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Notre Dame Journal of Formal Logic
Volume XV, Number 3, July 1974
NDJFAM

# ON A NONTHESIS OF CLASSICAL MODAL LOGIC

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Classical modal systems subjoined to the full classical statement calculus accept as a thesis the conditional

(1) $CLpp$.

Such systems, however, reject as a thesis

(2) $CpLp$.

The common argument given in the literature for rejecting (2) is that, in any modal system in which (1) is a thesis, accepting (2) would yield

(3) $ELpp$

and thus reduce such a modal system into the so-called "The Trivial System"—that is, into an assertoric one with the redundant notation $L$.

Let us look more closely into the argument for accepting (1) and rejecting (2). To block derivation of (3) from (1) and (2) one must reject either (1) or (2) though one needs not reject both. But it does not follow from this that one must accept or even reject (1) or that one must accept or reject (2). Nor does it follow that one must accept at least one of (1) or (2). Therefore, the argument for blocking (3) cannot be the argument for accepting (1) and rejecting (2). Now suppose one accepts (2) rather than (1). Under this supposition, I wish to establish a contention of Prior's in [2], p. 199, that the resultant statement modal system is formally consistent—that is, consistent without reference to any intended interpretation of modal functors.

Let M′(T′) be a modal system which is like Feys-von-Wright M(T) except that (2) in M′(T′) replaces (1), the Axiom of Necessity, in M(T) and that the Rule of Necessitation,

$$(\mathrm{Rn})\ \alpha/L\alpha$$

which is a primitive rule of inference in M(T) becomes a derived rule in M′(T′), as Cresswell has shown in [1], p. 31. I prove that for no $\alpha$ of M′(T′) are both $\alpha$ and $N\alpha$ theses—that is, M′(T′) is (simply) consistent. To

*Received October 22, 1972*

establish this result, it suffices to show that every thesis of M′(T′) is regular in the sense of Sobociński in [3]—that is, every axiom and every theorem of M′(T′) is regular. Clearly, every axiom of the statement calculus base for M′(T′) is regular, since each axiom of that base is *ipso facto* non-modal and a thesis in that calculus. Henceforth, I shall call classical statement calculus (the system) **P**. It is also apparent that both modal axioms of M′(T′), *CpLp* and *CLCpqCLpLq*, are regular, for after deleting the modal qualifier '*L*' in both, they reduce to *Cpp* and *CCpqCpq* respectively, and these are theses of **P**.

Now it remains to show that every theorem of M′(T′) is regular. We have so far derived no theorems of M′(T′). However, it is plain that, by the supposed likeness of M′(T′) to M(T), all theorems of M(T) are theorems of M′(T′), exceptions being those theorems of M(T) which require the Axiom of Necessity for their derivation. Of such exceptions, we single out the M(T) thesis, *ab esse ad posse valet consequentia,*

(4) *CpMp.*

It is well known that not only is (4) a theorem of M(T) but also that its converse

(4′) *CMpp*

is a non-thesis of M(T). However, (4′) is a thesis of M′(T′), for it is derivable in M′(T′) as follows:

(i) *CpLp*
(ii) *CNpLNp*
(iii) *CNLNpNNp*
(iv) *CMpp.*

For my purpose, I need not derive any more theorems in M′(T′). Nevertheless, I can show that every theorem of M′(T′) is regular.

For every wff $\alpha$ of M′(T′), let $\alpha'$ be a wff of **P** such that $\alpha'$ is the result of deleting every occurrence of '*L*' in $\alpha$. I show that if $\alpha$ is a theorem of M′(T′), $\alpha'$ is a theorem of **P**. For suppose $\alpha$ is a theorem of M′(T′). Then $\alpha$ is derived in M′(T′) either by *modus ponens* or by uniform substitution.

Case (i): $\alpha$ is derived in M′(T′) by *modus ponens*. Then there is in M′(T′) wffs $\beta$ and $C\beta\alpha$ such that, by *modus ponens*, $\alpha$ is derived in M′(T′). Hence, by hypothesis, there is in **P** wffs $\beta'$ and $C\beta'\alpha'$ such that, by *modus ponens*, $\alpha'$ is derived in **P**. Hence, if $\alpha$ is a theorem of M′(T′), $\alpha'$ is a theorem of **P**.

Case (ii): $\alpha$ is derived in M′(T′) by uniform substitution. Then there are wffs $\beta$, $\gamma$, $\delta$, (not necessarily distinct) of M′(T′) such that $\alpha$ results from $\beta$ by substituting every occurrence of $\gamma$ in $\beta$ by $\delta$. Hence, by hypothesis, there is, in **P** wffs $\beta'$, $\gamma'$, $\delta'$, (not necessarily distinct) such that $\alpha'$ results from $\beta'$ by uniform substitution of $\gamma'$ by $\delta'$. Hence, if $\alpha$ is a theorem of M′(T′), $\alpha'$ is a theorem of **P**.

By cases (i) and (ii) above, if $\alpha$ is a theorem of M′(T′), $\alpha'$ is a theorem of **P**. Hence, every theorem of M′(T′) is regular. It follows that for no wffs $\alpha$ of M′(T′) are both $\alpha$ and $N\alpha$ derivable in M′(T′), for if they were, so would both $\alpha'$ and $N\alpha'$ be derivable in **P**, which, as the consistency of **P** is well established, is impossible. QED

It should be noted that the proceeding consistency proof is established without reference to any intended interpretation of the modal qualifier '*L*'. To be sure, some interpretation of '*M*' trickles in over our proof of (4′). But the proof of (4′) is not a part of the consistency proof of M′(T′). Since (1) is not a thesis of M′(T′), (3) is not derivable in M′(T′). Hence, M′(T′) does not reduce to "The Trivial System."

## REFERENCES

[1] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen & Co., Ltd., London (1968).

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[3] Sobociński, B., "A note on the regular and irregular modal systems of Lewis," *Notre Dame Journal of Formal Logic*, vol. III (1962), pp. 109-113.

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