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INVOLUTION AS A BASIS FOR PROPOSITIONAL CALCULI

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Given two sets of propositions, S_1 and S_2, we may say that the relation of *involution* holds between S_1 and S_2—or simply that S_1 *involves* S_2—provided that, if all elements of S_1 are true, then at least one element of S_2 is true. The notion was introduced by Carnap [3], and treated at length by Kneale [7]. According to these authors it may profitably be taken, in place of *entailment* (of which it is a generalization), as the primary object of study in logic. In developing its properties, they treat it solely as a metalogical relation between sets of propositions in an involution-free system; nested involutions do not occur. It is interesting to enquire what happens if, on the model of existing implicational calculi, involution is treated rather as a primitive operator *within* a system. This has been done by Duthie [4]. His enquiry is, however, somewhat restricted in scope, being concerned almost exclusively with the question of avoiding the (so-called) paradoxes of implication. Apart from this, the nearest approach to an involutional calculus appears to be the 'deduction-logic' of Lorenzen [11] (also Kutschera [9]). This is formulated in terms of a primitive operator $\rightarrow$, on the left of which may appear a variable number of arguments; the consequent, however, is restricted to have exactly one formula.

In the present paper an attempt is made to develop a less restricted theory. It is shown that, by varying the inference rules, purely involutional calculi may be constructed that are substantially equivalent to the classical and intuitionistic propositional calculi, and to the modal systems T, S4, S5. Some of the applications are discussed in the final section.

1 *The formalism and its interpretation.* The basic vocabulary shall consist of:

i) A denumerable set $\mathcal{P}$ of *proposition letters* (or *atomic formulas*).
ii) A logical constant, denoted by '$\rightarrow$'.
iii) Auxiliary symbols: comma, parentheses. Further auxiliary symbols, used for special purposes: the star *, and the 'signs' T, F.

The set of formulas is defined inductively by:

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i) Every proposition letter is a *formula* (specifically, an *atomic* formula).
ii) If L and M are (possibly empty) lists of formulas, then $(L \rightarrow M)$ is a *formula.*

The formula-occurrences in a list (in case there are two or more) are assumed to be separated by commas. Outermost parentheses of formulas will usually be omitted. In certain contexts, the formulas defined so far are called *unstarred formulas*. Then we add: If X is an unstarred formula, X^* is a *starred formula.*

In certain contexts, the formulas defined so far (whether starred or not) are called *unsigned formulas*. Then we add: If X is an unsigned formula, $\mathsf{T}X$ and $\mathsf{F}X$ are *signed formulas.*

We define the *components* of a signed formula A as follows: If A is $\mathsf{T}X_1, \ldots, X_m \rightarrow Y_1, \ldots, Y_n$, the components of A are $\mathsf{F}X_1, \ldots, \mathsf{F}X_m, \mathsf{T}Y_1, \ldots, \mathsf{T}Y_n$; if A is $\mathsf{F}X_1, \ldots, X_m \rightarrow Y_1, \ldots, Y_n$, the components of A are $\mathsf{T}X_1, \ldots, \mathsf{T}X_m, \mathsf{F}Y_1, \ldots, \mathsf{F}Y_n$. The device of starring comes from Ackermann [1], by way of Hacking [6]. The technique of signed formulas is due to Smullyan [14].

A comparison of the present system with standard logical systems will be undertaken later in this section, and for this purpose we require *classical, modal,* and *intuitionistic* formulas, built up by means of $\wedge$, $\vee$, $\sim$ (classical negation), $\Box$ (necessity operator), $\neg$ (intuitionistic negation), and $\supset$ (intuitionistic implication) from $\mathscr{P}$ in the usual way. Also, it will be technically convenient to allow mixed formulas containing, say, $\rightarrow$ as well as classical connectives. To avoid confusion, formulas as originally defined (having $\rightarrow$ as the only connective) may be called *involutional* formulas, or simply *involutions*. Only involutional formulas can be starred or signed.

The interpretation of the formalism will be in terms of a Kripke-style modelling. Let H be a set (the set of *universes*) having a distinguished element G_0, R a reflexive relation defined on H, $\phi : \mathscr{P} \times H \rightarrow \{\mathsf{T}, \mathsf{F}\}$ a mapping which assigns a truth-value to each proposition letter for each universe. Then the quadruple $\langle \mathsf{G}_0, H, R, \phi\rangle$ is called a $\mathbf{T}$-*model*. Suppose that a $\mathbf{T}$-model satisfies, in addition, one of the following four conditions:

C (for 'classical'): G_0 is the only element of H;
S4: R is transitive;
I (for 'intuitionistic'): R is transitive; and for all G, G' such that GRG', and all $p \in \mathscr{P}$, if $\phi(p, G) = \mathsf{T}$ then $\phi(p, G') = \mathsf{T}$;
S5: R is transitive and symmetric.

Then the model will be called, respectively, a **C**-, **S4**-, **I**-, or **S5**-model. (In the case of **C**-models most of the apparatus is, of course, dispensible, but it is convenient to retain it for the sake of a unified treatment.)

We adopt the convention (Fitting [5]) that, when 'G' refers to a universe, 'G^*' is a variable which ranges over those universes G' such that GRG'. Given a model $\langle \mathsf{G}_0, H, R, \phi\rangle$, the relation $G \models X$ (read 'G realizes X'), for arbitrary unstarred formula X and universe $G \in H$, is defined inductively by (the appropriate selection from) the rules:

$G \models X$, *where* X *is atomic, iff* $\phi(X, G) = \mathsf{T}$
$G \models \sim X$ *iff* $G \not\models X$
$G \models X \wedge Y$ *iff* $G \models X$ *and* $G \models Y$
$G \models X \vee Y$ *iff* $G \models X$ *or* $G \models Y$
$G \models \Box X$ *iff* $G^* \models X$ *for all* G^*
$G \models \neg X$ *iff, for all* G^*, $G^* \not\models X$
$G \models X \supset Y$ *iff, for all* G^*, $G^* \not\models X$ *or* $G^* \models Y$
$G \models X_1, \ldots, X_m \rightarrow Y_1, \ldots, Y_n$ *iff, for every* G^*: $G^* \not\models X_i$ *for some* i $(1 \leqslant i \leqslant m)$ *or* $G^* \models Y_j$ *for some* j $(1 \leqslant j \leqslant n)$
$G \models \mathsf{T}X$ *iff* $G \models X$
$G \models \mathsf{F}X$ *iff* $G \not\models X$.

Stated in terms of components, the rules for signed involutional formulas are:

G realizes $\mathsf{T}X$ *iff, for every* G^*, G^* *realizes some component of* $\mathsf{T}X$
G realizes $\mathsf{F}X$ *iff, for some* G^*, G^* *realizes every component of* $\mathsf{F}X$.

Thus, in the terminology of Smullyan [15], an involution signed with T is a $\forall$-special β, while an involution signed with F is a $\exists$-special α.

