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A NEW EXTENSION OF S4

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In this paper it is shown that the addition to S4 of the axiom

Γ1 $MLp \to (LMp \to LMLp)$

generates a new system, to be called S4.01, that is contained in every known extension of S4 except S4.02 and S4.04. To prove this it suffices to derive **Γ1** in S4.1, S4.2, and Z1, since all S4-extensions other than S4.02 and S4.04 contain at least one of these three systems.

In the field of S4 there are a number of interesting formulae that are deductively equivalent to **Γ1**. These include

Γ2 $MLp \to L(LMp \to MLp)$
Γ3 $ML(p \to Lp) \to L(LMp \to MLp)$
Γ4 $(LMp \to MLp) \to L(LMp \to MLp)$

Proof:

(1)	$L(LMp \to MLp) \to (LMp \to LMLp)$	S4
(2)	$LML(p \to Lp) \to L(LMp \to MLp)$	S4
(3)	$LM(p \to Lp)$	S2
Γ2	$MLp \to L(LMp \to MLp)$	**Γ4**, PC
Γ1	$MLp \to (LMp \to LMLp)$	**Γ2**, (1)
(4)	$ML(p \to Lp) \to (LM(p \to Lp) \to LML(p \to Lp))$	**Γ1**, $p/p \to Lp$
Γ3	$ML(p \to Lp) \to L(LMp \to MLp)$	(2), (3), (4)
(5)	$(L \sim p \vee Lp) \to L(\sim p \vee Lp)$	S4
(6)	$M(Mp \to Lp) \to ML(p \to Lp)$	(5), C2
Γ4	$(LMp \to MLp) \to L(LMp \to MLp)$	**Γ3**, (6), C2

The substitution $p/\sim p$ in **Γ2**, and simple transformations show that yet another axiom for S4.01 is

Γ5 $LMp \vee L(LMp \to MLp)$

Γ1 is easily derivable from the S4.2 axiom

G2 $MLp \to LMLp$

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The derivation of **Γ1** from the **Z1** axiom

Z1 $\quad LMp \to (LMq \to (M(p\,.\,q) \to LM(p\,.\,q)))$

proceeds as follows

(7)	$LMp \to (LM(p \to Lp) \to (M(p\,.\,(p \to Lp)) \to LM(p\,.\,(p \to Lp))))$	**Z1**, $q/p \to Lp$
(8)	$MLp \to M(p\,.\,(p \to Lp))$	E2
(9)	$LM(p\,.\,(p \to Lp)) \to LMLp$	C2
Γ1	$LMp \to (MLp \to LMLp)$	(3), (7), (8), (9)

Now Sobociński [5], p. 372, has shown that

(10) $\quad ML(p \to Lp) \to LM(Mp \to Lp)$

is a theorem of **Z1**. But (10) is clearly deductively equivalent to **Γ3**, and so this yields an alternative proof that **Z1** contains S4.01.

The proof that **Γ1** is a consequence in S4 of the S4.1 axiom

N1 $\quad L(L(p \to Lp) \to p) \to (MLp \to p)$

is rather more complex. Letting $\nabla = (MLp \vee p) \to L(LMp \to MLp)$ we have[1]

