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SOME RESULTS AND ALGEBRAIC APPLICATIONS IN THE THEORY OF HIGHER-ORDER ULTRAPRODUCTS

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Introduction Perhaps the chief result of this paper is the higher-order extension, (Theorems 3.11, 3.12), via the ultraproduct construction, of a first-order embedding theorem of Robinson, *cf.* [7], p. 34, Theorem 2.4.1.

Section **1** summarises the higher-order ultraproduct construction and gives a partial answer to the question of necessary and sufficient conditions for the preservation of the 'fullness' property by that construction. Section **2** provides an extension of Łoś's theorem for a first-order ultraproduct and an associated formal language to a higher-order ultraproduct and an associated higher-order language involving a special class of formulae of infinite length. Section **3** develops a number of results involving subsystems of higher-order systems and leads to the embedding theorems. Section **4** illustrates some of these results in two algebraic situations. The first is Stone's representation theorem for non-finite boolean algebras and the second, properties of Sylow (maximal) p-subgroups of locally normal groups.

Terminology Let $\mathbf{T}$ be the class of finite types as described in Kreisel and Krivine [5], pp. 95-101. A (relational) system of order $\tau \in \mathbf{T}$, (hereafter called a τ-system), is a collection $M = \{\mathbf{E}^\sigma | \sigma \leqslant \tau\} \cup \{\epsilon^\sigma | \sigma \leqslant \tau, \sigma \neq 0\} \cup \langle R_1, . ., R_p, . .\rangle$, where $\{\mathbf{E}^\sigma | \sigma \leqslant \tau\}$ is a collection of non-empty, mutually disjoint classes; for each $\sigma \leqslant \tau$, $\sigma = (\sigma_1, . ., \sigma_n)$, ϵ^σ is an $n+1$-placed 'membership' relation defined on $(\mathbf{E}^{\sigma_1} \times . . \times \mathbf{E}^{\sigma_n}) \times \mathbf{E}^\sigma$; and each R_p is an n-placed relation on some $\mathbf{E}^{\sigma_1} \times . . \times \mathbf{E}^{\sigma_n}$, $\sigma_1, . ., \sigma_n \leqslant \tau$. Such an R_p is said to be of type $(\sigma_1, . ., \sigma_n)$. If $\sigma = (\sigma_1, . ., \sigma_n)$, $\sigma \leqslant \tau$ and R_p is a relation of type $(\sigma_1, . ., \sigma_n)$ then R_p may be regarded as a nominated member of $\mathbf{E}^\sigma$.[1] If $(a_1, . ., a_n) \in \mathbf{E}^{\sigma_1} \times . . \times \mathbf{E}^{\sigma_n}$ and $a \in \mathbf{E}^\sigma$ then $(a_1, . ., a_n)$ is said to 'belong' to a, written $(a_1, . ., a_n) \epsilon^\sigma a$, if, and only if, $\epsilon^\sigma (a_1, . ., a_n, a)$; that is if, and only if, $a_1, . ., a_n, a$ are related by ϵ^σ. $\mathbf{E}^0$ is the class of individuals of M. The

1. If R_p is 0-placed of type 0 it will be regarded as a nominated member of $\mathbf{E}^0$.

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members of the classes E^σ, $\sigma \leqslant \tau$, are the objects of M. The R_p's are called the constant relations of M.

If $N = \{\mathsf{F}^\sigma | \sigma \leqslant \tau_1\} \cup \{\epsilon^\sigma | \sigma \leqslant \tau_1, \sigma \neq 0\} \cup \langle S_1, \ldots, S_p, \ldots\rangle$ is a τ_1-system then M and N are said to be *similar* if $\tau = \tau_1$, and if, for every p, the corresponding relations R_p and S_p are both k-placed, for some integer k, and of type $(\sigma_1, \ldots, \sigma_k)$ for some $\sigma_1, \ldots, \sigma_k \leqslant \tau$. The class of all systems similar to M is called the similarity class of M.

M is called a *normal* structure if, for all $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, and for all $a, b \in \mathsf{E}^\sigma$, $a = b$ if, and only if, $\hat{a} = \hat{b}$, where $\hat{a} = \{(a_1, \ldots, a_n) | (a_1, \ldots, a_n) \epsilon^\sigma a\}$ and $\hat{b}$ is defined similarly. Unless otherwise stated all systems later discussed will be assumed normal.

$\mathsf{L}^\tau(M)$ is a formalized logic associated with the similarity class of M, where $\mathsf{L}^\tau(M)$ has, for each $\sigma \leqslant \tau$, a countable class of variable symbols of type σ, viz $\{x^\sigma, y^\sigma, \ldots x_1^\sigma, y_1^\sigma, \ldots\}$; for each $\sigma \leqslant \tau$, $\sigma \neq 0$, a 'membership' relation symbol ϵ^σ; and $\langle R_1, \ldots, R_p \ldots,\rangle$ a sequence of constant relation symbols. (No confusion will be caused by using 'ϵ^σ,' 'R_p' to denote both elements of M and of $\mathsf{L}^\tau(M)$.) For each $\sigma \leqslant \tau$, $\mathsf{L}^\tau(M)$ will have an identity symbol, $=$.

A standard interpretation of $\mathsf{L}^\tau(M)$ with respect to any member, N, of the similarity class of M will be one in which each symbol ϵ^σ of $\mathsf{L}^\tau(M)$ denotes the 'membership' relation, ϵ^σ, of N; each symbol R_p denotes the relation R_p of N; and each identity symbol of type σ of $\mathsf{L}^\tau(M)$ denotes the identity relation on F^σ of N.

Let $\mathfrak{a} = \langle a_1, a_2, \ldots\rangle$ be a sequence of objects of M. If ϕ is a formula with free variables $x_{i_1}^{\sigma_1}, \ldots, x_{i_n}^{\sigma_n}$, then $\mathfrak{a}$ is said to be ϕ-allowable if $a_{i_k} \in \mathsf{E}^{\sigma_k}$, $1 \leqslant k \leqslant n$. A ϕ-allowable sequence $\mathfrak{a}$ is said to *satisfy* ϕ in M, written $M \models \phi(\mathfrak{a})$, (or if ϕ is written $\phi(x_{i_1}^{\sigma_1}, \ldots, x_{i_n}^{\sigma_n})$ then $M \models \phi(\mathfrak{a})$ may be alternatively written as $M \models \phi(a_{i_1}, \ldots, a_{i_n})$), if the sequence $\langle a_{i_1}, \ldots, a_{i_n}\rangle$ satisifes ϕ in M under the assignment of a_{i_k} to $x_{i_k}^{\sigma_k}$, $1 \leqslant k \leqslant n$. ϕ holds in M, $M \models \phi$, if for all ϕ-allowable sequences $\mathfrak{a}$, $M \models \phi(\mathfrak{a})$.

1 *Higher Order Ultraproducts* Let $\{M_i | i \in I\}$ be a family of τ-systems belonging to the same similarity class. That is, for each $i \in I$, let $M_i = \{\mathsf{E}_i^\sigma | \sigma \leqslant \tau\} \cup \{\epsilon^\sigma | \sigma \leqslant \tau, \sigma \neq 0\} \cup \langle R_1, \ldots, R_p, \ldots\rangle$. If X is an ultrafilter over I then the ultraproduct of the family is the τ-system $\pi M_i/X = \{\pi \mathsf{E}_i^\sigma/X | \sigma \leqslant \tau\} \cup \{\epsilon^\sigma | \sigma \leqslant \tau, \sigma \neq 0\} \cup \langle R_1, \ldots, R_p, \ldots\rangle$. For each $\sigma \leqslant \tau$, $\pi \mathsf{E}_i^\sigma/X$ is the set of equivalence classes of the cartesian product $\underset{i \in I}{\pi} \mathsf{E}_i^\sigma = \left\{f \,|\, f: I \to \bigcup \{\mathsf{E}_i^\sigma | i \in I\}, f(i) \in \mathsf{E}_i^\sigma\right\}$ under the equivalence relation defined by: $f \sim g$ if, and only if, $\{i \,|\, f(i) = g(i)\} \in X$. The equivalence class of f is denoted by $\bar{f}$. For each $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, ϵ^σ is defined in $\pi M_i/X$ by: $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon^\sigma \bar{f}$ if, and only if, $\{i \,|\, (f_1(i), \ldots, f_n(i)) \epsilon^\sigma f(i)\} \in X$. Similarly each R_p, (where, for each M_i, R_p is k-placed and of type $(\sigma_1, \ldots, \sigma_k)$ say) is defined in $\pi M_i/X$ by, $R_p(\bar{f}_1, \ldots, \bar{f}_k)$ if, and only if, $\{i \,|\, R_p(f_1(i), \ldots, f_k(i))\} \in X$.

The necessary lemmas to support the above definitions are assumed. It is noted that $\pi M_i/X$ belongs to the same similarity class as the M_i. The requirement of similarity for the family $\{M_i \,|\, i \in I\}$ is not a necessary one for the definition of the ultra-product. A relaxation of the similarity condition in the case of the first-order ultraproduct construction is discussed in a paper by the author [6]. The method extends to higher order systems, if desired.

Theorem 1.1 *If each member of* $\{M_i \mid i \epsilon I\}$ *is a normal system then so is* $\pi M_i/X$.

Proof: Take $\bar{f}, \bar{g} \epsilon \mathbf{E}^\sigma$, $\sigma \leqslant \tau$. Let $F = \{i \mid f(i) = g(i)\}$. Assume $\bar{f} = \bar{g}$, that is $F \epsilon X$. Now $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon^\sigma \bar{f}$ if, and only if, $G \epsilon X$, where $G = \{i \mid (f_1(i), \ldots, f_n(i)) \epsilon^\sigma f(i)\}$. But each M_i is normal and so $H \supseteq G \cap F$, where $H = \{i \mid (f_1(i), \ldots, f_n(i)) \epsilon^\sigma g(i)\}$. Hence $H \epsilon X$ and so $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon^\sigma \bar{g}$. Similarly $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon^\sigma \bar{f}$ if $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon^\sigma \bar{g}$ and so $\hat{\bar{f}} = \hat{\bar{g}}$.