A formula X is said to be *true* in $\langle G_0, H, R, \phi\rangle$ iff $G_0 \models X$; *logically true* provided it is true in every model (of the appropriate kind). *Validity* will be defined for sequents $\Gamma \vdash \Delta$ (where Γ, Δ may be conceived as either *sets* or *lists* of formulas). A sequent $\Gamma \vdash \Delta$, then, is *valid* in the sense of **C**, **I**, **T**, **S4** or **S5** iff, for every model $\langle G_0, H, R, \phi\rangle$ of the appropriate kind, and every G_0^*: if every unstarred element of Γ is realized by G_0 and every formula X such that $X^* \in \Gamma$ is realized by G_0^*, then some unstarred element of Δ is realized by G_0 or some formula Y such that $Y^* \in \Delta$ is realized by G_0^*. It should be pointed out that starred formulas are used only in connection with T. When no starred formulas are present, the definition of validity reduces to the usual one.

We now consider a method of translation which assigns to each classical (propositional) formula, each modal formula of the form $\Box X$, and each intuitionistic formula of the form $\neg X$ or $X \supset Y$, a logically equivalent finite set of involutions. The method may be described, somewhat imprecisely, as follows:

For a classical formula Q, first reduce to conjunctive normal form $C_1 \wedge C_2 \wedge \ldots$. To each conjunct C_i, assign an involution $L \rightarrow M$ such that L (resp. M) is a list of all the proposition letters which occur negated (resp. unnegated) in C_i. The resulting set of involutions is the *involutional transform* of Q.

For modal and intuitionistic formulas we proceed inductively. If Q is atomic, the *involutional transform* of Q is $\{Q\}$. Let, then, $\Box Q$ be a modal formula such that the involutional transform has already been defined for all formulas of the form $\Box X$ properly contained in $\Box Q$. Let $\Box X_1, \Box X_2, \ldots$ be the maximal occurrences of this form in Q (i.e., $\Box X_1, \Box X_2, \ldots$, do not lie within the scope of any modal operator in Q). Replace each $\Box X_i$ $(i = 1, 2, \ldots)$ by the conjunction of the elements of its involutional transform.

Q is thus replaced by an expression Q' which is built up from involutional formulas by means of $\wedge$, $\vee$, and $\sim$. Treating the (maximal) involutional formulas occurring in Q' as atoms, apply the reduction already described for classical propositional formulas. This yields a set of involutions, which is taken as the *involutional transform* of $\Box Q$. Example: the transform of $\Box(\sim\Box P \equiv \Box\sim\Box P)$, i.e., $\Box((\sim\sim\Box P \vee \Box\sim\Box P) \wedge (\sim\Box\sim\Box P \vee \sim\Box P))$, is $\{\rightarrow(\rightarrow P), ((\rightarrow P)\rightarrow); (\rightarrow P), ((\rightarrow P)\rightarrow)\rightarrow\}$.

Finally, let X be an intuitionistic formula of the form $\neg P$ or $P \supset Q$, and suppose that the involutional transform has already been defined for all proper subformulas of X having either of these forms. Begin by replacing every such formula (-occurrence) in X by the conjunction of the elements of its involutional transform. If the result is $\neg P'$, this is replaced in turn by $P'\rightarrow$; if $P' \supset Q'$, by $P' \rightarrow Q'$. We now have an expression $P' \rightarrow Q''$ (Q'' empty, or $Q'' = Q'$) where P', Q'' are built up from involutions by means of $\wedge$ and $\vee$. It is easily seen that the following transformations:

$$
\begin{aligned}
&L \rightarrow M_1, A \wedge B, M_2 \Rightarrow L \rightarrow M_1, A, M_2;\ L \rightarrow M_1, B, M_2\\
&L \rightarrow M_1, A \vee B, M_2 \Rightarrow L \rightarrow M_1, A, B, M_2\\
&L_1, A \wedge B, L_2 \rightarrow M \Rightarrow L_1, A, B, L_2 \rightarrow M\\
&L_1, A \vee B, L_2 \rightarrow M \Rightarrow L_1, A, L_2 \rightarrow M;\ L_1, B, L_2 \rightarrow M
\end{aligned}
$$

in each of which a formula having a specified occurrence of $\wedge$ or $\vee$ is replaced by (at most two) formulas not having that occurrence—enable us to reduce $P' \rightarrow Q''$ to a set of involutions. This set is taken as the *involutional transform* of X. Example: $\neg(P \vee Q) \supset (\neg P \wedge \neg Q)$ reduces to

$$\{(P\rightarrow), (Q\rightarrow) \rightarrow (P\rightarrow);\ (P\rightarrow), (Q\rightarrow) \rightarrow (Q\rightarrow)\}.$$

Clearly, these instructions may be converted into a precise algorithm, yielding a unique result. Moreover, a trivial verification establishes the following result:

If S is the involutional transform of a formula X, then, for any model μ (of the appropriate kind), X is true in μ iff every element of S is true in μ.

It is perhaps worth remarking that a more comprehensive translation is possible for some of the standard systems. For example, intuitionistically, a disjunction $P \vee Q \vee \ldots$ is logically equivalent to $\rightarrow P, Q, \ldots$ (This depends on the fact that, in an **I**-model, a formula X is realized by a universe G iff X is realized by all G^*.) By making use of this equivalence, one can extend the translation to all intuitionistic formulas. Again, by utilizing certain modal equivalences one can extend the translation of modal formulas in **S4** and **S5** (though not to a complete translation).

For logical truth/validity a complete (and natural) reduction is possible even for the modal calculi. As to logical truth, we need only point out that a formula X is logically true iff $\Box X$ is logically true. For validity, the proof is almost as straightforward: Let a modal sequent $\Gamma \vdash \Delta$ be given. Begin by replacing each maximal formula-occurrence of the form $\Box X$ by the conjunction of the elements of its involutional transform. Then apply Gentzen-like rules to 'reduce' the sequent to a set of sequents in which no truth-functional connectives occur—and which must, therefore, be purely involutional sequents.

2 *Axiomatics.* In this section we set up involutional calculi corresponding to the classical and intuitionistic propositional calculi, **T**, **S4** and **S5**, and establish their correctness and completeness. The methods used are mainly due to Kripke, Hintikka, Smullyan, and Fitting.

'Formula' is now understood to mean 'involutional formula'. A formula of the form $\mathsf{T}X$, where X is non-atomic (and may be starred) is called a T-*involution*; F-*involution* is defined similarly. The involutional calculi take the form of deductive systems for proving the inconsistency of sets of signed formulas. In the statement of the rules, S is an (arbitrary) finite set of signed formulas, X an unsigned formula, D a signed formula, A a T-involution, and B an F-involution. To simplify the notation one writes, e.g., 'S, A' instead of '$S \cup \{A\}$'. For $\mathbf{C}_{\mathrm{inv}}$ ('classical' involution) the rules are:

O $\qquad\{\mathsf{T}X, \mathsf{F}X\}$

W (Weakening) $\qquad\dfrac{S}{S, D}$

T-intr $\qquad\dfrac{S, A_1; \ldots; S, A_n}{S, A}$ where $A_1, \ldots, A_n$ are the components of A.

F-intr $\qquad\dfrac{S, B_1, \ldots, B_n}{S, B}$ where each $B_i (i = 1, \ldots, n)$ is a component of B.

That is, a set S' is considered as derivable in $\mathbf{C}_{\mathrm{inv}}$ iff it is either of the form $\{\mathsf{T}X, \mathsf{F}X\}$ or can be constructed from sets of this form by a series of applications of the transformations **W**, T-intr, F-intr.