(11)	$L(p \to q) \to (LMp \to Mq)$	E2
(12)	$L\nabla \to (LM(Mp \vee p) \to ML(LMp \to MLp))$	(11), $p/Mp \vee p$, $q/L\ (LMp \to MLp)$
(13)	$LMp \to LM(Mp \vee p)$	C2
(14)	$ML(LMp \to MLp) \to (LMp \to MLp)$	S4
(15)	$L\nabla \to (LMp \to (LMp \to MLp))$	(12), (13), (14)
(16)	$L\nabla \to (LMp \to MLp)$	(15), PC
(17)	$(\sim MLp\,.\sim p) \to \nabla$	PC
(18)	$((\nabla \to L\nabla)\,.\sim MLp\,.\sim p) \to L\nabla$	(17), PC
(19)	$(\nabla \to L\nabla) \to (\sim MLp \to (\sim p \to (LMp \to MLp)))$	(16), (18), PC
(20)	$L(\nabla \to L\nabla) \to (L \sim MLp \to (M \sim p \to M(LMp \to MLp)))$	(19), C2
(21)	$M(LMp \to MLp) \to (LMp \to MLp)$	S4
(22)	$L \sim MLp \to M \sim p$	E2
(23)	$\sim MLp \to L \sim MLp$	S4
(24)	$L(\nabla \to L\nabla) \to (\sim MLp \to (LMp \to MLp))$	(20), (21), (22), (23)
(25)	$L(\nabla \to L\nabla) \to (LMp \to MLp)$	(24), PC
(26)	$L(L(\nabla \to L\nabla) \to \nabla)$	(25), S4, PC, T°
(27)	$ML\nabla \to \nabla$	(26), **N1** p/∇
(28)	$MLp \to (LMp \to MLp)$	PC
(29)	$MLp \to MLL(LMp \to MLp)$	(28), S4

1. This result was first obtained by Mr. K. E. Pledger, using essentially the same substitution but a rather different method. Mr. Pledger also informs me that he has established that **Γ1** is the only instance of the schema $Ap \to (Bp \to Cp)$, where A, B, and C are affirmative S4 modalities, that is not deductively equivalent to the axiom of an already known S4-extension.

(30)	$MLp \rightarrow ML\nabla$	(29), C2
(31)	$MLp \rightarrow ((MLp \vee p) \rightarrow L(LMp \rightarrow MLp))$	(27), (30)
Γ1	$MLp \rightarrow (LMp \rightarrow LMLp)$	(1), (31), **PC**

Using the matrices of Sobociński [4], p. 350, we observe

1. $\mathfrak{M}8$ verifies **Γ1**, but rejects **Z1** ([5], p. 373).
2. $\mathfrak{M}7$ verifies **Γ1**, but rejects S4.02 ([6], p. 381) and hence also S4.04 and S4.1.
3. $\mathfrak{M}5$ verifies **Z1** ([5], p. 374) and hence **Γ1**, but rejects S4.2 ([4], p. 354).
4. $\mathfrak{M}11$ verifies S4.04 ([4], p. 354) and hence S4.02, but rejects **Γ1**. For $p = 10$, $ML10 \rightarrow (LM10 \rightarrow LML10) = 4 \rightarrow (1 \rightarrow 12) = 9$.

These considerations show that S4.01 is a proper extension of S4, properly contained in Z1, S4.1, and S4.2, and independent of S4.02 and S4.04.

We turn now to a characterisation of S4.01 in terms of Kripke-style frames, or model-structures. Extensive use will be made of filtration theory, as developed by Segerberg in [1], [2], and [3]. The reader is assumed to be familiar with the terminology, methods, and results of these three references.

We recall that $\langle X, R\rangle$ is an S4-frame if X is a non-empty set, and R is a reflexive, transitive, binary relation on X. A non-empty subset Y of X is a *cluster* if the relation R is universal over Y, and no extension of Y has this property. Y is a *proper* cluster if it contains at least two elements, and *simple* otherwise. x precedes y (y succeeds x) if xRy and not yRx. Y is a *final* cluster if no element of X succeeds any member of Y, and is a *last* cluster if every element of X either is in Y or precedes every member of Y.

Metatheorem. S4.01 *is characterised by the class of finite* S4-*frames in which every proper final cluster is last.*

Proof:

Soundness. Suppose that **Γ1** is false at a point $x \in X$ for some model on a frame as described. Then

$$MLp \text{ is true at } x \tag{a}$$
$$LMp \text{ is true at } x \tag{b}$$

and

$$LMLp \text{ is false at } x \tag{c}$$

From (a), there is some point z such that xRz and

$$Lp \text{ is true at } z \tag{d}$$

From (c), there is some y such that xRy and

$$MLp \text{ is false at } y \tag{e}$$

Since the frame is reflexive, transitive, and finite, there is a final cluster that either contains y or succeeds y ([1], p. 19). Let this cluster be called Y, and suppose $t \in Y$. Then yRt and so by (e)

$$Lp \text{ is false at } t \tag{f}$$

By transitivity, xRt, and so from (b)

$$Mp \text{ is true at } t \tag{g}$$

If Y is a simple cluster then it contains only t and, since it is final we have

$$tRw \text{ implies } t = w \tag{h}$$

From (g) and (h) we see that p is true at t and then from (h) again that Lp is true at t, which contradicts (f). On the other hand, if Y is a proper final cluster then by hypothesis it is last. Hence zRt and so by (d) (and transitivity) Lp is true at t, contrary to (f) again. Thus it follows that every finite S4-frame in which every proper final cluster is last must verify S4.01.

Completeness. Let A be any wff not derivable in S4.01. Then A is false at some point t in the canonical model for S4.01. It follows that A is false in the submodel $\mathfrak{U}$ generated from the canonical model by t ([3], p. 307). Let Ψ be the closure under modalities of the set of all subwff of A, and $\mathfrak{U}'$ a Lemmon filtration of $\mathfrak{U}$ through Ψ. Then $\mathfrak{U}'$ is finite ([1], section 3) and reflexive and transitive ([2], Chapter I, Theorem 7.6). Furthermore, by the Filtration Theorem ([3], p. 303) A is false at the point $[t]$ in $\mathfrak{U}'$. It remains only to show that every proper final cluster in $\mathfrak{U}'$ is last, and so the model verifies S4.01.

Suppose then that Y is a proper final cluster in $\mathfrak{U}'$. Then Y must contain at least two distinct points, say $[x]$ and $[y]$. Since $[x] \neq [y]$ there is some wff $C \in \Psi$ such that

$$C \text{ is true in } \mathfrak{U}' \text{ at } [x], \text{ but false in } \mathfrak{U}' \text{ at } [y] \tag{j}$$

Assume further that Y is not last in $\mathfrak{U}'$. Then there is some $[z]$ in $\mathfrak{U}'$ that is not in Y and does not precede Y. By the method of [2], Chapter II, Lemma 2.1 we can construct a Boolean combination B of members of Ψ such that

$$B \text{ is true in } \mathfrak{U} \text{ at } u \text{ iff } [u] \notin Y \tag{k}$$

We observe also that since $\mathfrak{U}$ is transitive and generated by t,

$$tRu \text{ for all } u \text{ in } \mathfrak{U} \tag{l}$$

Now if zRu in $\mathfrak{U}$, then $[z]\,R'\,[u]$ in $\mathfrak{U}'$, hence, since $[z]$ does not precede Y, $[u] \notin Y$. By (k) this gives B true at u. Thus LB, and so $L(B \vee C)$ is true at z, whence by (l),

$$ML(B \vee C) \text{ is true at } t \tag{m}$$

Now for any point u in $\mathfrak{U}$, if $[u] \notin Y$, then by (k) B is true at u. But $\mathfrak{U}$ is reflexive, so $M(B \vee C)$ is true at u. On the other hand, if $[u] \in Y$ then $[u]\,R'\,[x]$ and so by (j), MC is true in $\mathfrak{U}'$ at $[u]$. But $MC \in \Psi$ so by the

Filtration Theorem MC and hence $M(B \vee C)$ is true in $\mathfrak{U}$ at u. Thus $M(B \vee C)$ holds at every point in $\mathfrak{U}$ and so by (l)

$$LM(B \vee C) \text{ is true at } t \tag{n}$$

Now if $ML(B \vee C)$ is true at x, then there is some u such that xRu and

$$L(B \vee C) \text{ is true at } u \tag{o}$$

But $[x]\ R'\ [u]$ so $[u] \in Y$, as Y is final. Now if uRw, by (o) $(B \vee C)$ is true at w. But again by the finality of Y, $[w] \in Y$, whence by (k) B is false at w, and so C must be true at w. This implies that LC must be true in $\mathfrak{U}$ at u and therefore by the Filtration Theorem LC is true in $\mathfrak{U}'$ at $[u]$. Since $[u] \in Y$, $[u]\ R'\ [y]$, and so C is true at $[y]$, which contradicts (j). The upshot of all this is that $ML(B \vee C)$ cannot be true at x, and so by (l)