Conversely, assume $\bar{f} \neq \bar{g}$, that is $F \notin X$ and so $\complement F \epsilon X$. Now as each M_i is normal there exists for each $i \epsilon \complement F$, $(a_1^i, \ldots, a_n^i) \epsilon \mathbf{E}_i^{\sigma_1} \times \ldots \times \mathbf{E}_i^{\sigma_n}$ such that $(a_1^i, \ldots, a_n^i)$ 'belongs' to one, and only one, of $f(i)$, $g(i)$. For each $i \epsilon \complement F$ define $f_j(i) = a_j^i$, $1 \leqslant j \leqslant n$. Thus $\bar{f}_j$, $1 \leqslant j \leqslant n$, are well defined as $\complement F \epsilon X$. Let $F_0 = \{i \mid (f_1(i), \ldots, f_n(i)) \epsilon^\sigma f(i)\}$ and $G_0 = \{i \mid (f_1(i), \ldots, f_n(i)) \epsilon^\sigma g(i)\}$. Now $(\complement F \cap F_0) \cup (\complement F \cap G_0) = \complement F$ and $(\complement F \cap F_0) \cap (\complement F \cap G_0) = \emptyset$. Therefore one, and only one, of F_0, G_0 belongs to X; that is $(\bar{f}_1, \ldots, \bar{f}_n)$ 'belongs' to one, and only one, of $\bar{f}, \bar{g}$. Thus $\hat{\bar{f}} \neq \hat{\bar{g}}$. Q.E.D.

A τ-system M is termed *full* if, for each $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, and for each subclass $\mathbf{K}$ of $\mathbf{E}^{\sigma_1} \times \ldots \times \mathbf{E}^{\sigma_n}$, there exists an object $a \epsilon \mathbf{E}^\sigma$ such that $\hat{a} = \mathbf{K}$. The next three theorems discuss the fullness of the ultraproduct of a family of full systems.

Theorem 1.2 *Let* $\{M_i \mid i \epsilon I\}$ *be a family of similar and full* τ*-systems. If* X *is a given ultrafilter over* I *and* $\pi M_i/X$ *the ultraproduct then for each* $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, *and for each subclass* K *of* $\mathbf{E}^{\sigma_1} \times \ldots \times \mathbf{E}^{\sigma_n}$, *there exists some* $\bar{f} \epsilon \mathbf{E}^\sigma$ *such that* $\mathbf{K} \subseteq \hat{\bar{f}}$.

Proof: For each $i \epsilon I$, let $K_i = \{(f_1(i), \ldots, f_n(i)) \mid (\bar{f}_1, \ldots, \bar{f}_n) \epsilon \mathbf{K}\}$. But each M_i is full and so there exists some object $a_i \epsilon \mathbf{E}_i^\sigma$ such that $\hat{a}_i = K_i$. Define $\bar{f} \epsilon \mathbf{E}^\sigma$ by $f(i) = a_i$, for each $i \epsilon I$. Take any $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon \mathbf{K}$. Hence $\{i \mid (f_1(i), \ldots, f_n(i)) \epsilon^\sigma f(i)\} = I$. But $I \epsilon X$ and so $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon^\sigma \bar{f}$. Thus $\mathbf{K} \subseteq \hat{\bar{f}}$. Q.E.D.

Theorem 1.3 *Let* $\{M_i \mid i \epsilon I\}$ *be a family of similar and full* τ*-systems. Let* X *be an ultrafilter over* I *and* $\pi M_i/X$ *is the resulting ultraproduct. If* $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, *and if* $\mathbf{K}$ *is a subclass of* $\mathbf{E}^{\sigma_1} \times \ldots \times \mathbf{E}^{\sigma_n}$ *such that* $|\mathbf{K}| = \beta$, (that is the cardinality of $\mathbf{K}$ is β), *then there exists no* $\bar{f} \epsilon \mathbf{E}^\sigma$ *such that* $\hat{\bar{f}} = \mathbf{K}$ *only if* X *is* β*-incomplete.*

Proof: Let the members of $\mathbf{K}$ be indexed by β, that is $\mathbf{K} = \{(\bar{g}_1, \ldots, \bar{g}_n)_j \mid j < \beta\}$. Further, for each $j < \beta$, let $(g_1, \ldots, g_n)$ be an arbitary but fixed representation of $(\bar{g}_1, \ldots, \bar{g}_n)$. Let $K_i = \{(g_1(i), \ldots, g_n(i))_j \mid j < \beta\}$, each $i \epsilon I$ and as in Theorem 1.2 let $\bar{f} \epsilon \mathbf{E}^\sigma$ be defined such that $f(i) = K_i$, each $i \epsilon I$. Thus $K \subseteq \hat{\bar{f}}$.

Assume there exists no $\bar{g} \epsilon \mathbf{E}^\sigma$ such that $\hat{\bar{g}} = \mathbf{K}$ and so there exists some $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon \mathbf{E}^{\sigma_1} \times \ldots \times \mathbf{E}^{\sigma_n}$ such that $(\bar{f}_1, \ldots, \bar{f}_n) \epsilon \hat{\bar{f}}$ but is not a member of $\mathbf{K}$. Let $F = \{i \mid (f_1(i), \ldots, f_n(i)) \epsilon^\sigma f(i)\}$ and so $F \epsilon X$. Let $F_j = F \cap \{i \mid (f_1(i), \ldots, f_n(i)) =$

$(g_1(i), \ldots, g_n(i))_j\}$, for all $j < \beta$. Now $F_j \notin X$, $j < \beta$, as $(\bar{f}_1, \ldots, \bar{f}_n) \notin \mathbf{K}$. Further, $\bigcup\{F_j \mid j < \beta\} = F$ as, for all $i \in I$, $f(i) = K_i$, and the K_i have been defined using only the fixed representations of the members of $\mathbf{K}$. Therefore $\bigcap\{\complement F_j \mid j < \beta\} \cap F = \emptyset$ and hence X is β-incomplete. Q.E.D.

The question as to whether the incompleteness of the ultrafilter X guarantees the non-fullness of the ultraproduct does not seem to have an immediate answer. The next theorem is a possible step towards such an answer.

Theorem 1.4 *Let $\{M_i \mid i \in I\}$ be a family of similar and full τ-systems. Let X be a β-incomplete ultrafilter over I. $\pi M_i/X$ is the resulting ultraproduct. If for $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, there exists some $\mathbf{K} \subseteq \mathbf{E}^{\sigma_1} \times \ldots \times \mathbf{E}^{\sigma_n}$, say $\mathbf{K} = \{(\bar{g}_1, \ldots, \bar{g}_n)_j \mid j < \alpha\}$, $\beta \leqslant \alpha$, such that $G \in X$, where $G = \bigcap\{\complement F_{m,n} \mid m, n < \beta, m \neq n\}$, and $F_{m,n} = \{i \mid (g_1(i), \ldots, g_n(i))_m \hat{=} (g_1(i), \ldots, g_n(i))_n\}$, all $m, n < \beta$, $m \neq n$, then there exists no $\bar{f} \in \mathbf{E}^\sigma$ such that $\hat{\bar{f}} = \mathbf{K}$.*

Proof: As X is β-incomplete let $\{H_k \mid k < \beta\}$ be a family of members of X such that $\bigcap\{H_k \mid k < \beta\} = \emptyset$. Assume there exists $\bar{f} \in \mathbf{E}^\sigma$ such that $\hat{\bar{f}} = \mathbf{K}$. Thus, for each $j < \alpha$, $(\bar{g}_1, \ldots, \bar{g}_n)_j \in^\sigma \bar{f}$ if, and only if, $(\bar{g}_1, \ldots, \bar{g}_n)_j \in \mathbf{K}$.

For each $j < \beta$ put $G_j = \{i \mid (g_1(i), \ldots, g_n(i))_j \in^\sigma f(i)\}$, $G_j' = G_j \cap G$ and $H_j' = G_j' \cap H_j$. Thus G_j', $H_j' \in X$ and $\bigcup\{\complement' H_j' \mid j < \beta\} = G$, where $\complement' H_j' = G \cap \complement H_j'$. Now define $(\bar{f}_1, \ldots, \bar{f}_n) \in \mathbf{E}^{\sigma_1} \times \ldots \times \mathbf{E}^{\sigma_n}$ as follows: For all $i \in \complement' H_0'$ put $(f_1(i), \ldots, f_n(i)) = (g_1(i), \ldots, g_n(i))_0$. Assume $(f_1(i), \ldots, f_n(i))$ has been defined for all $i \in \bigcup\{\complement' H_j' \mid j < \delta\}$, for some $\delta < \beta$, and define $(f_1(i), \ldots, f_n(i)) = (g_1(i), \ldots, g_n(i))_\delta$, for all $i \in \bigcap\{H_j' \mid j < \delta\} - H_\delta'$. By transfinite induction $(f_1(i), \ldots, f_n(i))$ is defined for all $i \in G$, as $\bigcup\{\complement' H_j' \mid j < \beta\} = G$. Hence $(\bar{f}_1, \ldots, \bar{f}_n)$ is well defined as $G \in X$.

But $(\bar{g}_1, \ldots, \bar{g}_n)_j \neq (\bar{f}_1, \ldots, \bar{f}_n)$, for any $j < \beta$, as $\{i \mid (g_1(i), \ldots, g_n(i))_j = (f_1(i), \ldots, f_n(i))\} \cap G = \bigcap\{H_k' \mid k < j\} \cap \complement H_j'$, and $\complement H_j' \notin X$. Hence $(\bar{f}_1, \ldots, \bar{f}_n) \notin \mathbf{K}$. But $(\bar{f}_1, \ldots, \bar{f}_n) \in^\sigma \hat{\bar{f}}$ as $\{i \mid (f_1(i), \ldots, f_n(i)) \in^\sigma f(i)\} \supseteq G$. This contradicts the assumption that $\hat{\bar{f}} = \mathbf{K}$ and hence the theorem is established. Q.E.D.