Next, we must state the modifications needed for the modal and intuitionistic systems. In modal contexts, a *strict* formula is a formula that contains at least one occurrence of $\rightarrow$. For $\mathbf{S4}_{\mathrm{inv}}$, then, the system is changed in just one respect: in the rule F-intr, S is restricted to contain only T-involutions. The system $\mathbf{I}_{\mathrm{inv}}$ is like $\mathbf{S4}_{\mathrm{inv}}$ except that now, in the statement of F-intr, S may contain any formulas signed with T (and only such formulas).

In $\mathbf{S5}_{\mathrm{inv}}$, we have **O**, **W** and T-intr as in $\mathbf{C}_{\mathrm{inv}}$. F-intr is now subject to the restriction: either every member of S is strict, or every B_i is strict (or both). We add the special rule:

T-intr$_{\mathbf{S5}}$ $$\frac{S, A^0; S, A_1^s; \ldots; S, A_n^s}{S, A}$$

where $A_1^s, \ldots, A_n^s$ are the strict components of A, and A^0 is the formula obtained by omitting all strict components from A (that is, if A is $\mathsf{T}L \rightarrow M$, then A^0 is $\mathsf{T}L^0 \rightarrow M^0$, where L^0, M^0 are the lists obtained by omitting all strict formulas from L, M).

In $\mathsf{T}_{\mathrm{inv}}$, formulas may be starred. We have the rule of star-introduction:

Star-intr $\dfrac{S}{\{D* \mid D \in S\}}$ where no element of S is starred.

O, **W** and T-intr are as for $\mathbf{C}_{\text{inv}}$, and in addition we have the following starred version of T-intr:

T-intr* $\dfrac{S, A_1^*; \ldots; S, A_n^*}{S, A}$ where $A_1, \ldots, A_n$ are the components of A.

F-intr is replaced by:

F-intr* $\dfrac{S, B_1^*, \ldots, B_n^*}{S, B}$ where each B_i is a component of B, and no element of S is starred.

This completes the statement of the rules for $\mathbf{T}_{\text{inv}}$.

A set S of signed formulas is said to be *satisfied* by the model $\langle G_0, H, R, \phi\rangle$ iff there is some G_0 such that: every unstarred element of S is realized by G_0, and every formula D such that $D* \in S$ is realized by G_0^*. We will show that, relative to the appropriate class of models, each of the systems just described is correct and complete w.r.t. unsatisfiability of (finite) sets of signed formulas.

There is a natural correspondence between finite sets of signed formulas and sequents of unsigned formulas (which is (1,1) if the two parts of a sequent are regarded as *sets* of formulas). Namely, the sequent $\Gamma \vdash \Delta$ corresponds to the set $\{\mathsf{T}X \mid X \in \Gamma\} \cup \{\mathsf{F}X \mid X \in \Delta\}$. The sequent is valid iff the corresponding set of signed formulas is unsatisfiable. Hence the results in the present section immediately yield similar results for sequents.

Lemma *In each of the involutional systems: in any application of an inference rule (excluding* **O**), *if the conclusion is satisfiable, then at least one premise is satisfiable.*

Proof: For **W** this is trivial. For T-intr, in any of its versions, it follows immediately from the definitions that any model which satisfies the conclusion satisfies at least one premise. This holds also for F-intr in $\mathbf{C}_{\text{inv}}$ and for F-intr* (in $\mathbf{T}_{\text{inv}}$). Consider, then, an application of F-intr in $\mathbf{S4}_{\text{inv}}$, such that the conclusion S, B is satisfied by the model $\langle G_0, H, R, \phi\rangle$. Since G_0 realizes B, there is some G_0^* which realizes all the components of B, including $B_1, \ldots, B_n$. Since each element of S is realized by G_0, and S contains only T-involutions, each element of S is also realized at G_0^*. Hence by taking G_0 as the new distinguished universe, and restricting (in the obvious sense) H, R and ϕ to the R-successors of G_0^* we obtain a model satisfying $S, B_1, \ldots, B_n$. The proof for $\mathbf{I}_{\text{inv}}$ is similar. Finally, consider an application of F-intr in $\mathbf{S5}_{\text{inv}}$, under the same conditions. If all formulas of S are strict, we obtain a model satisfying the premise $S, B_1, \ldots, B_n$ by the same construction as in $\mathbf{S4}_{\text{inv}}$. Suppose, instead, that $B_1, \ldots, B_n$ are all strict. Since G_0 realizes B, there is some G_0^* which realizes $B_1, \ldots, B_n$. Since the latter formulas are strict, it follows by familiar properties of $\mathbf{S5}$-models that G_0 realizes $B_1, \ldots, B_n$. Hence the model $\langle G_0, H, R, \phi\rangle$, assumed to satisfy S, B, also satisfies $S, B_1, \ldots, B_n$.

Theorem 1 *Each of the involutional systems is correct w.r.t. unsatisfiability in the appropriate class of models.*

Proof: A set of the form **O** is trivially unsatisfiable. By contraposing the lemma we find that, in any application of one of the rules of transformation, unsatisfiability of the premises implies unsatisfiability of the conclusion. The result follows by induction.

The role of the starred rules in $\mathbf{T}_{\text{inv}}$ is particularly noteworthy. Let S, A be a set of formulas satisfied by the **T**-model $\langle G_0, H, R, \phi\rangle$. By the interpretation assigned to T-involutions, some component of A must be realized by G_0; this is mirrored in the rule T-intr. But, *in addition,* some component of A must be realized at an arbitrarily chosen G_0^*. Hence if none of $S, A_1^*; \ldots; S, A_n^*$ is satisfied by the model, then S, A cannot be satisfied by it; this explains the presence of the rule T-intr*. A similar explanation can be given for the rule F-intr*. From this point of view, the use of starred formulas is a direct reflection of the fact that the realization of a strict formula by a universe G has 'effects', not only at G, but at neighbouring G^*. It is perfectly feasible to set up all the modal systems in terms of this device (*cf.* [6]). But we have preferred to avoid its use as far as possible—at the cost, it must be admitted, of some artificiality in the rules for $\mathbf{S5}_{\text{inv}}$.

By inspection of the transformation rules, we find that, in each involutional system, if S is a finite set of signed formulas, then

(i) there is only a finite number of collections $C_1, C_2, \ldots$, such that S can serve as the conclusion in some application of a rule in which C_i is the collection of premises ($i = 1, 2, \ldots$);
(ii) the possible premises (i.e., elements of some C_i) are, in an obvious sense, simpler than S. (We do not quite have the subformula principle, on account of the special T-intr rule in $\mathbf{S5}_{\text{inv}}$, and the starred rules in $\mathbf{T}_{\text{inv}}$.)

The construction of trees of possible premises therefore constitutes a decision-procedure. Let us say that a finite set of formulas is *consistent* w.r.t. an involutional system iff it is not derivable in the system. Then the construction (decision-procedure) just mentioned, applied to a consistent set, yields a model which satisfies the set; this is, roughly, the content of the completeness proofs which follow.

First, the notions of *reduced set* and *associated set* (Fitting [5]) will be adapted to the present systems. Let S be a consistent set. Then, in $\mathbf{S4}_{\text{inv}}$, $\mathbf{I}_{\text{inv}}$ or $\mathbf{T}_{\text{inv}}$, a consistent set S' is called a *reduced set* for S iff conditions (i)-(iii) are satisfied:

(i) $S \subset S'$.
(ii) Every element of S' is a subformula[1] of some element of S.

