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SET-VALUED SET THEORY: PART ONE

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1 The Informal Systems In this section, we will consider the original exposition of Zadeh [3] and one of the more recent versions of the theory by Brown [1], both for the purpose of introducing the reader to the intuitive ideas to be formalized and so that the reader may see first hand some of the difficulties involved in the earlier formulations of the theory.

The Original System of Zadeh In [3], Zadeh defines *fuzzy sets* to be functions from some ordinary set X to the unit interval $[0, 1]$. (Strictly speaking, Zadeh says that fuzzy sets are characterized by these functions, but for our purposes, we may identify the set and the function that characterizes it.) Thus these functions are generalizations of the ordinary characteristic functions of the subsets of X. Zadeh tells us that such fuzzy sets are to represent classes of elements "with a continuum of grades of membership" ([3], p. 339), "classes of objects encountered in the real physical world [which] do not have precisely defined criteria of membership" ([3], p. 338). Among his examples is the "class of all real numbers which are much greater than one"; he indicates that such classes "play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction" ([3], p. 338). He further notes that "the notion of a fuzzy set is completely nonstatistical in nature" ([3], p. 340) and that the concept of fuzzy set "provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables" ([3], p. 339).

Zadeh continues ([3], pp. 340-341) with the following definitions:

A fuzzy set is *empty* if it is the constant function zero.
Two fuzzy sets are *equal* if they are equal as functions.
The *complement* of a fuzzy set f is the function defined by: $f'(x) = 1 - f(x)$.
The fuzzy set f is a *subset* of the fuzzy set g if and only if, for all x in X, $f(x) \leqq g(x)$.
The *union* of the fuzzy sets f and g is the function $f \cup g$ defined by $(f \cup g)(x) = \max[f(x), g(x)]$ (equivalently, the smallest fuzzy set having both f and g as subsets).

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The *intersection* of the fuzzy sets f and g is the function $f \cap g$ given by $(f \cap g)(x) = \min[f(x), g(x)]$ (equivalently, the largest fuzzy set which is a subset of both f and g).

Using these definitions, both DeMorgan's laws and the usual distributive laws hold:

$$(f \cup g)' = f' \cap g',\ (f \cap g)' = f \cup g',$$
$$f \cap (g \cup h) = (f \cap g) \cup (f \cap h),$$
$$f \cup (g \cap h) = (f \cup g) \cap (f \cup h)$$

([3], pp. 342-343)

However, although the fuzzy sets constitute a distributive lattice with 0 (the empty set) and 1 (the constant function 1) ([3], p. 343), the operation of complementation does not make this lattice into a complemented lattice. While it is true that such pleasant properties as DeMorgan's laws and the law $f'' = f$ hold, it is certainly not always the case that $f \cup f' = 1$ or that $f \cap f' = 0$. For example if $f(x) = a$ and a is neither 0 nor 1, then, as can be verified from the definitions $(f \cup f')(x) = \max[a, 1 - a] \neq 1$ and $(f \cap f')(x) = \min[a, 1 - a] \neq 0$. Exactly how this situation is to be interpreted is not totally clear. Evidently, the three seemingly natural definitions of union, intersection, and complement do not mesh in the usual manner. Since the usual laws for union and intersection are obeyed, the problem would seem to lie with the complementation process.

In fact, Zadeh implicitly speaks to this situation by defining the following further operations ([3], pp. 344-345).

The *algebraic product* of fuzzy sets f and g is the fuzzy set fg defined by $fg(x) = f(x)g(x)$.
The *algebraic sum* of fuzzy sets f and g is the fuzzy set $f + g$ defined by $(f + g)(x) = f(x) + g(x)$ provided that this sum is less than or equal to 1 for all x in X. Otherwise, $f + g$ is not defined.
The *absolute difference* of fuzzy sets f and g is the fuzzy set $|f - g|$ defined by $|f - g|(x) = |f(x) - g(x)|$. (This corresponds to the operation of symmetric difference in classical set theory.)
The *convex combination* of the fuzzy sets f, g, and k is the fuzzy set $(f, g; k)$ given by $kf + k'g$. (This is to correspond to a linear combination $\lambda f + (1 - \lambda) g$, $0 \leqq \lambda \leqq 1$.) Zadeh observes that, for all k,

$$f \cap g \subseteq (f, g; k) \subseteq f \cup g$$

and that any set h lying between $f \cap g$ and $f \cup g$ can be obtained as a convex combination $(f, g; k')$ by choosing

$$k'(x) = \frac{h(x) - g(x)}{f(x) - g(x)}$$

using a suitable interpretation when the denominator is 0 (the value of $k(x)$ at such points can in fact be chosen at will in the interval $[0, 1]$).

The *sum* of two fuzzy sets f and g is the fuzzy set $f \oplus g$ given by

$$(f \oplus g)(x) = f(x) + g(x) - f(x)g(x).$$

For ordinary sets (functions that take on values 0 and 1 at most), algebraic product and intersection are identical, as are sum and union ([3], p. 344). Zadeh uses some of these definitions in his theory of convex fuzzy sets (see below), but from our point of view, the simple fact that alternate definitions are possible (that coincide with the usual ones in the case of ordinary sets) is important, together with the interesting observation that $f + f' = 1$. We still, however, do not have an operation $^\circ$ that is well-behaved in other respects and satisfies $f \circ f' = 0$. At this point Zadeh continues with the following definitions ([3], pp. 345-346) which we shall certainly have to take into account in our axiomatized theory.