2 *Ultraproducts and an Associated Higher-Order Language.* Let $\{M_i \mid i \in I\}$ be a family of τ-systems of the same similarity class. $\mathcal{L}^\tau$ is the formalized language associated with this similarity class, as described in the introduction. If $\mathfrak{a} = \langle \bar{f}_1, \bar{f}_2, \ldots \rangle$ is a sequence of elements where each member of the sequence is an object of $\pi M_i/X$, for some ultrafilter X, then $\mathfrak{a}(i) = \langle f_1(i), f_2(i), \ldots \rangle$ is the associated sequence of objects of M_i, each $i \in I$. The first theorem of this paragraph is the natural extension of Łoś's theorem for a first order ultraproduct and associated language.

Theorem 2.1 *Let X be a given ultrafilter over I. If ϕ is any well formed formula (wff) of $\mathcal{L}^\tau$ and $\mathfrak{a} = \langle \bar{f}_1, \bar{f}_2, \ldots \rangle$ any ϕ-allowable sequence then $\pi M_i/X \vDash \phi(\mathfrak{a})$ if, and only if, $\{i \mid M_i \vDash \phi(\mathfrak{a}(i))\} \in X$.*

Proof: The details of proof are straightforward extensions of those for the first order theorem—for which see Kochen [3], pp. 226-229, Theorem 5.1.

Corollary *If* ϕ *is a sentence of* $\mathcal{L}^{\tau}$ *then* $\pi M_i/X \models \phi$ *if, and only if,* $\{i \mid M_i \models \phi\} \in X$.

Proof: Immediate from Theorem 2.1.

For the purpose of later application (see footnote on page 15), the language $\mathcal{L}^{\tau}$ is extended to include a wider class of formulae, developed relative to $\pi M_i/X$ as follows: Let $\{\phi_t \mid t \in \alpha\}$ be any class of wff's of $\mathcal{L}^{\tau}$ such that (i) only a finite number of distinct free variables occur in all of the ϕ_t, $t \in \alpha$; (ii) for any ϕ_t-allowable sequence, $\mathfrak{a}$, (because of (i) any sequence allowable for one ϕ_t will be allowable for all), and for all $k \in \alpha$, if there exists some $j \in I$ such that $M_j \models \phi_k(\mathfrak{a}(j))$ then $\{i \mid M_i \models \phi_k(\mathfrak{a}(i))\} \in X$. The infinite disjunction $\bigvee_{t \in \alpha} \phi_t$ will be a $(\pi M_i/X)$ allowable formula.

Formulae generated by the rules of formation of $\mathcal{L}^{\tau}$ from the wffs of $\mathcal{L}^{\tau}$ together with the $(\pi M_i/X)$ allowable disjunctions will comprise the wider class of formulae of $\mathcal{L}^{\tau}$. $\mathcal{L}'^{\tau}$ $(\pi M_i/X)$, or in context just $\mathcal{L}'^{\tau}$, will denote $\mathcal{L}^{\tau}$ with this wider class of formulae.

Theorem 2.2 *Let* X *be a given ultrafilter over* I. *If* ϕ *is any wff of* $\mathcal{L}'^{\tau}$ *and* $\mathfrak{a}$ *any* ϕ*-allowable sequence of objects of* $\pi M_i/X$ *then* $\pi M_i/X \models \phi(\mathfrak{a})$ *if, and only if,* $\{i \mid M_i \models \phi(\mathfrak{a}(i))\} \in X$.

Proof: In view of the inductive procedures of the proof of Theorem 2.1 it is necessary only to consider the case where ϕ is of the form $\bigvee_{t \in \alpha} \phi_t$ as described above. First assume that $\pi M_i/X \models \bigvee_{t \in \alpha} \phi_t(\mathfrak{a})$ and so by the semantical rules for a disjunction there exists some $k \in \alpha$ such that $\pi M_i/X \models \phi_k(\mathfrak{a})$. Hence from Theorem 2.1, $\{i \mid M_i \models \phi_k(\mathfrak{a}(i))\} \in X$ and so $\left\{i \mid M_i \models \bigvee_{t \in \alpha} \phi_t(\mathfrak{a}(i))\right\} \in X$.

Conversely, assume that $\left\{i \mid M_i \models \bigvee_{t \in \alpha} \phi_t(\mathfrak{a}(i))\right\} \in X$. Hence there exists some $j \in I$ such that $M_j \models \bigvee_{t \in \alpha} \phi_t(\mathfrak{a}(i))$ and thus some $k \in \alpha$ such that $M_j \models \phi_k(\mathfrak{a}(j))$. Therefore $\{i \mid M_i \models \phi_k(\mathfrak{a}(i))\} \in X$ and so $\pi M_i/X \models \phi_k(\mathfrak{a})$. Therefore $\pi M_i/X \models \bigvee_{t \in \alpha} \phi_t(\mathfrak{a})$. Q.E.D.

Corollary *If* ϕ *is a sentence of* $\mathcal{L}'^{\tau}$ *then* $\pi M_i/X \models \phi$ *if, and only if,* $\{i \mid M_i \models \phi\} \in X$.

Proof: Immediate from Theorem 2.2.

Theorem 2.3 *Let* X *be a given ultrafilter over* I. *If* K *is a class of wffs of* $\mathcal{L}'^{\tau}$ *and* $\mathfrak{a}$ *any sequence of objects of* $\pi M_i/X$, *(allowable for all members of* K*), then* $\pi M_i/X \models \mathsf{K}(\mathfrak{a})$ *if* $\{i \mid M_i \models \mathsf{K}(\mathfrak{a}(i))\} \in X$.

Proof: Assume $\{i \mid M_i \models \mathsf{K}(\mathfrak{a}(i))\} \in X$. Thus for all $\phi \in \mathsf{K}$, $\{i \mid M_i \models \phi(\mathfrak{a}(i))\} \in X$ and hence $\pi M_i/X \models \phi(\mathfrak{a})$. That is $\pi M_i/X \models \mathsf{K}(\mathfrak{a})$. Q.E.D.

Corollary *If* K *is a collection of sentences of* $\mathcal{L}''^{\tau}$ *then*

$$\pi M_i/X \models \mathsf{K} \text{ if } \{i \mid M_i \models \mathsf{K}\} \epsilon X.$$

Proof: Immediate from Theorem 2.3.

The final theorem of this paragraph is a partial converse to Theorem 2.3.

Theorem 2.4 *Let* X *be a given ultrafilter over* I. *If* K *is a set of wffs of* $\mathcal{L}''^{\tau}$ *such that* $|K| = \beta$ *and* X *is* β*-complete then, for any allowable sequence* $\mathfrak{a}$, $\pi M_i/X \models K(\mathfrak{a})$ *only if* $\{i \mid M_i \models K(\mathfrak{a}(i))\} \epsilon X$.

Proof: Assume $\pi M_i/X \models K(\mathfrak{a})$ and so, for all $\phi \epsilon K$, $\pi M_i/X \models \phi(\mathfrak{a})$. Let $F_\phi = \{i \mid M_i \models \phi(\mathfrak{a}(i))\}$, for all $\phi \epsilon K$. Now $\{i \mid M_i \models K(\mathfrak{a}(i))\} \supseteq \bigcap\{F_\phi \mid \phi \epsilon K\}$. But each $F_\phi \epsilon X$ and X is β-complete. Thus $\{i \mid M_i \models K(\mathfrak{a}(i))\} \epsilon X$. Q.E.D.

Corollary *If* K *is a class of sentences of* $\mathcal{L}''^{\tau}$ *such that* $|\mathsf{K}| = \beta$ *and* X *is* β*-complete then* $\pi M_i/X \models \mathsf{K}$ *only if* $\{i \mid M_i \models \mathsf{K}(\mathfrak{a}(i))\} \epsilon X$.

Proof: Immediate from Theorem 2.4.

3 *Substructures and Embeddings* Let $M = \{\mathsf{E}^\sigma \mid \sigma \leqslant \tau\} \cup \{\epsilon^\sigma \mid \sigma \leqslant \tau, \sigma \neq 0\} \cup \langle R_1, \ldots\rangle$, and $N = \{\mathsf{F}^\sigma \mid \sigma \leqslant \tau\} \cup \{\epsilon^\sigma \mid \sigma \leqslant \tau, \sigma \neq 0\} \cup \langle S_1, \ldots\rangle$ be two normal τ-systems. N is called a *subsystem* of M if (i) $\mathsf{F}^0 \subseteq \mathsf{E}^0$; (ii) for each $\sigma \leqslant \tau$, $\sigma \neq 0$, there exists a surjective map $p : \mathsf{E}^\sigma \to \mathsf{F}^\sigma$, and for $\sigma = 0$; $p : \mathsf{F}^0 \to \mathsf{F}^0$ is the identity map, such that for all $(p(a_1), \ldots, p(a_n)) \epsilon \mathsf{F}^{\sigma_1} \times \ldots \times \mathsf{F}^{\sigma_n}$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, and all $p(a) \epsilon \mathsf{F}^\sigma$, $(p(a_1), \ldots, p(a_n)) \epsilon^\sigma p(a)$ if, and only if, there exists some $(a_1', \ldots, a_n') \epsilon \mathsf{E}^{\sigma_1} \times \ldots \times \mathsf{E}^{\sigma_n}$ such that $p(a_k) = p(a_k')$, $1 \leqslant k \leqslant n$, and $(a_1', \ldots, a_n') \epsilon^\sigma a$; (iii) there exists a surjective map $p : \langle R_1, \ldots\rangle \to \langle S_1, \ldots\rangle$ such that if R_t is a relation of type $(\sigma_1, \ldots, \sigma_n)$, $\sigma_1, \ldots, \sigma_n \leqslant \tau$, then $p(R_t)$ is of the same type and for all $(p(a_1), \ldots, p(a_n)) \epsilon \mathsf{F}^{\sigma_1} \times \ldots \times \mathsf{F}^{\sigma_n}$, $p(R_t)(p(a_1), \ldots, p(a_n))$ if, and only if, there exists $(a_1', \ldots, a_n') \epsilon \mathsf{E}^{\sigma_1} \times \ldots \times \mathsf{E}^{\sigma_n}$ such that $p(a_k) = p(a_k')$, $1 \leqslant k \leqslant n$, and $R_t(a_1', \ldots, a_n')$. The family of maps is denoted by $\mathfrak{p} : M \to N$ and called the canonical projection of M to the subsystem N.