1. 'Subformula (of)' may here be taken to represent the reflexive and transitive closure of the relation *component* (*of*).

(iii) If a T-involution A belongs to S', then at least one component of A belongs to S'.

In $\mathbf{C}_{\text{inv}}$ it is required that S' satisfy (i)-(iii) and, in addition,

(iv) If an F-involution B belongs to S', then every component of B belongs to S'.

In $\mathbf{S5}_{\text{inv}}$ it is required that S' satisfy (i)-(iii) and also (v), (vi):

(v) If an F-involution B belongs to S', then every strict component of B belongs to S'.
(vi) If a T-involution A belongs to S', then either some strict component of A belongs to S', or else A^0 belongs to S', where A^0 is the formula obtained by omitting strict components from A (*cf.* the statement of T-intr$_{S5}$).

Lemma *In each system, if S is a consistent set, there exists a reduced set for S.*

Proof: Consider first the systems $\mathbf{T}_{\text{inv}}$, $\mathbf{S4}_{\text{inv}}$, $\mathbf{I}_{\text{inv}}$. Suppose that S' is a consistent set which satisfies conditions (i) and (ii), but not condition (iii): i.e., there is a T-involution A in S' such that no component of A is in S'. Let $A_1, \ldots, A_n$ be the components of A. At least one of the sets S, A_i ($i = 1, \ldots, n$) is consistent; for if all these sets were derivable, S, A would be derivable by virtue of T-intr. Hence S' can be properly extended to a new consistent set which also satisfies conditions (i), (ii). Since there are only finitely many subformulas of members of S, a finite sequence of such extensions, starting from S (which, of course, satisfies conditions (i), (ii)), must yield a reduced set for S. For $\mathbf{C}_{\text{inv}}$ and $\mathbf{S5}_{\text{inv}}$ we have to take account of extensions of S' corresponding to conditions (iv), (v) and (vi) (as well as (iii)). That these extensions preserve consistency is seen by considering the rules F-intr (in $\mathbf{C}_{\text{inv}}$), F-intr (in $\mathbf{S5}_{\text{inv}}$) and T-intr$_{S5}$, respectively. The argument is then completed as before. Thus, for each consistent set S, there is a finite, non-empty collection of reduced sets. At least one element of this collection is minimal, i.e., has no proper subset that is a reduced set for S. We shall assume that, by some arbitrary rule, a single, minimal element is chosen from each collection, so that we can speak of *the* reduced set S' for S.

We now define the *associated sets* of a consistent set S, for each system except $\mathbf{C}_{\text{inv}}$. There will be one associated set for each F-involution in S. In the definition, there occurs the symbol 'S^+', whose meaning depends on the system under consideration: in $\mathbf{S4}_{\text{inv}}$, S^+ is the set of all T-involutions belonging to S; in $\mathbf{I}_{\text{inv}}$, S^+ is the set of all formulas signed with T which belong to S; in $\mathbf{S5}_{\text{inv}}$, S^+ is the set of all strict formulas belonging to S. For $\mathbf{S4}_{\text{inv}}$, $\mathbf{I}_{\text{inv}}$ and $\mathbf{S5}_{\text{inv}}$, the definition is: If B is an F-involution belonging to S and $B_1, \ldots, B_n$ are the components of B, then $\{B_1, \ldots, B_n\} \cup S^+$ is an *associated set* of S. For $\mathbf{T}_{\text{inv}}$, the construction of the associated sets is a little more complex. Let S be a consistent set of unstarred formulas, and B an F-involution in S. Let $A^1, \ldots, A^m$ be all the

T-involutions in S. Construct a sequence of sets as follows: Set $S^0 = S \cup \{B_1^*, \ldots, B_n^*\}$, where $B_1, \ldots, B_n$ are the components of B. S^0 is consistent, since F-intr* enables one to derive S from S^0. Suppose that S^{k-1} $(1 \leq k \leq m)$ has been defined, as a consistent extension of S. In virtue of T-intr*, at least one set $S^{k-1} \cup \{A_i^{k*}\}$, where A_i^k is a component of A^k, is consistent; take a particular such set as S^k. This construction produces a sequence $S^0, \ldots, S^m$. Then the set $\{D \mid D^* \in S^m\}$ is an *associated set* of S. We note that, since S^m is consistent, the presence of the rule star-intr ensures the consistency of the associated set.

Theorem 2 *For each involutional system: if S is consistent and contains no starred formulas, S is satisfiable.*

Proof: The simplest case, $\mathsf{C}_{\mathsf{inv}}$, is treated first. In constructing the model, we ignore G_0, H and R. Let S' be the reduced set of S. For each proposition letter P, let $\phi(P) = \mathsf{T}$ if $\mathsf{T}P \in S'$, $\phi(P) = \mathsf{F}$ otherwise. ϕ determines a model (in an obvious way); we show, by induction on the degree of D, that if $D \in S'$ then D is true in this model. If X is atomic, this is trivial. Suppose that X is a T-involution, and that the assertion is true for all formulas of degree less than that of X. If $X \in S'$, at least one component of X, say X_i, also belongs to S'. By the induction hypothesis, X_i is true in the model. But then (since there is only one universe) X is true in the model. The case in which X is an F-involution is dealt with similarly. Thus each element of S', and a fortiori of S, is true in the model.

For each of the other systems, we construct the required model with the aid of a certain directed graph, to each node of which is attached a set of formulas. The construction of the graph begins with a single node G_0, to which is attached S' (the reduced set of S). Then the graph is extended by steps of the following kind: Let N be a node already present, U the set attached to N. Let U' be the reduced set of any associated set of U. If U' is already attached to some node of the graph, insert an arc leading from N to this node (provided such an arc is not present already); if not, insert a new node N', attach U' to N', and add an arc leading from N to N'. Since only a finite number of distinct sets of formulas can arise in the course of this construction, we must reach, in a finite number of steps, a graph Z for which no further extension is possible.

We take the nodes of Z as universes, i.e., set H = set of nodes of Z. Define ϕ by: for any $G \in H$, and any proposition letter P,

$$\phi(P, G) = \begin{cases} \mathsf{T} \text{ if } \mathsf{T}P \text{ occurs at } G, \\ \mathsf{F} \text{ otherwise.} \end{cases}$$

For the rest of the construction—and its justification—the various systems are treated separately.

$\mathsf{T}_{\mathsf{inv}}$: Define GRG' (where $G, G' \in H$) to mean that either $G = G'$ or there is an arc leading from G to G'. Thus we have a T-model $\langle \mathsf{G}_0, H, R, \phi \rangle$. Trivially, if X is an atomic formula occurring at a node G in Z, then X is realized by G in $\langle \mathsf{G}_0, H, R, \phi \rangle$. Further, the method of constructing Z has ensured that (i) if an F-involution X occurs at node G, then there is a node G' such that GRG' and every component of X occurs at G', and (ii) if a T-involution X occurs at G, then at least one component of X occurs at each G' such that GRG'. By induction on the degree of X it follows that every formula occurring at a node G in Z is realized by G in $\langle \mathsf{G}_0, H, R, \phi\rangle$. In particular, every element of S is realized by G_0; i.e., S is satisfied by $\langle \mathsf{G}_0, H, R, \phi\rangle$.

