is finite, but *infinite*, if the set ν is infinite. Define the *distance* $d(x, y)$ between the vertices x and y of Q^ν as card$(\rho_x \oplus \rho_y)$, i.e., as the number of components in which x and y differ, when they are interpreted as sequences of zeros and ones. If ν and N are finite, there is for every vertex x of Q^ν a unique *opposite* vertex y, i.e., a vertex y such that $d(x, y)$ assumes its maximal value, namely N. On the other hand, if ν and N are infinite, we have $\{d(x, y) \in \varepsilon \mid y \in 2^\nu\} = \varepsilon$, so that x has no opposite vertex. If we define a *diagonal* of Q^ν as a "line-segment" whose endpoints are vertices of Q^ν, but not of any r-dimensional face of Q^ν with $r < N$, then Q^ν has diagonals iff ν is finite, i.e., iff $N \in \varepsilon$. In fact, if N is finite, Q^ν has 2^{N-1} diagonals, since any two opposite vertices determine the same diagonal.

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