Theorem 3.1 *If* N *is a subsystem of the* τ*-system* M *then the canonical projection* $\mathfrak{p} : M \to N$ *is unique.*

Proof: Let $\mathfrak{p}_1 : M \to N$, $\mathfrak{p}_2 : M \to N$ be two canonical projections. Now, for $\sigma = 0$, $p_1 = p_2$ as both are the identity map on F^0. Assume for all $\sigma_i < \sigma$, $\sigma \leqslant \tau$, that $p_1 = p_2$ on E^{σ_i}, (on F^0 if $\sigma_i = 0$). Now to show $p_1 = p_2$ on E^σ.

For any $a \epsilon \mathsf{E}^\sigma$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, consider $p_1(a)$, $p_2(a)$. Take $(p_1(a_1), \ldots, p_1(a_n)) \epsilon^\sigma p_1(a)$. Therefore there exists some $(a_1', \ldots, a_n') \epsilon \mathsf{E}^{\sigma_1} \times \ldots \times \mathsf{E}^{\sigma_n}$ such that $(a_1', \ldots, a_n') \epsilon^\sigma a$ and $p_1(a_k) = p_1(a_k')$, $1 \leqslant k \leqslant n$. Hence $(p_2(a_1'), \ldots, p_2(a_n')) \epsilon^\sigma p_2(a)$. But $p_1 = p_2$ on E^{σ_i}, $\sigma_i < \sigma$. Hence $p_2(a_k') = p_1(a_k') = p_1(a_k)$, $1 \leqslant k \leqslant n$. That is $(p_1(a_1), \ldots, p_1(a_n)) \epsilon^\sigma p_2(a)$. Similarly if $(p_2(a_1), \ldots, p_2(a_n)) \epsilon^\sigma p_2(a)$ then $(p_2(a_1), \ldots, p_2(a_n)) \epsilon^\sigma p_1(a)$. Thus $\widehat{p_1(a)} = \widehat{p_2(a)}$ and as N is normal then $p_1(a) = p_2(a)$. By a similar argument it can be shown that for all constant relations R_t of M, $p_1(R_t) = p_2(R_t)$. Hence $\mathfrak{p}_1 = \mathfrak{p}_2 : M \to N$. Q.E.D.

If N is a subsystem of M then N can be regarded as being in the same similarity class as M. For if $\langle R_1, \ldots\rangle$ is the sequence of constant relations of M then $\langle p(R_1), \ldots\rangle$ can be taken as the corresponding sequence of relations of N. As p is surjective all of the relations of N will be included in this sequence, although there may be repetitions. This will not matter.

Canonical subsystems M is a τ-system and F^0 a given, non-empty, subset of E^0. A τ-system, N, can be built inductively on F^0 as follows (*cf.* Kreisel and Krivine, [5], p. 98, Theorem 16):

(i) F^0 comprises the individuals of N.
(ii) Take $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, and assume F^{σ_i} is defined for all $\sigma_i < \sigma$, together with surjective maps $p : \mathsf{E}^{\sigma_i} \to \mathsf{F}^{\sigma_i}$, $\sigma_i \neq 0$, and $p : \mathsf{F}^0 \to \mathsf{F}^0$ the identity map. For each $a \in \mathsf{E}^{\sigma}$, define $p(a) = \{(p(a_1), \ldots, p(a_n)) \mid$ there exists $a_1', \ldots, a_n'$, such that $p(a_j) = p(a_j')$, $1 \leqslant j \leqslant n$, and $(a_1', \ldots, a_n') \epsilon^{\sigma} a\}$. Let $\mathsf{F}^{\sigma} = \{p(a) \mid a \in \mathsf{E}^{\sigma}\}$ and $p : \mathsf{E}^{\sigma} \to \mathsf{F}^{\sigma}$ is thus defined. Now for all $(p(a_1), \ldots, p(a_n)) \in \mathsf{F}^{\sigma_1} \times \ldots \times \mathsf{F}^{\sigma_n}$, and all $p(a) \in \mathsf{F}^{\sigma}$, define $(p(a_1), \ldots, p(a_n)) \epsilon^{\sigma} p(a)$ if $(p(a_1), \ldots, p(a_n)) \in p(a)$. That is ϵ^{σ} in N is the ordinary membership relation.
(iii) For each relation R_t of M of type $(\sigma_1, \ldots, \sigma_n)$, $\sigma_1, \ldots, \sigma_n \leqslant \tau$, define $p(R_t)$ by: $p(R_t)(p(a_1), \ldots, p(a_n))$ if there exists $(a_1', \ldots, a_n')$ such that $p(a_k') = p(a_k)$, $1 \leqslant k \leqslant n$, and $R_t(a_1', \ldots, a_n')$.

Theorem 3.2 *N as constructed above is a normal τ-system.*

Proof: Immediate from a direct checking of the definitions.

Theorem 3.3 *If M is a τ-system and N is constructed as above on a subset F^0 of E^0 then N is a substructure of M.* (N is termed a canonical substructure.)

Proof: Immediate from the details of the construction and where the maps $\mathfrak{p}$ form the canonical projection of M to the subsystem N.

If M, N, are two similar τ-systems then M and N are said to be *isomorphic* if there exists, for each $\sigma \leqslant \tau$, a bijective map $\psi : \mathsf{E}^{\sigma} \to \mathsf{F}^{\sigma}$ such that (i) if $\sigma = (\sigma_1, \ldots, \sigma_n)$ then for all $(a_1, \ldots, a_n) \in \mathsf{E}^{\sigma_1} \times \ldots \times \mathsf{E}^{\sigma_n}$ and all $a \in \mathsf{E}^{\sigma}$, $(a_1, \ldots, a_n) \epsilon^{\sigma} a$ if, and only if, $(\psi(a_1), \ldots, \psi(a_n)) \epsilon^{\sigma} \psi(a)$; (ii) for all R_t of type $(\sigma_1, \ldots, \sigma_n)$ and all $(a_1, \ldots, a_n) \in \mathsf{E}^{\sigma_1} \times \ldots \times \mathsf{E}^{\sigma_n}$, $R_t(a_1, \ldots, a_n)$ if, and only if, $S_t(\psi(a_1), \ldots, \psi(a_n))$. (Note: The correspondence between the R_t's and the S_t's could be varied by permutations of either the relations of M or the relations of N, but compatible with the similarity requirements.)

Theorem 3.4 *If M is a τ-system and N_1, N_2 are two subsystems of M such that $\mathsf{F}_1^0 = \mathsf{F}_2^0$ then N_1 and N_2 are isomorphic.*

Proof: Let $\mathfrak{p}_1 : M \to N_1$, $\mathfrak{p}_2 : M \to N_2$ be the canonical projections of M to N_1, and N_2 respectively.

It is first established by induction that for all $\sigma \leqslant \tau$ and all $p_1(a)$, $p_1(b) \in \mathsf{F}_1$, that $p_1(a) = p_1(b)$ if, and only if, $p_2(a) = p_2(b)$. If $\sigma = 0$ then the result is immediate as p_1, p_2 are identity maps on $\mathsf{F}_1^0 = \mathsf{F}_2^0$. For $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, assume that the result is true for all $\sigma_i < \sigma$. Take $p_1(a)$, $p_1(b) \in \mathsf{F}_1^{\sigma}$ and assume $p_1(a) = p_1(b)$. Take any $(p_2(a_1), \ldots, p_2(a_n)) \epsilon^{\sigma} p_2(a)$.

Hence there exists some $(a'_1, \ldots, a'_n) \epsilon^\sigma a$ such that $p_2(a'_k) = p_2(a_k)$, $1 \leqslant k \leqslant n$. Therefore $(p_1(a'_1), \ldots, p_1(a'_n) \epsilon^\sigma p_1(b)$, as $p_1(a) = p_1(b)$, and so there exists $(a''_1, \ldots, a''_n) \epsilon^\sigma b$, where $p_1(a'_k) = p_1(a''_k)$, $1 \leqslant k \leqslant n$. Thus $(p_2(a''_1), \ldots, p_2(a''_n)) \epsilon^\sigma p_2(b)$. But from the induction assumption $p_2(a''_k) = p_2(a_k)$, $1 \leqslant k \leqslant n$, and so $(p_2(a_1), \ldots, p_2(a_n)) \epsilon^\sigma p_2(b)$. Similarly if $(p_2(a_1), \ldots, p_2(a_n)) \epsilon^\sigma p_2(b)$ then $(p_2(a_1), \ldots, p_2(a_n)) \epsilon^\sigma p_2(a)$. Therefore $\widehat{p_2(a)} = \widehat{p_2(b)}$ and so $p_2(a) = p_2(b)$. Conversely if $p_2(a) = p_2(b)$ then $p_1(a) = p_1(b)$. Hence this first result is established.

Now define $\psi(p_1(a) = p_2(a)$. ψ is thus well defined, for if $p_1(a) = p_1(b)$ then $p_2(a) = p_2(b)$. It is now necessary to show that ψ is an isomorphism between N_1 and N_2. First to show that for each $\sigma \leqslant \tau$, $\psi : \mathbf{F}_1^\sigma \to \mathbf{F}_2^\sigma$ is bijective. If $\sigma = 0$ then $\psi : \mathbf{F}_1^0 \to \mathbf{F}_2^0$ is the identity map as $\mathbf{F}_1^0 = \mathbf{F}_2^0$ and p_1, p_2 are the identity maps on $\mathbf{F}_1^0$, $\mathbf{F}_2^0$ respectively. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$, and assume that for all $\sigma_i < \sigma$, $\psi : \mathbf{F}_1^{\sigma_i} \to \mathbf{F}_2^{\sigma_i}$ is bijective. For any $p_1(a), p_1(b) \epsilon \mathbf{F}_1^\sigma$, if $\psi(p_1(a)) = \psi(p_1(b))$ then $p_2(a) = p_2(b)$ and so $p_1(a) = p_1(b)$. That is ψ is injective. If $p_2(a) \epsilon \mathbf{F}_2^\sigma$ then $a \epsilon \mathbf{E}^\sigma$ and so $p_1(a) \epsilon \mathbf{F}_1^\sigma$ and $\psi(p_1(a)) = p_2(a)$. That is $\psi : \mathbf{F}_1 \to \mathbf{F}_2$ is surjective. Hence by induction $\psi : \mathbf{F}_1^\sigma \to \mathbf{F}_2^\sigma$ is bijective for each $\sigma \leqslant \tau$.