S4$_{\mathrm{inv}}$: Define GRG' to mean that there is a sequence of zero or more arcs leading from G to G'. Evidently, $\langle \mathsf{G}_0, H, R, \phi\rangle$ is an **S4**-model. The construction of Z has ensured that, if a T-involution X occurs at node G, then X, and hence at least one component of X, occurs at each G' such that GRG'. Apart from this, the argument proceeds as before (**T**$_{\mathrm{inv}}$).

I$_{\mathrm{inv}}$: The construction is the same as for **S4**$_{\mathrm{inv}}$. That the extra condition demanded of **I**-models—namely, $G \models \mathsf{T}X$ implies $G^* \models \mathsf{T}X$ even when X is atomic—is fulfilled, follows from the way in which 'associated set' has been defined for **I**$_{\mathrm{inv}}$.

S5$_{\mathrm{inv}}$: The argument here is more involved. The main burden of the proof is to show that, if X is a strict formula occurring at some node G in Z, then X occurs at every node in Z. Let G' be a successor of G (i.e., there is an arc from G to G'). It is trivial that any strict formula which occurs at G occurs also at G'. We prove the converse by reductio ad absurdum. Suppose, if possible, that X is a strict formula which occurs at G', but not at G, and that X is of maximal degree among formulas having this property. Let S, S' be the sets attached to G, G', so that S' is the reduced set of an associated set of S. By the minimal property of the reduced set, one of the following cases (i), (ii) must obtain:

(i) X is a component of some F-involution belonging to S;
(ii) X is a component of some T-involution A belonging to S'.

(Otherwise $S'-\{X\}$ would satisfy the conditions for a reduced set.) But (i) can be ruled out since, by the definition of 'reduced set' for **S5**, every strict component of an F-involution in S must itself be in S. Turning to (ii), we can assume, without loss of generality, that X is the *only* component of A that belongs to S'; for if *every* T-involution in S' that has X as a component also has other components which belong to S', then, as is easily seen, $S'-\{X\}$ must satisfy the conditions for a reduced set. Since A is strict and of higher degree than X, $A \in S$. But no strict component of A belongs to S (for every strict element of S is an element of S'). Hence (clause (vi) in the definition of 'reduced set') $A^0 \in S$, where the components of A^0 are exactly the atomic components of A. Then A^0—and therefore at least one component of A^0—belongs to S'. But this contradicts the assumption that X is the only component of A in S'. Thus case (ii) is ruled out, and we conclude that there is no strict formula in S' that is not also in S. Let R be the relation on H such that GRG' holds for every pair of nodes G, G'. Since Z is connected, the preceding argument shows that every

T-involution that occurs at G also occurs at each G' such that GRG' (i.e., at every node G'). As with $\mathbf{S4}_{\mathrm{inv}}$, it follows that S is satisfied by the $\mathbf{S5}$-model $\langle \mathrm{G}_0, H, R, \phi\rangle$. This completes the proof of the theorem.

An example of a set whose derivation in $\mathbf{S5}_{\mathrm{inv}}$ requires the use of $\mathsf{T}\text{-}\mathsf{intr}_{\mathbf{S5}}$ is $\{\mathsf{T}p \rightarrow (q \rightarrow r),\ \mathsf{T}q,\ \mathsf{F}r,\ \mathsf{F}p\}$ (p, q, r proposition letters). If sets of signed formulas are represented by the corresponding sequents, the derivation may be set out in tree form as follows:

$$\dfrac{\dfrac{p\rightarrow\ \vdash p\rightarrow}{p\rightarrow;\ q\vdash r;\ q\rightarrow}\,\mathsf{W} \qquad \dfrac{\dfrac{q\vdash q}{q\vdash r;\ p\rightarrow;\ q}\,\mathsf{W} \quad \dfrac{r\vdash r}{r;\ q\vdash r;\ p\rightarrow}\,\mathsf{W}}{q\rightarrow r;\ q\vdash r;\ p\rightarrow}\,\mathsf{T\text{-}intr}}{p\rightarrow(q\rightarrow r);\ q\vdash r;\ p\rightarrow}\,\mathsf{T\text{-}intr}_{\mathbf{S5}}$$

There is no way of deriving this conclusion if the rule $\mathsf{T}\text{-}\mathsf{intr}_{\mathbf{S5}}$ is dropped.

3 *Applications.*

3.1 *The Problem of Entailment.* The formula

$$p \wedge {\sim} p \rightarrow q \tag{1}$$

where the arrow may, for the moment, be taken to represent strict implication, enshrines a standard implicational 'paradox'. According to the scheme of translation into the (classical or) modal involutional systems, (1) is to be construed as

$$p \rightarrow p,\ q \tag{2}$$

The occurrence of q here is 'redundant': if it is dropped, we obtain the 'stronger', but still valid, involution

$$p \rightarrow p \tag{3}$$

It would seem that the redundancy of q provides the only possible ground for objecting to (2). But before attempting to generalize this observation, we need to consider whether, in replacing (1) by (2) and (3), we have not unduly trivialized the problem. A less trivial rendering of the principle which (1) is intended to express would be, for example,

$$\rightarrow p;\ p\rightarrow\ \dot{\rightarrow}\ \rightarrow q \tag{4}$$

or, more generally,

$$t \rightarrow p;\ t,\ p\rightarrow\ \dot{\rightarrow}\ t \rightarrow q \tag{5}$$

(Parentheses have here been suppressed in favour of dots, as in [11].)
(5) may be read as: something which entails p, and at the same time is incompatible with p, entails anything. A slight variation of (5) gives us

$$t \rightarrow p,\ q;\ t,\ p\rightarrow\ \dot{\rightarrow}\ t \rightarrow q \tag{6}$$

which may be regarded as a version of the principle of disjunctive syllogism (say, D).

It is apparent that the occurrence of q on the right of (5) is redundant, but not so the parallel occurrence in (6): if the second occurrence of q in (6) is omitted, the resulting involution is invalid. Thus, according to the criterion—so far only vaguely indicated—that entailments should be irredundant, D seems to be acceptable. This agrees with the views of most theorists of entailment. An apparent exception is provided by Anderson and Belnap [2], who maintain that—at least if disjunction is understood purely truth-functionally—D is invalid. In rejecting D, they have in mind a formulation such as

$$(p \vee q) \wedge \sim p \rightarrow q \tag{7}$$

Now, to construe the disjunction (and negation) in (7) 'purely truth-functionally' may be taken to mean, on the present approach, that (7) is to be dissolved into involutions in accordance with the scheme of translation in section 1. One finds that (7) reduces to the pair of involutions

$$p \rightarrow p, q \qquad q \rightarrow p, q$$

both of which have redundant constituents. Anderson and Belnap qualify their rejection of D in case the connectives are understood *intensionally*. In the involutional systems, the intensional disjunction of p, q may be rendered by $\rightarrow p, q$; and the intensional negation of p by $p \rightarrow$ (*cf.* Duthie, [4]). This involves construing (7) as

$$\rightarrow p, q; p \rightarrow \mathrel{\dot{\rightarrow}} q \ ,$$

a restricted version of (6). We have already indicated that (6) is acceptable. In this way the view of Anderson and Belnap on the status of D can perhaps be reconciled with the involutional approach.