$$LML(B \vee C) \text{ is false at } t \tag{p}$$

But (m), (n), and (p) together contradict the fact that every substitution-instance of $\Gamma 1$ is true in $\mathfrak{U}$ at t. We conclude that in $\mathfrak{U}'$ every proper final cluster is last. Thus for any non-theorem of S4.01 we can find a falsifying model on a frame that verifies the system.

Corollary: S4.01 *is decidable.*

Proof: The model $\mathfrak{U}'$ of the above theorem has at most 2^{14n} elements, where n is the number of subwff of the non-theorem A (S4.01 has the same fourteen distinct modalities as S4). Thus any non-theorem of S4.01 is falsifiable in a model whose size is determined by the degree of complexity of the formula. So S4.01 has the finite model property, and being finitely axiomatisable is therefore decidable.

Since S4.01 is independent of S4.02 and S4.04 the question naturally arises as to whether any new systems result from the addition of $\Gamma 1$ to either of these last two systems. However this can be answered in the negative.

Segerberg [2], Chapter II, Lemmata 3.6 and 3.7 has established that S4.1 is characterised by the class of finite S4-frames in which

(i) every proper final cluster is last

and

(ii) there are no non-final proper clusters.

As we have seen, S4.01 corresponds to condition (i) above. The system corresponding to condition (ii) is in fact S4.02, whose proper axiom is

Ⱡ1 $\quad L(L(p \to Lp) \to p) \to (LMLp \to p)$

An adaptation of the proof of Lemma 3.1 of [2], Chapter II shows that any non-theorem of S4.02 is falsifiable in a model on a finite S4-frame in which every non-final cluster is simple. On the other hand the reader may check that any such frame verifies Ⱡ1.

These observations indicate that in S4, the combination of **Ł1** and **Γ1** is equivalent to the S4.1 axiom **N1**. That **Ł1** and **Γ1** are consequences of **N1** has been shown in [6] and this paper. For the converse:

(32)	$L \sim p \rightarrow (L(p \rightarrow Lp) . \sim p)$	E2
(33)	$(L(p \rightarrow Lp) \rightarrow p) \rightarrow \sim L \sim p$	(32), PC
(34)	$L(L(p \rightarrow Lp) \rightarrow p) \rightarrow LMp$[2]	(33), C2

Then reasoning by deduction from hypotheses we have

(35)	$L(L(p \rightarrow Lp) \rightarrow p)$	Hyp.
(36)	MLp	Hyp.
(37)	$LMp \rightarrow LMLp$	**Γ1**, (36), PC
(38)	LMp	(34), (35), PC
(39)	$LMLp$	(37), (38), PC
(40)	$LMLp \rightarrow p$	(35), **Ł1**, PC
(41)	p	(39), (40), PC

Thus by the deduction theorem in S4, **N1** is provable in the system S4.02 + **Γ1**. Whence S4.02 + **Γ1** = S4 + **N1** = S4.1. Since S4.04 contains S4.02 it is immediate that S4.04 + **Γ1** = S4.04 + **N1** = S4.1.2 and no new systems result from the addition of **Γ1** to any proper extension of S4.

We end this paper with a diagram[3] that displays the relationships between S4.01 and the other extensions of S4 that have appeared in the literature at the time of writing. A straight line joining two systems, whether horizontal or sloping, indicates that the leftmost system properly contains the rightmost.

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[4] Sobociński, B., "Certain extensions of modal system S4," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 347-368.

2. Formula (34) is due to Professor G. E. Hughes, to whom I am indebted for some valuable discussions on the considerations of this paper. He had previously produced a formula,

$$L(L(LMp \rightarrow MLp) \rightarrow p) \rightarrow (MLp \rightarrow p),$$

which he has subsequently established is equivalent in S4 to my **Γ1**.