Take $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and any $(p_1(a_n), \ldots, p_1(a_n) \epsilon^\sigma p_1(a)$. Therefore there exists $(a'_1, \ldots, a'_n) \epsilon^\sigma a$ such that $p_1(a'_k) = p_1(a_n)$, $1 \leqslant k \leqslant n$. Hence $(p_2(a'_1), \ldots, p_2(a'_n)) \epsilon^\sigma p_2(a)$. But as $p_1(a'_k) = p_1(a_k)$, $1 \leqslant k \leqslant n$, then $p_2(a'_k) = p_2(a_k)$. Hence $(p_2(a_1), \ldots, p_2(a_n)) \epsilon^\sigma p_2(a)$. That is $(\psi(p_1(a_1)), \ldots, \psi(p_1(a_n))) \epsilon^\sigma \psi(p_1(a))$. Conversely, take $(\psi(p_1(a_1)), \ldots, \psi(p_1(a_n))) \epsilon^\sigma \psi(p_1(a))$. That is $(p_2(a_1), \ldots, p_2(a_n)) \epsilon^\sigma p_2(a)$ and so, by a similar argument to that above, $(p_1(a_1), \ldots, p_1(a_n)) \epsilon^\sigma p_1(a)$.

Finally, by a similar argument as above, it can be shown that if $p_1(R_t)$ is any n-placed relation on $\mathbf{F}_1^{\sigma_1} \times \ldots \times \mathbf{F}_n^{\sigma_n}$, $\sigma_1, \ldots, \sigma_n \leqslant \tau$, then for all $(p_1(a_1), \ldots, p_1(a_n))$, $p_1(R_t)(p_1(a_1), \ldots, p_1(a_n))$ if, and only if, $p_2(R_t)(\psi(p_1(a_1)), \ldots, \psi(p_1(a_n)))$. Hence $\boldsymbol{\psi} : N_1 \to N_2$ is an isomorphism between N_1 and N_2. Q.E.D.

Corollary *If M is a τ-system then every subsystem of M is isomorphic to a canonical subsystem of M.*

Proof: Let N_1 be any substructure of M. Let N_2 be the canonical substructure of M constructed on $\mathbf{F}_1^0 \subseteq \mathbf{E}^0$. Hence from Theorem 3.4 N_1 is isomorphic to N_2.

Theorem 3.5 *N_1, N_2 are two subsystems of a τ-system M. If $\mathbf{F}_1^0 \subseteq \mathbf{F}_2^0$ then N_1 is a subsystem of N_2 and if $\mathfrak{p}_3 : N_2 \to N_1$ is the canonical projection of N_2 to N_1, then for $\sigma \leqslant \tau$, and all $p_2(a) \epsilon \mathbf{F}_2^\sigma$, $p_3 p_2(a) = p_1(a)$, where $\mathfrak{p}_1$, $\mathfrak{p}_2$ are the canonical projections from M to N_1, N_2 respectively.*

Proof: By an argument similar to that in the proof of Theorem 3.4 it can be shown (i) that for any $\sigma \leqslant \tau$, and all $p_1(a), p_1(b) \epsilon \mathbf{F}_1^\sigma$, if $p_2(a) = p_2(b)$ then $p_1(a) = p_1(b)$; (ii) that if for each $\sigma \leqslant \tau$, $p_3 : \mathbf{F}_2^\sigma \to \mathbf{F}_1^\sigma$ is defined by putting $p_3(p_2(a)) = p_1(a)$, for all $p_2(a) \epsilon \mathbf{F}_2^\sigma$, and if for each constant relation $p_2(R_t)$ of N_2, $p_3(p_2(R_t))$ is defined as $p_1(R_t)$, then $\mathfrak{p}_3 : N_2 \to N_1$ is the canonical projection defining N_1 as a subsystem of N_2. Further, by definition $p_3 p_2 = p_1 : \mathbf{E}^\sigma \to \mathbf{F}_1^\sigma$, for each $\sigma \leqslant \tau$, $\sigma \neq 0$. Q.E.D.

Theorem 3.6 *If N_1 is a subsystem of a τ-system M such that* $(p_1(a_1), \ldots,$

$p_1(a_n)) \not\epsilon^\sigma p_1(a)$, $\sigma \leq \tau$, $\sigma = (\sigma_1, . ., \sigma_n)$ then in any subsystem N_2 of M, which contains N_1, $(p_2(a_1), . ., p_2(a_n)) \not\epsilon^\sigma p_2(a)$. Similarly if $p_1(R_t)(p_1(a_1), . ., p_1(a_n))$ does not hold in N_1 then $p_2(R_t)(p_2(a_1), . ., p_2(a_n))$ does not hold in N_2.

Proof: Let $\mathfrak{p}_3 : N_2 \to N_1$ be the canonical projection as in Theorem 3.5. Assume $(p_2(a_1), . ., p_2(a_n)) \epsilon^\sigma p_2(a)$ and so $(p_3(p_2(a_1)), . ., p_3(p_2(a_n))) \epsilon^\sigma p_3(p_2(a))$. Hence $(p_1(a_1), . ., p_1(a_n)) \epsilon^\sigma p_1(a)$, as $p_3 p_2 = p_1$. That is if $(p_1(a_1), . ., p_1(a_n)) \not\epsilon^\sigma p_1(a)$ then $(p_2(a_1), . ., p_2(a_n)) \not\epsilon^\sigma p_2(a)$. The second part follows likewise.Q.E.D.

Theorem 3.7 *Any subsystem, N, of a full τ-system M is itself full.*

Proof: Take $\sigma \leq \tau$, $\sigma = (\sigma_1, . ., \sigma_n)$ and $\mathbf{K}$ any subclass of $\mathbf{F}^{\sigma_1} \times . . \times \mathbf{F}^{\sigma_n}$. It is required to find some $p(a) \in \mathbf{F}^\sigma$ such that $\mathbf{K} = \widehat{p(a)}$, where $\mathfrak{p} : M \to N$ is the canonical projection associated with N. Let $K' = \{(a_1, . ., a_n) \mid (p(a_1), . ., p(a_n)) \in \mathbf{K}\}$. Now M is full and K' is a subclass of $\mathbf{E}^{\sigma_1} \times . . \times \mathbf{E}^{\sigma_n}$ and so there exists $a \in E^\sigma$ such that $\hat{a} = K'$.

Now to show that $K = \widehat{p(a)}$. Take $(p(a_1), . ., p(a_n)) \in \mathbf{K}$. Hence $(a_1, . ., a_n) \in K'$ and so $(a_1, . ., a_n) \epsilon^\sigma a$. Therefore $(p(a_1), . ., p(a_n)) \epsilon^\sigma p(a)$. That is $\mathbf{K} \subseteq \widehat{p(a)}$. Now take $(p(a_1), . ., p(a_n)) \epsilon^\sigma p(a)$. Therefore there exists $(a_1', . ., a_n') \epsilon^\sigma a$, where $p(a_k') = p(a_k)$, $1 \leq k \leq n$. Hence $(a_1', . ., a_n') \in K'$, as $\hat{a} = K'$, and so $(p(a_1'), . ., p(a_n')) \in \mathbf{K}$. That is $(p(a_1), . ., p(a_n)) \in \mathbf{K}$. Thus $\widehat{p(a)} \subseteq \mathbf{K}$ and so $\widehat{p(a)} = \mathbf{K}$. Q.E.D.

Theorem 3.8 *Let $\{M_i \mid i \in I\}$ be a family of similar τ-systems. For each $i \in I$, let N_i be a subsystem of M_i. If X is any ultrafilter over I then $\pi N_i/X$ is a subsystem of $\pi M_i/X$.*

Proof: For each $i \in I$ let $\mathfrak{p}_i : M_i \to N_i$ be the canonical projection associated with each N_i. Define $\mathfrak{p} : \pi M_i/X \to \pi N_i/X$ as follows: For $\sigma = 0$ and for all $\bar{g} \in \mathbf{F}^0$ define $p(\bar{g}) = \overline{p(g)}$, where $p(g)$: $I \to \pi_{i \in I} \mathbf{F}_i^0$ is defined by $p(g)(i) = g(i)$, all $i \in I$. Hence $p : \mathbf{F}^0 \to \mathbf{F}^0$ is the identity map.

For $\sigma \neq 0$, $\sigma \leq \tau$, for all $\bar{f} \in \mathbf{E}^\sigma$ put $p(\bar{f}) = \overline{p(f)}$, where $p(f) : I \to \pi_{i \in I} \mathbf{F}_i^\sigma$ is defined by $p(f)(i) = p_i(f(i))$, all $i \in I$. If $f_1 \sim f$ then $p(f_1) \sim p(f)$, as $\{i \mid f_1(i) = f(i)\} \in X$; and so $p : \mathbf{E}^\sigma \to \mathbf{F}^\sigma$ is well defined. Further p is surjective as each $p_i : \mathbf{E}_i^\sigma \to \mathbf{F}_i^\sigma$ is surjective.

Take $\sigma = (\sigma_i, . ., \sigma_n)$, $\sigma \leq \tau$ and consider $(p(\bar{f}_1), . ., p(\bar{f}_n)) \epsilon^\sigma p(\bar{f})$. Hence $G \in X$, where $G = \{i \mid (p(f_1)(i), . ., p(f_n)(i)) \epsilon^\sigma p(f)(i)\}$. That is, for each $i \in G$, $(p_i(f_1(i)), . ., p_i(f_n(i))) \epsilon^\sigma p_i(f(i))$ and so there exists $(a_1^i, . ., a_n^i) \epsilon^\sigma f(i)$, where $p_i(a_k^i) = p_i(f_k(i))$, $1 \leq k \leq n$. Now define $\bar{g}_k \in \mathbf{F}^{\sigma k}$, $1 \leq k \leq n$, by putting $g_k(i) = a_k^i$, all $i \in G$. Thus $\bar{g}_k$, $1 \leq k \leq n$, are well defined as $G \in X$. Now $\{i \mid (g_1(i), . ., g_n(i)) \epsilon^\sigma f(i)\} \supseteq G$ and so $(\bar{g}_1, . ., \bar{g}_n) \epsilon^\sigma \bar{f}$. And further, $p(\bar{g}_k) = p(\bar{f}_k)$, $1 \leq k \leq n$, as $\{i \mid p_i(g_k(i)) = p_i(f_k(i))\} \supseteq G$. Conversely, take $(\bar{g}_1, . ., \bar{g}_n) \epsilon^\sigma \bar{f}$ such that $p(\bar{g}_k) = p(\bar{f}_k)$, $1 \leq k \leq n$. Let $G_k = \{i \mid p_i(g_k(i)) = p_i(f(i))\}$, $1 \leq k \leq n$, and $G_0 = \{i \mid (g_1(i), . ., g_n(i)) \epsilon^\sigma f(i)\}$. Thus $G \in X$, where $G = \bigcap \{G_k \mid 0 \leq k \leq n\}$. Hence $(p(\bar{f}_1), . ., p(\bar{f}_n)) \epsilon^\sigma p(\bar{f})$ as $\{i \mid (p_i(f_1(i)), . ., p_i(f_n(i))) \epsilon^\sigma p_i(f(i))\} \supseteq G$.