Related to the criterion of irredundancy is a second criterion having to do with methods of proof: it requires that, in the course of proving an entailment (or involution), no items should appear that are not *relevantly used* in the proof. That the two criteria (which will be stated more precisely in a moment) are non-equivalent is shown by an example of Smiley [12]:

$$p \rightarrow q; q \rightarrow p \ ; p \rightarrow q \mathrel{\dot{\rightarrow}} p \rightarrow q \tag{8}$$

According to the first criterion, (8) is clearly unacceptable. Yet we can easily give a natural-deductive proof of (8), in which all items which appear are relevantly used:

$$(2)\,(3)\,(4)\ \frac{(1)\ \dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{p} \qquad \overset{(2)}{p \rightarrow q}}{q} \qquad \overset{(3)}{q \rightarrow p}}{p} \qquad \overset{(4)}{p \rightarrow q}}{q}}{p \rightarrow q}}{p \rightarrow q;\ q \rightarrow p;\ p \rightarrow q \mathrel{\dot{\rightarrow}} p \rightarrow q}$$

It should be noted that (8), although unacceptable by the first criterion, is a substitution-instance of

$$p \rightarrow q; q \rightarrow r; r \rightarrow s \dashrightarrow p \rightarrow s \ ,$$

which is acceptable.

For the systematic treatment, we again use signed formulas. But now we shall be concerned with *lists* rather than sets of such formulas. For simplicity, we consider only versions of the *classical* involutional system. Let L be a list of signed formulas, P an arbitrary proposition letter not occurring in L. Then L is said to be *irredundantly unsatisfiable* (*i.u.*) iff

(i) no C-model satisfies L,
(ii) if any unsigned formula which occurs in L (i.e., as a subformula of an item in L) is replaced, at any one occurrence, by P, the resulting list is satisfiable.

For the precise version of the second criterion, we set up a modified version of $\mathsf{C}_{\mathsf{inv}}$, in which only 'relevant' deductions of (the inconsistency of) lists of signed formulas can be carried out. A list is *relevantly deducible* (*r.d.*), then, iff it can be derived from lists of the form

O: $\mathsf{T}X$, $\mathsf{F}X$ by the transformation rules.
P (Permutation): A list may be permuted in any manner.

T-intr: $\dfrac{L_1, A_1; \ldots; L_n, A_n}{L_1, \ldots, L_n, A}$ where $A_1, \ldots, A_n$ is the list of components of the T-involution A; i.e., if A is $\mathsf{T}X_1, \ldots, X_k \rightarrow Y_1, \ldots, Y_l$, then $n = k + l$, and $A_1 = \mathsf{F}X_1, \ldots, mA_k = \mathsf{F}X_k, A_{k+1} = \mathsf{T}Y_1, \ldots, A_n = \mathsf{T}Y_l$.

F-intr: $\dfrac{L, B_1, \ldots, B_n}{L, B}$ where $B_1, \ldots, B_n$ is the list of components of B.

As already hinted, the two criteria are closely related:

Theorem 3 *Every r.d. list is a substitution-instance of an i.u. list.*

Proof: Trivially, a list of the form O is a substitution-instance of an i.u. list; and, if the premise of an application of P or F-intr has this property, so does the conclusion. It remains to consider T-intr. Suppose that the lists $L_1', A_1'; \ldots; L_n', A_n'$ are substitution-instances of i.u. lists $L_1', A_1'; \ldots; L_n', A_n'$. We assume that the latter lists are chosen in such a way that no proposition-letter occurs in more than one list. Let P be a proposition-letter not occurring in any of the lists. Let A' be the T-involution whose list of components is $A_1', \ldots, A_n'$. Then the list $L_1', \ldots, L_n', A'$ is unsatisfiable (by the correctness of T-intr), and has $L_1, \ldots, L_n, A$ as a substitution-instance. We have to show that, if some formula-occurrence in $L_1', \ldots, L_n', A'$ is replaced by P, then the resulting list $L_1', \ldots, L_{i-1}', L_i'', L_{i+1}', \ldots, L_n', A'$ (supposing the replacement made in L_i'), or $L_1', \ldots, L_n', A''$ (supposing the replacement made in A') is satisfiable. Take the former case (replacement in L_i') first. Since each of $L_1', A_1'; \ldots; L_n', A_n'$ is i.u., each of $L_1'; \ldots; L_{i-1}'; L_i'', A_i'; L_{i+1}'; \ldots; L_n'$ is satisfiable. Since no proposition letter occurs in more than one of the latter lists, we can construct a model μ which simultaneously satisfies them all. It is clear that μ satisfies the list

$L_1', \ldots, L_{i-1}', L_i'', L_{i+1}', \ldots, A'$, as required. A similar argument is easily supplied for the case that the replacement is made in A'. This completes the induction-step required for the proof of the theorem.

The converse of this theorem is false. Example: The list whose only item is

$$\mathsf{F}p; p \to q; p \to (q \to r) \dot{\to} r \tag{9}$$

is i.u. but, as is easily checked, not r.d. We can get an r.d. list by repeating the first p in (9):

$$\mathsf{F}p; p; p \to q; p \to (q \to r) \dot{\to} r \ .$$

It seems plausible that every i.u. list can be obtained from an r.d. list by deleting repeated items in sublists; but we do not have a proof of this at present.

3.2 *Operative Logic.* By restricting the rules of formation of I_{inv} so that only involutions having exactly one formula in the consequent are permitted—a restriction which will be presupposed in the rest of this sub-section—one obtains a system essentially equivalent to the 'consequence-logic' of Lorenzen [11] and the system of 'positive S-formulas' of Kutschera [9]. The completeness of these systems, relative to I-models, is easily proved. However, the following question arises: Is the interpretation in terms of Kripke's models consonant with the meaning which Lorenzen and Kutschera attribute to their formalisms?

Lorenzen's interpretation (which we can here only hastily summarize) is in terms of the *admissibility* of rules in calculi. Roughly speaking, a first-order rule is admissible with respect to a given calculus iff, when applied to derivable formulas as 'premises', it yields a 'conclusion' that is also derivable in the calculus. Rules of arbitrary finite order are introduced, and the definition extended to these rules as follows: a rule of order n is *admissible* iff, when applied to admissible rules of order n - 1 as premises, it yields an admissible rule (of order n - 1) as conclusion. Let P be an involution of degree n. P is said to be *generally admissible* provided that, for any calculus K, if formulas of K are substituted for the proposition letters in P, the result is an admissible rule of order n over K. The system of generally admissible involutions gives us *deduction-logic.* Note: the present restriction to propositional logic means that we can treat only of 'rules' in which no variables (ranging over expressions of K) occur.

Kutschera [9], citing Lorenz [10], argues that, on constructivist grounds, one should work with *derived* rules rather than admissible rules. A first-order rule R is *derivable* w.r.t. calculus K provided that, if the premises of R are added to K as axioms, the conclusion of R is a derivable formula in the extended calculus. The definition may be repeated for second-order rules, since the premises of a second-order rule can be added to K as *rules* (instead of axioms).

It is not immediately evident how to extend this kind of definition to rules of arbitrary order. Kutschera's method involves the construction of a special hierarchy of calculi on the basis of K. An alternative, perhaps simpler, procedure is as follows: A formula of K is regarded as a *rule of order* 0. For rules of order 0, i.e., formulas, the definition of *derivable* is as usual. Then, a rule $A_1, \ldots, A_n \rightarrow B$ of order $n > 0$ is said to be *derivable* iff, in every extension of K for which the rules $A_1, \ldots, A_n$ (of order $n > 1$) are derivable, B is also derivable. It is readily seen that this agrees with the usual definition in the case of 1'st- and 2'nd-order rules.