3. See p. 574 below.

[5] Sobociński, B., "A new class of modal systems," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 371-377.

[6] Sobociński, B., "A proper subsystem of S4.04," *Notre Dame Journal of Formal Logic*, vol. XII (1971), pp. 381-384.

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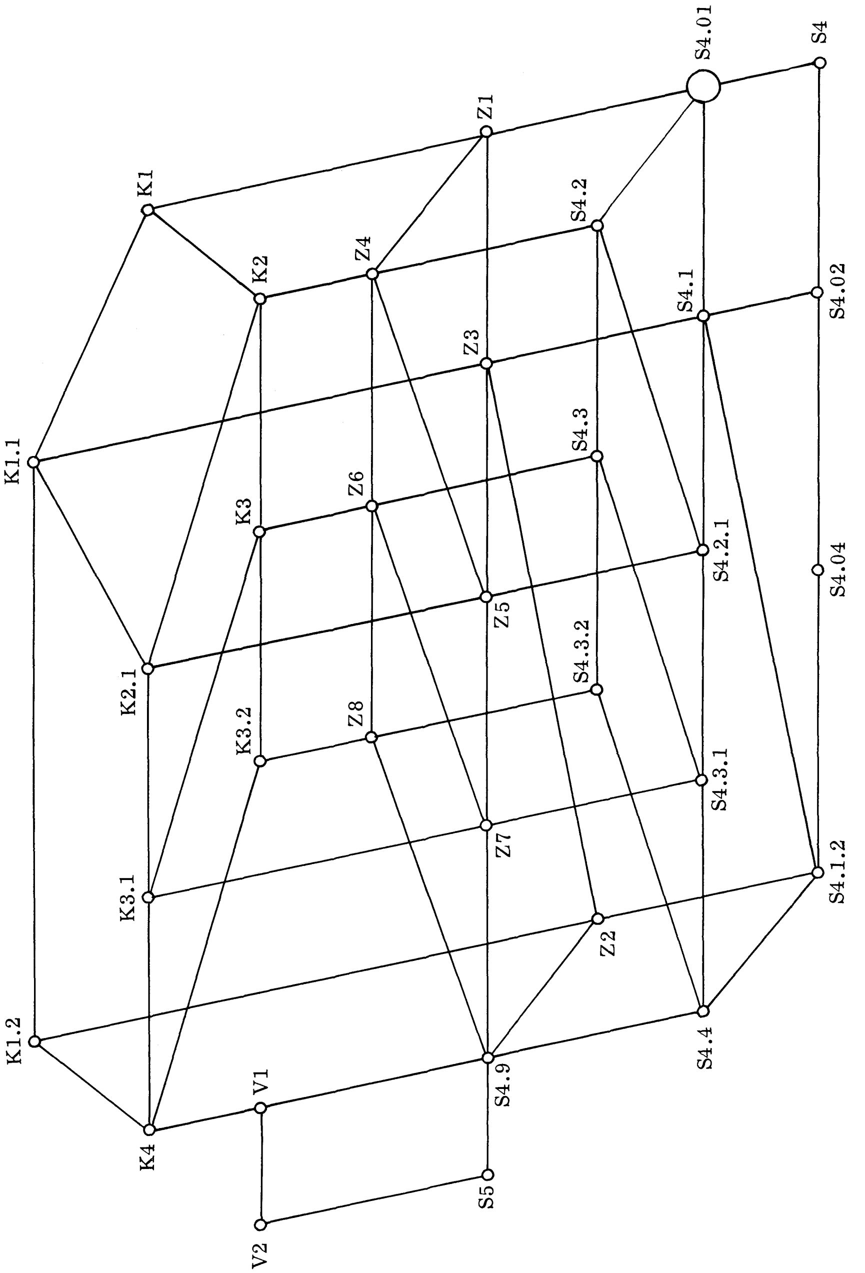
S4
S4.01
S4.02
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Z1
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Z7
S4.1.2
K3.1
Z2
K1.2
S4.4
V1
S4.9
K4
S5
V2

DIAGRAM