By a similar argument it can be shown that $p(R_t)$ can be defined in $\pi N_i/X$ by reference to $p_i(R_t)$ for each M_i and that $p(R_t)(p(\bar{f}_1), . ., p(\bar{f}_n))$ if, and only if, there exists $(\bar{g}_1, . ., \bar{g}_n)$ such that $R_t(\bar{g}_1, . ., \bar{g}_n)$, where $p(\bar{g}_k) = p(\bar{f}_k)$, $1 \leq k \leq n$. Hence $\pi N_i/X$ is a subsystem of $\pi M_i/X$ with canonical projection $\mathfrak{p} : \pi M_i/X \to \pi N_i/X$ as defined. Q.E.D.

α-embeddings A τ_1-system M_1 is said to be α-embedded in a τ_2-system M_2, where $\tau_1 \leqslant \tau_2$, by an α-embedding map ψ^α, if (i) α is an injective map from $|\tau_1|$ to $|\tau_2|$ ($|\tau_1| = \{\sigma | \sigma \leqslant \tau_1\}$) such that for each $\sigma \leqslant \tau_1$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\alpha(\sigma) = (\alpha(\sigma_1), \ldots, \alpha(\sigma_n))$; (ii) for each $\sigma \leqslant \tau_1$, ψ^α is an injective map from E_1^σ to $\mathsf{E}_2^{\alpha(\sigma)}$ such that for all $(a_1, \ldots, a_n) \in \mathsf{E}_1^{\sigma_1} \times \ldots \times \mathsf{E}_n^{\sigma_n}$, $a \in \mathsf{E}_1^\sigma$, $(a_1, \ldots, a_n) \epsilon^\sigma a$ if, and only if, $(\psi^\alpha(a_1), \ldots, \psi^\alpha(a_n)) \epsilon^{\alpha(\sigma)} \psi^\alpha(a)$; (iii) for each R_t, n-placed and of type $(\sigma_1, \ldots, \sigma_n)$, $\psi^\alpha(R_t)$ is an n-placed relation of M_2 of type $(\alpha(\sigma_1), \ldots, \alpha(\sigma_n))$, such that $R_t(a_1, \ldots, a_n)$ if, and only if, $\psi^\alpha(R_t)(\psi^\alpha(a_1), \ldots, \psi^\alpha(a_n))$. If $\alpha : |\tau_1| \to |\tau_2|$ is such that $\alpha(0) = 0$ then the α-embedding is referred to simply as an embedding and the α is omitted from the ψ^α's.

Local family of subsystems (*cf.* Kurosh [4], vol. II, §55, p. 166.) A family of subsystems $\{N_i | i \in I\}$, of a τ-system M, is called a *local family* of M if (i) every member of E^0 belongs to at least one N_i, $i \in I$; and (ii) for every $i, j \in I$ (and hence for any finite number of elements of I) there exists $k \in I$ such that N_i and N_j are subsystems of N_k.

Theorem 3.9 *If* L *is the class of all finite subsystems of a* τ-*system* M *then* L *is a local family of* M.

Proof: Immediate.

L-*finitary systems* A τ-system M, with a given local family $\mathfrak{L} = \{N_i | i \in I\}$, is said to be L-*finitary* if (i) for each $\sigma \leqslant \tau$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, if $(a_1, \ldots, a_n) \not\epsilon^\sigma a$ then there exists some member, N, of $\mathfrak{L}$ such that $(p(a_1), \ldots, p(a_n)) \not\epsilon^\sigma p(a)$, where $p : M \to N$ is the canonical projection; (ii) similarly if $R_t(a_1, \ldots, a_n)$ does not hold in M then for some $N \in \mathfrak{L}$, $p(R_t)(p(a_1), \ldots, p(a_n))$ does not hold in N. If $\mathfrak{L}$ is the family of all finite subsystems of M and if M is L-finitary then M will be simply termed finitary.

Theorem 3.10 *If* M *is any first order system and* $\mathfrak{L}$ *any local family of* M *then* M *is* L-*finitary.*

Proof: Let R_t be a n-placed constant relation of type $(0, \ldots, 0)$. Take $a_1, \ldots, a_n \in \mathsf{E}^0$ such that $R_t(a_1, \ldots, a_n)$ does not hold in M. Let N be some member of $\mathfrak{L}$ which contains $a_1, \ldots, a_n$. Hence $p(R_t)(p(a_1), \ldots, p(a_n))$ does not hold in N, where $p : M \to N$ is the canonical projection. Q.E.D.

L-*associated ultrafilter* Let $\mathfrak{L} = \{N_i | i \in I\}$ be a local family of a τ-system M. For each $N \in \mathfrak{L}$ let $F_N = \{i | N_i \supseteq N\}$. Now $\{F_N | N \in \mathfrak{L}\} = B$ is such that the intersection of any finite set of members of B is non-empty. The ultrafilter X formed on B as sub-basis is called the L-*associated ultrafilter.*

Theorem 3.11 $\mathfrak{L} = \{N_i | i \in I\}$ *is a local family of a* L-*finitary* τ_1-*system* M *such that, for each* $i \in I$, N_i *can be* α-*embedded in a* τ_2-*system,* N_i', $\tau_1 \leqslant \tau_2$, *where each such* N_i' *is a model of a class of sentences,* K, *of* $\mathcal{L}'^{\tau_2}(\pi N_i'/X)$, X *being the* L-*associated ultrafilter. Then* M *can be* α-*embedded in a model of* K.

Proof: For each $i \in I$, let $\psi_i^\alpha : N_i \to N_i'$ be an α-embedding. Define $\psi^\alpha : M \to \pi N_i'/X$ by (i) if $a \in \mathsf{E}^0$ put $\psi^\alpha(a) = \bar{f}_a$, where $f_a(i) = \psi_i^\alpha(a)$, for all $i \in F_N$, where

N is a member of $\mathfrak{L}$ containing a; (ii) for each $\sigma \leq \tau_1$, $\sigma \neq 0$, and each $a \in \mathsf{E}^\sigma$, put $\psi^\alpha(a) = \bar{f}_a$, where $f_a(i) = \psi_i^\alpha(p_i(a))$, for all $i \in I$, and where $\mathfrak{p}_i : M \to N_i$ is the canonical projection. The N_i', $i \in I$, can be regarded as belonging to the same similarity class as M by ignoring all constant relations in N_i' other than those connected to N_i by ψ_i^α. It is now required to show that $\psi^\alpha : M \to \pi N_i'/X$ is an α-embedding. Take $\sigma \leq \tau_1$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $(a_1, \ldots, a_n) \epsilon^\sigma a$. Let J be the subset of $\{1, 2, \ldots, n\}$ such that if $j \in J$ then $\sigma_j = 0$. Now for all $i \in F_N$, where N is some member of $\mathfrak{L}$ which contains each $a_j, j \in J$, $(p_i(a_1), \ldots, p_i(a_n)) \epsilon^\sigma p_i(a)$. Hence $\{i \mid (\psi_i^\alpha(p_i(a_1)), \ldots, \psi_i^\alpha(p_i(a_n))) \epsilon^\sigma \psi_i^\alpha(p_i(a))\} \in X$. That is $\{i \mid (f_{a_1}(i), \ldots, f_{a_n}(i)) \epsilon^\sigma f_a(i)\} \in X$ and so $(\bar{f}_{a_1}, \ldots, \bar{f}_{a_n}) \epsilon^\sigma \bar{f}_a$. That is $(\psi^\alpha(a_1), \ldots, \psi^\alpha(a_n)) \epsilon^\sigma \psi^\alpha(a)$. Conversely, assume that $(a_1, \ldots, a_n) \not\epsilon^\sigma a$. Now M is L-finitary and so there exists some member N of $\mathfrak{L}$ such that $(p(a_1), \ldots, p(a_n)) \not\epsilon^\sigma p(a)$, where $\mathfrak{p} : M \to N$ is the canonical projection. Let $F_N = \{i \mid N_i \supseteq N$ and $N_i \in \mathfrak{L}\}$ and so $F_N \in X$. Now from Theorem 3.6, $\{i \mid (p_i(a_1), \ldots, p_i(a_n)) \not\epsilon^\sigma p_i(a)\} \supseteq F_N$ and so $\{i \mid (\psi_i^\alpha(p_i(a_1)), \ldots, \psi_i^\alpha(p_i(a_n))) \not\epsilon^\sigma \psi_i^\alpha(p_i(a))\} \in X$. Hence $(\psi^\alpha(a_1), \ldots, \psi^\alpha(a_n)) \not\epsilon^\sigma \psi^\alpha(a)$. Similarly, if R_t is a constant relation of type $(\sigma_1, \ldots, \sigma_n)$, $\sigma_1, \ldots, \sigma_n \leq \tau_1$, then $R_t(a_1, \ldots, a_n)$ if, and only if, $\psi^\alpha(R_t)(\psi^\alpha(a_1), \ldots, \psi^\alpha(a_n))$.