In its systematic use of *extensions* of calculi, the method closely resembles Kripke's 'provability' interpretations of intuitionism [8]. Explicitly: Let H be a set of extensions of the calculus K (all with the same vocabulary and rules of formation as K). Take $\mathsf{G}_0 = K$, and define R by: GRG' iff G' is an extension of G. Let a mapping $\mathfrak{B}: \mathcal{P} \rightarrow E$ be given, where E is the set of formulas of K, and define ϕ by: $\phi(P, G) = \mathsf{T}$ if $\mathfrak{B}(P)$ is derivable in G, F otherwise. With these specifications $\langle \mathsf{G}_0, H, R, \phi \rangle$ is an I-model. Consider the special case in which H is the set of all extensions of K. For any involution X, let $\mathfrak{B}(X)$ be the result of replacing each proposition letter P in X by $\mathfrak{B}(P)$. Then we have: an involution X is true in the model just defined iff $\mathfrak{B}(X)$ is derivable as a rule over K. It is apparent that, with a different specialization of H, we obtain also the *admissible* rules. Namely, let $H = \{K\}$—in effect, reducing the model to a C-model. In this model, X is true iff $\mathfrak{B}(X)$ is admissible over K. Thus, on the present analysis, Lorenzen's method presupposes a classical, Kutschera's an intuitionistic, modelling.

3.3 *Syllogistic.* Quantificational logic lies outside the scope of this paper. Yet involution is quite naturally thought of as a relation between *properties*, not just propositions; and to illustrate this point of view, the involutional version of a fragment of predicate logic, in which variables and quantifiers are dispensable, will now be sketched.

Throughout the section, involutional formulas are restricted to be of degree at most one. In view of the application to traditional syllogistic proposition letters are usually referred to as *terms*. By an S-*model* is meant an ordered triple $\langle \mathsf{D}, U, \theta \rangle$, where U is an arbitrary non-empty set, D a distinguished element of U, and $\theta: \mathcal{P} \rightarrow 2^U$ a mapping which assigns a subset of U to each term. Truth in such a model is defined thus: an atomic formula P is *true* iff $\mathsf{D} \in \theta(P)$; a formula $P_1, \ldots, P_m \rightarrow Q_1, \ldots, Q_n$, of degree one, is *true* iff $\bigcap_{i=1}^{m} \theta(P_i) \subset \bigcup_{j=1}^{n} \theta(Q_j)$. A sequent $\Gamma \vdash \Delta$ is S-*valid* iff, in every S-model in which every element of Γ is true, at least one element of Δ is true. A formal system for deriving S-valid sequents may be constructed along the lines of the modal and intuitionistic systems of section **2**. Indeed the restriction of any of these systems to formulas of degree at most one would suffice, in view of:

Theorem 4 *For sequents composed of (unstarred) formulas of degree at most one,* S-, T-, $\mathsf{S4}$-, I-, *and* $\mathsf{S5}$-*validity are equivalent.*

Proof: The proof depends on the following observation: an involution

$P_1, \ldots, P_m \to Q_1, \ldots, Q_n$, of degree one, is true in the **T**-model (or **S5**-model) $\langle G_0, H, R, \phi\rangle$ iff

$$\bigcap_{i=1}^{m} \{G_0^* \mid \phi(P_i, G_0^*) = \mathsf{T}\} \subset \bigcup_{j=1}^{n} \{G_0^* \mid \phi(Q_j, G_0^*) = \mathsf{T}\}.$$

To establish the theorem, it suffices to show that every **S5**-valid sequent is **S**-valid, and that every **S**-valid sequent is **T**-valid. As to the first, suppose that $\langle \mathsf{D}, U, \theta\rangle$ is an **S**-countermodel for $\Gamma \vdash \Delta$—i.e., an **S**-model in which every element of Γ is true, every element of Δ false. An **S5**-countermodel for the sequent may be obtained as follows: Take $G_0 = \mathsf{D}$, $H = U$, R as the relation which holds between every pair of elements of U, and define ϕ by

$$\phi(P, G) = \mathsf{T} \text{ if } G \in \theta(P), \; \mathsf{F} \text{ otherwise.}$$

As to the second, suppose that $\langle G_0, H, R, \phi\rangle$ is a **T**-countermodel for $\Gamma \vdash \Delta$. An **S**-countermodel is obtained by taking $\mathsf{D} = G_0$, $U = H$, and, for $P \in \mathcal{P}$,

$$\theta(P) = \{G_0^* \mid \phi(P, G_0^*) = \mathsf{T}\}.$$

The next theorem shows, in effect, that we can restrict attention to sequents having at most one formula in the consequent.

Theorem 5 *If no element of* Γ *is atomic, and the sequent* $\Gamma \vdash \Delta$ *is* **S**-*valid, then, for some* $Y \in \Delta$, $\Gamma \vdash Y$ *is* **S**-*valid.*

Proof: The proof rests on a simple observation concerning the algebra of sets: Suppose that, for $j = 1, \ldots, n$, $\langle S_1^j, \ldots, S_l^j, T_1^j, \ldots, T_m^j\rangle$ is an $l + m$-tuple of subsets of a set U_j (l, m fixed). Then, provided that $U_1, \ldots, U_n$ are pairwise disjoint,

$$\bigcap_{i=1}^{l}\left(\bigcup_{j=1}^{n} S_i^j\right) \subset \bigcup_{i=1}^{m}\left(\bigcup_{j=1}^{n} T_i^j\right)$$

iff, for $j = 1, \ldots, n$,

$$\bigcap_{i=1}^{l} S_i^j \subset \bigcup_{i=1}^{m} T_i^j.$$

Let $\Delta = \{Y_1, \ldots, Y_n\}$, and suppose that each $\Gamma \vdash Y_j$ is invalid ($j = 1, \ldots, n$). Choose countermodels $\langle \mathsf{D}_j, U_j, \theta_j\rangle$ for these sequents, such that U_j ($j = 1, \ldots, n$) are pairwise disjoint. Let D be any object not in the set $\bigcup_{j=1}^{n} U_j$. Let $U = \bigcup_{j=1}^{n} U_j \cup \{\mathsf{D}\}$. Define $\theta : \mathcal{P} \to 2^U$ by

$$\theta(P) = \bigcup_{j=1}^{n} \theta_j(P).$$

Then it follows from the observation on set-algebra that $\langle \mathsf{D}, U, \theta\rangle$ is a countermodel for $\Gamma \vdash \Delta$. Hence, if $\Gamma \vdash \Delta$ is valid, some $\Gamma \vdash Y_j$ is valid.

From this point on we shall be concerned only with sequents in which all formulas are of exactly degree one. For studying the validity of these sequents, the **S**-models are unnecessarily complicated. All that is necessary is to consider *assignments*, that is, (partial) mappings from terms to sets.

$P_1, \ldots, P_m \to Q_1, \ldots, Q_n$ is *true* under the assignment θ iff

$$\bigcap_{i=1}^{m} \theta(P_i) \subset \bigcup_{j=1}^{n} \theta(Q_j)\,.$$

Definitions of validity, etc., are as usual. When all formulas are of degree one, the two definitions of validity are in agreement. An assignment is said to be *restricted* iff all its values are non-empty sets. For the interpretation of traditional syllogistic, it is required that all assignments be restricted. This involves a change in the notion of validity. Let us temporarily distinguish validity in the original sense (unrestricted assignments) from validity w.r.t. restricted assignments by means of the subscripts u, r.

Theorem 6 *The sequent* $\Gamma \vdash \Delta$ *is valid*$_r$ *iff either*

(i) *for some* $Y \in \Delta$, $\Gamma \vdash Y$ *is valid*$_u$,

or

(ii) *for some term* P *occurring in* Γ *or* Δ, $\Gamma \vdash P \to$ *is valid*$_u$.

Proof: An assignment θ is restricted iff, for all P for which θ is defined, $P\to$ is false under θ. Hence $\Gamma \vdash \Delta$ is valid$_r$ iff $\Gamma \vdash \Delta;\ P_1\to;\ \ldots;\ P_n\to$ is valid$_u$, where $P_1, \ldots, P_n$ are all the terms which occur in Γ or Δ. The result follows by Theorem 5.

The preceding results provide the basis for a treatment of syllogisms with an arbitrary number of premises and with arbitrarily complex terms (built up from simple terms by means of $\wedge$, $\vee$ and $\sim$). For a syllogistic sequent may be reduced to a set of involutional sequents (and thus tested for validity) by the following operations: First, replace

S a P	by	$S \to P$
S e P	by	$S, P\to$
S i P	by	$\sim(S, P\to)$
S o P	by	$\sim(S \to P)$

Next, reduce each expression of the form $S \to P$ or $S, P\to$, in which S or P is compound, to a conjunction of purely involutional formulas (of degree one) by methods familiar from section 1. Finally, reduce the resulting sequent to involutional sequents in the natural way (*cf.* remarks at end of section 1).

If there are no compound terms, the syllogistic sequent reduces to a single involutional sequent, in which all formulas are of the form $S \to P$ or $S, P\to$.[2] This case will be treated in somewhat more detail; we begin with some definitions: Let Γ be a set of formulas each of which is of the form

2. The reduction of syllogisms to sequents of this form, and the method of enumerating valid syllogistic moods outlined below, was suggested to me by Mr. L. Jackson.

$S \to P$ or $S, P \to$. For any terms Q, R, we say that Q is a *predecessor* of R, and that R is a *successor* of Q, in Γ, iff there is a sequence $P_1, \ldots, P_k$ of terms such that $Q = P_1$, $R = P_k$ and, for each i such that $1 \leq i < k$, $P_i \to P_{i+1} \in \Gamma$. Q is said to be *separated* from R in Γ iff Q, R have successors Q', R' resp., such that $Q', R' \to \in \Gamma$ or $R', Q' \to \in \Gamma$. (For a given Γ, these relations may be conveniently displayed in a directed graph.)[3] Finally, Γ is *consistent* iff, for all terms Q, R, if $Q, R \to \in \Gamma$ then Q and R have no common predecessor in Γ.

In the following theorem, and subsequent comments, it is presupposed that all involutional formulas are of the form $S \to P$ or $S, P \to$, that all assignments are restricted, and that 'valid' means 'valid$_r$'.

Theorem 7 (i) $\Gamma \vdash$ *is valid iff* Γ *is inconsistent. If* Γ *is consistent, then* (ii) $\Gamma \vdash P \to Q$ *is valid iff* Q *is a successor of* P *in* Γ; *and* (iii) $\Gamma \vdash P, Q \to$ *is valid iff* P *is separated from* Q *in* Γ.

Proof: (i) Suppose that Γ is inconsistent. We show that, if θ is any assignment, the elements of Γ are not all true under θ—in short, θ does not satisfy Γ. For suppose that θ satisfies Γ. Since Γ is inconsistent, there are terms P, Q with the following properties: P and Q have a common predecessor R in Γ; and $P, Q \to \in \Gamma$. The first property entails that $\theta(R) \subset \theta(P)$ and $\theta(R) \subset \theta(Q)$, hence that $\theta(P) \cap \theta(Q)$ is non-empty; the second entails that $\theta(P) \cap \theta(Q)$ is empty. Thus it is impossible that θ satisfy Γ; in fact, $\Gamma \vdash$ is valid.

Next, suppose that Γ is consistent. For each term P occurring in Γ, let $\theta(P)$ be the set of predecessors of P in Γ. Then θ satisfies Γ. (θ will be called the *canonical* assignment determined by Γ.)

(ii) It is trivial that $\Gamma \vdash P \to Q$ is valid if Q is a successor of P in Γ. Suppose, then, that Q is not a successor of P in Γ. By hypothesis, Γ is consistent. The canonical assignment θ therefore satisfies Γ, but—since P is not a predecessor of Q—not $P \to Q$. That is, θ is a 'counter-assignment' for $\Gamma \vdash P \to Q$.

(iii) Sufficiency of the condition (for validity) is again straightforward. Suppose, then, that P is not separated from Q in Γ. Let R be any term not occurring in Γ. Then the set $\Gamma' = \Gamma \cup \{R \to P\} \cup \{R \to Q\}$ is consistent; and the canonical assignment determined by Γ' refutes $\Gamma \vdash P, Q \to$.

It follows from Theorem 7 that a sequent $\Gamma \vdash Z$ (where Z is either empty or a single formula) is valid iff Γ has a subset whose elements may be listed as follows:

(i) $P_i \to P_{i-1}; \ldots; P_2 \to P_1$; $P_i \to P_{i+1}; \ldots; P_{n-1} \to P_n$; $P_1, P_n \to$ (where $1 \leq i \leq n$)—which we shall abbreviate as $P_i \dashrightarrow P_1$; $P_i \dashrightarrow P_n$; $P_1, P_n \to -$; or
(ii) $P_1 \dashrightarrow P_n$; or
(iii) $P_1 \dashrightarrow P_i$; $P_n \dashrightarrow P_{i+1}$; $P_i, P_{i+1} \to$ (where $1 \leq i < n$).

If $\Gamma \vdash Z$ is *irredundantly* valid, then Γ has no other elements than those

3. A detailed treatment of syllogistic in terms of directed graphs is given in [16].

listed. In case (i), Z is empty, and there are n possible choices for i, giving us n 'moods' for the antilogism. Case (ii) (with $Z = P_1 \rightarrow P_n$) provides one mood. In case (iii), Z is either $P_1, P_n \rightarrow$ or $P_n, P_1 \rightarrow$, and there are $n - 1$ choices for i: hence $2(n - 1)$ moods.

Each involutional sequent $\Gamma \vdash Z$ is equivalent, under the translation process described earlier, to several distinct syllogistic sequents. This enables us to enumerate the valid moods of the n-term syllogism, *cf.* [12]. Roughly speaking, we can arrange for any one of the n formulas in Γ, Z to appear, suitably transformed, on the right of the sequent; e.g., when $P_1 \rightarrow P_2 \in \Gamma$, we can get a syllogism $\ldots \vdash P_1 \circ P_2$. Thus each of the $n + 1 + 2(n - 1)$ involutional moods gives rise to n syllogistic moods, yielding $n(3n - 1)$ moods altogether.

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