Further, it needs to be shown that ψ^α is injective at each level $\sigma \leq \tau_1$. Let $\sigma = 0$ and take $a, b \in \mathsf{E}^0$ such that $\psi^\alpha(a) = \psi^\alpha(b)$. Thus $\bar{f}_a = \bar{f}_b$. Let $N_1, N_2 \in \mathfrak{L}$ such that $f_a(i) = \psi_i^\alpha(a)$, for all $i \in F_{N_1}$, $f_b(i) = \psi_i^\alpha(b)$, for all $i \in F_{N_2}$. If $G = \{i \mid f_a(i) = f_b(i)\}$ then $G \cap F_{N_1} \cap F_{N_2}$ is non-empty and so there exists an $i \in I$ such that $\psi_i^\alpha(a) = \psi_i^\alpha(b)$. Hence $a = b$, as ψ_i^α is injective. Now take $\sigma \leq \tau_1$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $a, b \in \mathsf{E}^\sigma$ such that $a \neq b$. Hence there exists $(a_1, \ldots, a_n)$ which 'belongs' to one, and only one, of a and b. Therefore $(\psi^\alpha(a_1), \ldots, \psi^\alpha(a_n))$ 'belongs' to one, and only one, of $\psi^\alpha(a)$ and $\psi^\alpha(b)$. That is $\psi^\alpha(a) \neq \psi^\alpha(b)$. Thus $\boldsymbol{\psi}^\alpha : M \to \pi N_i/X$ is an α-embedding.

Finally it remains to comment that $\pi N_i'/X$ is a model of K as each N_i', $i \in I$, is a model of K (Theorem 2.3). The theorem is therefore established. Q.E.D.

Corollary *If every finite subsystem of a finitary system M can be α-embedded in a model of* K *then M can be so embedded.*

Proof: Immediate from Theorems 3.10 and 3.11.

The first order case of the above corollary, (with $\alpha(0) = 0$), is proved by Robinson, *cf.* [7], p. 34, Theorem 2.4.1, by the method of diagrams. Grätzer, *cf.* [2], p. 243, Theorem 4, and p. 261, Theorem 7, gives a proof of this first order result using ultraproducts.

Theorem 3.12 *If $\mathfrak{L} = \{N_i \mid i \in I\}$ is a local family of a* L*-finitary τ-system M then M can be embedded in $\pi N_i/X$ where X is the* L*-associated ultrafilter.*

Proof: In Theorem 3.11, put $\alpha(0) = 0$, $K = \emptyset$ and, for each $i \in I$, put $N_i' = N_i$ and ψ_i^α as the identity map on N_i.

4 *Some algebraic applications of higher order ultraproducts* This final paragraph illustrates the presence and application of the higher order ultraproduct construction in two known algebraic situations.

Non-finite Boolean algebras

Theorem 4.1 *Any infinite Boolean algebra is isomorphic to a subset subalgebra of a second order ultraproduct* (*cf.* Stone's representation theorem).

Proof: Let $\mathfrak{M}$ be the infinite Boolean algebra regarded as a first order system. E^0 is the set of elements of the algebra. $\langle ', \vee, \wedge\rangle$ is the sequence of constant relations, where $'$ is the complement operator regarded as a two-place relation of type $(0, 0)$; $\vee, \wedge$ are the wedge and join operators of the algebra regarded as three-place relations of type $(0, 0, 0)$. Now let $\tau_2 = (0)$ and $\langle \complement, \cup, \cap\rangle$ be a sequence of constant relation symbols, $\complement$ being of type $((0), (0))$, and $\cup, \cap$ each of type $((0), (0), (0))$. ($\complement(x^{(0)}, y^{(0)})$ will be written $\complement x^{(0)} = y^{(0)}$ and similarly for $\cap$ and $\cup$.)

Let K be the set of sentences of $\mathcal{L}^{\tau_2}$ as follows:

(i) $\forall x^{(0)}\forall y^{(0)}(\complement x^{(0)} = y^{(0)} \Longleftrightarrow \forall x^0(x^0 \in^{(0)} x^{(0)} \Longleftrightarrow x^0 \notin^{(0)} y^{(0)}))$.
(ii) $\forall x^{(0)}\forall y^{(0)}\forall z^{(0)}(x^{(0)} \cup y^{(0)} = z^{(0)} \Longleftrightarrow \forall x^0(x^0 \in^{(0)} z^{(0)} \Longleftrightarrow (x^0 \in^{(0)} x^{(0)} \vee x^0 \in^{(0)} y^{(0)})))$.
(iii) $\forall x^{(0)}\forall y^{(0)}\forall z^{(0)}(x^{(0)} \cap y^{(0)} = z^{(0)} \Longleftrightarrow \forall x^0(x^0 \in^{(0)} z^{(0)} \Longleftrightarrow (x^0 \in^{(0)} x^{(0)} \wedge x^0 \in^{(0)} y^{(0)})))$.

Now $\mathfrak{L} = \{N_i \mid N_i$ is a finitely generated subalgebra of $M\}$ forms a local family of subsystems of $\mathfrak{M}$. Moreover each such N_i is finite and so can be α-embedded in N_i', a model of K, where $\alpha(0) = (0)$. Thus, by Theorem 3.11, $\mathfrak{M}$ can be α-embedded in $\pi N_i'/X$, where X is the L-associated ultrafilter. (Note: $\mathfrak{M}$ is L-finitary.) As K consists only of universal sentences, the image of $\mathfrak{M}$ in $\pi N_i'/X$ under the embedding is also a model of K. Finally, as in general $\pi N_i'/X$ is not a full system, the universal quantifiers of K of type (0) will not include all possible subsets of the individuals of the ultraproduct, and so the image of $\mathfrak{M}$ will not be a full subset algebra. Q.E.D.

Locally normal groups The following results from the theory of finite groups are assumed:

(α) For every two Sylow p-subgroups, P, Q, of a finite group $\mathfrak{G}$ there exists an inner automorphism of $\mathfrak{G}$ which when restricted to P is an isomorphism between P and Q.
(β) If H is a normal subgroup of a finite group G, P a Sylow p-subgroup of $\mathfrak{G}$, then $P \cap H$ is a Sylow p-subgroup of H.
(γ) If P is a p-subgroup of a finite group $\mathfrak{G}$, N a normal subgroup of $\mathfrak{G}$, such that $N \supseteq P$, and Q a Sylow p-subgroup of N containing P, then there exists a Sylow p-subgroup, Q', of $\mathfrak{G}$ which contains P and such that $Q' \cap N = Q$.

Let $\mathfrak{M}$ be a locally normal group. Regard M as a τ-system, where $\tau = ((0, 0), (0))$. E^0 is the individuals of the group $\mathfrak{M}$.[2] $E^{(0,0)}$ is the set of

2. $E^{(0)}$ is the set of all subsets of E^0.

all subsets of $E^0 \times E^0$ and $E^\tau = \{\emptyset\}$. The 'membership' relations are the ones of ordinary set membership, and will be written without the type prefixes. $\langle \mathsf{S}, \mathsf{e}, \subset \rangle$ is the sequence of constant relations, where S is of type $(0, 0, 0)$, representing the binary operation of $\mathfrak{M}$, e is the 0-placed relation, of type 0, denoting the identity element of the group. $\subset$ is of type $((0), (0))$, and denotes the strict inclusion relation.

Let $\mathfrak{L} = \{N_i \mid i \in I\}$ be the family of finite normal subgroups of $\mathfrak{M}$. Hence $\mathfrak{L}$ can be regarded as a local family of subsystems of M, as $\mathfrak{M}$ is locally normal. Further, M is L-finitary. Let X be the L-associated ultrafilter on I.

Take the following sentences and formulae of $\mathcal{L}'^{\tau}(\pi N_i/X)$. K_0 is the conjunction of sentences characterising group structure with respect to a binary operation S and identity e. (We adopt the usual shorthand that $x \circ y = z$ stands for $\mathsf{S}(x, y, z)$, $y = x^{-1}$ stands for $\mathsf{S}(x, y, \mathsf{e})$ and so on.) $\mathsf{G_s}(x^{(0)})$, ($x^{(0)}$ is a subgroup), is the formula

$$\forall x^0 \forall y^0 (x^0 \in x^{(0)} \wedge y^0 \in x^{(0)} \Longrightarrow x^0 \circ {y^0}^{-1} \in x^{(0)}) \wedge K_0.$$

$\mathsf{S}_1(y^0)$ is the formula $y^0 = \mathsf{e}$.
$\mathsf{S}_n(y^0)$, (y^0 is of order n, n an integer), is the formula

$$ {y^0}^n = \mathsf{e} \wedge \neg \mathsf{S}_{n-1}(y^0) \vee \ldots \vee \mathsf{S}_1(y^0)).$$

$\mathsf{S}_{op}(y^0)$, (y^0 has order some power of p, p a prime integer), is the formula $\bigvee_{k \in N} \mathsf{S}_{p}k(y^0)$, where N is the set of integers.
$\mathsf{G}_{ps}(x^{(0)})$, ($x^{(0)}$ is a p-subgroup), is the formula

$$\mathsf{G_s}(x^{(0)}) \wedge \forall y^0 (y^0 \in x^{(0)} \Longrightarrow \mathsf{S}_{op}(y^0)).$$

$x^{(0)} \cong y^{(0)}(w^{(0,0)})$, ($w^{(0,0)}$ is an isomorphism between subgroups $x^{(0)}$ and $y^{(0)}$), denotes the conjunction of the following formulae:

(i) $\mathsf{G_s}(x^{(0)}) \wedge \mathsf{G_s}(y^{(0)})$,
(ii) $\forall z^0 (z^0 \in x^{(0)} \Longrightarrow \exists' u^0 (u^0 \in y^{(0)} \wedge (z^0, u^0) \in w^{(0,0)}))$,
(iii) $\forall x^0 \forall y^0 \forall z^0 (x^0 \in x^{(0)} \wedge y^0 \in x^{(0)} \wedge z^0 \in y^{(0)} \wedge (x^0, z^0) \in w^{(0,0)} \wedge (y^0, z^0) \in w^{(0,0)} \Longrightarrow x^0 = y^0)$,
(iv) $\forall z^0 (z^0 \in y^{(0)} \Longrightarrow \exists x^0 (x^0 \in x^{(0)} \wedge (x^0, z^0) \in w^{(0,0)}))$,
(v) $\forall x^0 \forall y^0 \forall u^0 \forall v^0 (x^0 \in x^{(0)} \wedge y^0 \in x^{(0)} \wedge u^0 \in y^{(0)} \wedge v^0 \in y^{(0)} \wedge \langle x^0, u^0 \rangle \in w^{(0,0)} \wedge \langle y^0, v^0 \rangle \in w^{(0,0)} \Longrightarrow \langle x^0 \circ y^0, u^0 \circ v^0 \rangle \in w^{(0,0)})$.

Theorem 4.2 *If* $\mathfrak{M}$ *is a locally normal group as described above then* a) $\pi N_i/X$ *is a group*; b) P *is a subgroup of* $\mathfrak{M}$ *if, and only if,* $\psi(P)$ *is a subgroup of* $\pi N_i/X$; c) P *is a* p*-subgroup of* $\mathfrak{M}$ *if, and only if,* $\psi(P)$ *is a* p*-subgroup of* $\pi N_i/X$, (*where* $\psi : M \to \pi N_i/X$ *is the embedding of* Theorem 3.12), (see footnote on page 15.)

Proof: a) $\pi N_i/X \models K_0$, as $\{i \mid N_i \models K_0\} = I$, and so $\pi N_i/X$ is a group with respect to the binary operation $\psi(\mathsf{S})$.

b) Let P be a subgroup of $\mathfrak{M}$ and so for each i, $P_i = N_i \cap P$ is a subgroup of N_i. But $P_i = p_i(P)$, where $\mathfrak{p}_i : M \to N_i$ is the canonical projection associated with the subgroup N_i, regarded as a subsystem of M. Now

$\{i \mid N_i \models \mathsf{G_s}(P_i)\} = I$ and therefore $\pi N_i/X \models \mathsf{G_s}(\psi(P))$. That is $\psi(P)$ is a subgroup of $\pi N_i/X$. Conversely, assume $\psi(P)$ is a subgroup of $\pi N_i/X$. Let $\psi(\hat{M}) = \{\bar{f}_a \mid a \in M\}$ and $\psi(\hat{P}) = \{\bar{f}_a \mid a \in P\}$, where $\bar{f}_a$ is defined as in Theorem 3.12. Therefore $\psi(\hat{P}) = \psi(P) \cap \psi(\hat{M})$, $\psi(\hat{M})$ is isomorphic to M and $\psi(\hat{P})$ is isomorphic to P. But $\psi(\hat{P}) = \psi(P) \cap \psi(\hat{M})$ and so $\psi(\hat{P})$ is a subgroup of $\psi(\hat{M})$; that is P is a subgroup of $\mathfrak{M}$.

c) Let P be a p-subgroup of $\mathfrak{M}$. Hence $\{i \mid N_i \models \mathsf{G}_{ps}(P_i)\} = I$ and so $\pi N_i/X \models \mathsf{G}_{ps}(\psi(P))$; that is $\psi(P)$ is a p-subgroup of $\pi N_i/X$. Conversely, assume $\psi(P)$ is p-subgroup of $\pi N_i/X$. Hence $\psi(\hat{P})$ is a p-subgroup, and so P is a p-subgroup. Q.E.D.

The final two theorems are results first proved by Baer in [1], p. 604, Theorem 4.1 and p. 608, Theorem 4.4. Alternative proofs, via an ultraproduct construction, are here provided. Kurosh, *cf.* [4], vol. II, pp. 167-170, §55, records a proof of these results by the method of projection sets, (inverse limits). Gräzter, *cf.* [2], p. 160, Exercise 100, details the relationship between an ultraproduct of a family of algebras and the inverse limit of an associated family of algebras.

Theorem 4.3 *If $\mathfrak{M}$ is a locally normal group and P a given Sylow p-subgroup of $\mathfrak{M}$ then the intersection of P with an arbitary finite normal subgroup H of $\mathfrak{M}$ is a Sylow p-subgroup of H.*

Proof: Assume that $P \cap H$ is not a Sylow p-subgroup of H. Let Q' be a Sylow p-subgroup of H containing $P \cap H$. Put $G = \{i \mid N_i \supseteq H\}$. Hence $G \in X$, where $\mathfrak{L}$ is the local family of M as described above and X is the L-associated ultrafilter. But from property (β) above, for each $i \in G$, $P_i = N_i \cap P$ is not a Sylow p-subgroup of N_i. Hence for each $i \in G$ a Sylow p-subgroup, Q_i, of N_i can be chosen so that $P_i \subset Q_i$ and $Q_i \cap H = Q'$. (Property (γ).) Take $\overline{g} \in \pi N_i/X$, such that $g(i) = Q_i$, all $i \in G$. Hence $\overline{g}$ is a subgroup of $\pi N_i/X$ and $\psi(P) \subset \overline{g}$. But $\psi(\hat{P})$ is a Sylow p-subgroup of $\psi(\hat{M})$ as P is a Sylow p-subgroup of $\mathfrak{M}$. Also $\overline{g} \cap \psi(\hat{M})$ is a p-subgroup of $\psi(\hat{M})$ and so $\psi(\hat{P}) = \overline{g} \cap \psi(\hat{M})$. But there exists some $a \in Q'$ such that $a \notin P$. Therefore, for all $i \in G$, $a \in Q_i$ but $a \notin P_i$. Hence $\bar{f}_a \in \overline{g}$, but $\bar{f}_a \notin \psi(\hat{P})$, where $f_a(i) = a$, all $i \in G$. But $\bar{f}_a \in \psi(\hat{M})$, and so $\psi(\hat{P}) \neq \overline{g} \cap \psi(\hat{M})$. From the contradiction it is established that $P \cap H$ is a Sylow p-subgroup of $\mathfrak{M}$. Q.E.D.

Theorem 4.4 *Any two Sylow p-subgroups of a locally normal group $\mathfrak{M}$ are isomorphic and locally conjugate.*

Proof: Let M be the τ-system as above with $\mathfrak{L} = \{N_i \mid i \in I\}$ the local family of normal, finite subgroups. Let P, Q be two given Sylow p-subgroups of $\mathfrak{M}$. By Theorem 4.3, for each $i \in I$, $P_i = P \cap N_i$, $Q_i = Q \cap N_i$, are Sylow p-subgroups of N_i. Hence, by property (α), for each $i \in I$, there exists an inner automorphism, w_i, of N_i taking P_i to Q_i. Let $\overline{w} \in \pi N_i/X$ be defined by $w(i) = w_i$, all $i \in I$. Now $\{i \mid N_i \models P_i \cong Q_i(w_i)\} = I$ and so $\pi N_i/X \models \psi(P) \cong \psi(Q)(\overline{w})$. That is $\overline{w}$ is an isomorphism between $\psi(P)$ and $\psi(Q)$. It is now required to show that $\overline{w}$ restricted to $\psi(\hat{P})$ is an isomorphism between $\psi(\hat{P})$ and $\psi(\hat{Q})$. For this it is sufficient to show that if $\overline{w}(\bar{f}) = \overline{g}$, (as $\langle \bar{f}, \overline{g} \rangle \in \overline{w}$ will be now written), and $\bar{f} \in \psi(\hat{P})$ then $\overline{g} \in \psi(\hat{Q})$.

Take $\bar{f}_a$, such that $a \in P$. Let $F = \{i \mid f_a(i) = a\}$ and so $F \in X$. Let k be some member of F and put $F' = \{i \mid N_i \supseteq N_k$ and $i \in F\}$. Thus $F' \in X$. Now, for all $i \in F'$, if $w_i(a) = b_i$ then $b_i \in Q_k$, as N_k is normal in N_i and w_i is an inner automorphism of N_i. Let the individuals of Q_k be $b_1, \ldots, b_n$, and let $F_j = \{i \mid w_i(a) = b_j$ and $i \in F'\}$, $1 \leq j \leq n$. Now $F_1 \cup \ldots \cup F_n = F'$ and so one, and only one, of the F_j's, say F_m, belongs to X. Therefore $\bar{g} = \bar{f}_{b_m}$ and so $\bar{g} \in \psi(\hat{Q})$.

Finally, it is required to show that $\bar{w}$ restricted to an isomorphism between $\psi(P)$ and $\psi(\hat{Q})$ is locally an inner automorphism. Take $\bar{f}_{a_1}, \ldots, \bar{f}_{a_n} \in \psi(\hat{P})$, that is $a_1, \ldots, a_n \in P$. Let $\bar{w}(\bar{f}_{a_j}) = \bar{f}_{b_j}$, $b_j \in Q$, $1 \leq j \leq n$. It is required to find some $\bar{f}_a \in \psi(\hat{M})$ such that $\bar{f}_a^{-1} \circ \bar{f}_{a_j} \circ \bar{f}_a = \bar{f}_{b_j}$, $1 \leq j \leq n$. Let $G_j = \{i \mid f_{a_j}(i) = a_j\}$, $1 \leq j \leq n$, and $H_j = \{i \mid g_{b_j} = b_j\}$, $1 \leq j \leq n$. Let $D_j = \{i \mid w_i(a_j) = b_j\}$, $1 \leq j \leq n$. Thus $G \in X$, where $G = \bigcap \{G_j \cap H_j \cap D_j \mid 1 \leq j \leq n\}$. Take some $m \in G$ and let $D = \{i \mid N_i \supseteq N_m\}$. Therefore $D \cap G \in X$. Now w_m is an inner automorphism of N_m taking P_m to Q_m. Therefore there exists some $a \in N_m$ such that $w_m(a_j) = a^{-1} \circ a_j \circ a$, all $1 \leq j \leq n$. But for all $i \in D \cap G$, $w_i(a_j) = b_j = w_m(a_j)$, $1 \leq j \leq n$. That is $\{i \mid w_i(a_j) = a^{-1} \circ a_j \circ a\} \in X$, $1 \leq j \leq n$. Therefore $\bar{f}_a^{-1} \circ \bar{f}_{a_j} \circ \bar{f}_a = \bar{f}_{b_j}$, all $1 \leq j \leq n$. Hence the required result. Q.E.D.

Footnote added at proof stage: It was initially thought by the author that the formula $S_{op}(y^0)$ was $(\pi N_i/X)$ allowable. This is not so. Thus Theorem 4.2, part c) must be restricted to the 'if' statement alone. Counter-examples exist for the 'only if' portion.

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