A fuzzy *relation* is a fuzzy subset of $X \times X$.
The *composition* of two fuzzy relations $f(x, y)$ and $g(x, y)$ is the fuzzy relation

$$h(x, y) = \sup_{v \in x} \min[\, f(x, v), g(v, y)].$$

Thus for any particular pairs (x, v) in f and (v, y) in g indicating that (x, y) should belong to h to some degree, we assign (x, y) the minimum of the corresponding degrees of (x, v) and (v, y) and then assign as a final value to (x, y) supremum of these minima. This maximum of the minima concept will prove quite useful in the axiomatized theory.
If T is an (ordinary) mapping from fuzzy set d to fuzzy set r, and f is a fuzzy subset of r, the *inverse image* $T^{-1}(f)$ is given by the fuzzy set $(T^{-1}(f))(x) = f(y)$ for all x in d that T maps onto y. (If $d = 1$, i.e., the domain of the relation is the ordinary set X (the only case that Zadeh mentions), this definition is reasonable, assigning to an x in the inverse image the same degree of membership that $T(x)$ has in f. However, if d is an arbitrary fuzzy set, it need not be the case that $T^{-1}(f) \subseteq d$ since there may be elements that belong to f to a higher degree than their inverse x images belong to d.)
If T is an (ordinary) mapping from fuzzy set d to fuzzy set r (Zadeh takes the case r an ordinary set only), and f is a fuzzy subset of d, then $T(f)$ is the fuzzy set given by $(T(f))(y) = \mathrm{Max}\, f(x)$ where the maximum is taken over all x mapped by T to y. The intuitive interpretation here is again clear, but unless r is an ordinary set, it need not be the case that $T(f) \subseteq r$.

The difficulties here seem to be associated with the mixture of ordinary functions, fuzzy sets and relations, ordinary domains and ranges, and fuzzy domains and ranges. A theory having all of the objects involved of some uniform kind would seem profitable (*cf.*, section **2** ff.).

The rest of the Zadeh paper [3] will be discussed here in less detail, since, although it is quite interesting and presents several useful results, an understanding of those results is not necessarily germain for the purpose at hand. Zadeh wishes to study the idea of convex fuzzy sets and in particular, the question of separating such sets in some sense by hyperplanes. For this purpose, he restricts himself to the case that the original ordinary space X is a real Euclidean space E^n for some n. Typical of his definitions and theorems are the following ([3], pp. 347-353).

A fuzzy set f is *convex* if and only if the sets Γ_a defined by

$$\Gamma_a = \{x | f(x) \geqq a\}$$

are convex for all a in the interval, $[0, 1]$ (equivalently if and only if

$$f(\lambda x_1 + (1 - \lambda)x_2) \geqq \min[f(x_1), f(x_2)]$$

for all x_2 and x_2 in X and all λ in $[0, 1]$).
A fuzzy set f is *bounded* if and only if the sets Γ_a defined above are all bounded. Thus if we define f to be 1 on all points x within the unit circle and the reciprocal of the distance from x to the origin for points on or outside the unit circle, we have f as a sort of bounded plane.
If f and g are bounded fuzzy sets and H is a hyperplane in E^n defined by $h(x) = 0$, by the two sides of H, we mean, as usual, the set of all x such that $h(x) \leqq 0$ and the set of all x such that $h(x) \geqq 0$. Let $M(H)$ be the infimum of all k such that $f(x) \leqq k$ on one side of H and $g(x) \leqq k$ on the other side of H. (Clearly, 1 is an example of such a k, and all such k are greater than or equal to 0.) Then $1 - M(H)$ is the *degree of separation* of f and g by H and $1 - \inf_H M(H)$ (where the infimum is over all hyperplanes) is the degree of separation of f and g.

Theorem: *Let M be $\sup_{x \epsilon X}(f \cap g)(x)$ for f and g two fuzzy functions. Then M is also the degree of separation of f and g.*

This theorem is in some sense the best possible, for it would seem unlikely that we could separate two sets to a greater degree than the degree some one element belonged to their intersection. For details about this proof and other facts about the convexity of fuzzy sets, the reader is referred to the original paper [3].

The System of Brown In [1], Brown defines *fuzzy sets* as mappings from sets into a Boolean lattice. For this reason, his theory must be considered as a modification of the theory of Zadeh rather than as a generalization of that theory, since there are theorems of each version which are false in the other. Let the operations in the Boolean lattice in question be $\vee$, $\wedge$, $'$, and $\leqq$ (sup, inf, complement, order relation). Then Brown makes the following definitions ([1], pp. 33-34) parallel to those of Zadeh.

A fuzzy set is *empty* if and only if it is the zero function.
Two fuzzy sets are *equal* if they are identical as functions.
The *complement* of a fuzzy set A is the fuzzy set A' given by $A'(x) = (A(x))'$.
The *union* of two fuzzy sets A and B is the fuzzy set $(A \cup B)(x) = A(x) \vee B(x)$.
The *intersection* of two fuzzy sets A and B is the fuzzy set $(A \cap B)(x) = A(x) \wedge B(x)$.
Fuzzy set A is a *subset* of fuzzy set B provided that $A(x) \leqq B(x)$, for all x.

These definitions allow Brown to prove the various theorems of elementary set theory that Zadeh does. In particular, union and intersection are both commutative, both associative, each distributes over the other, and DeMorgan's law holds ([1], p. 34-5). However, in addition, both of the following are true.

$$A \cup A' = \mathbf{U},\ A \cap A' = \emptyset$$

where $\emptyset$ is the empty set and $\mathbf{U}$ is the constant function 1. As we noted above, such theorems need not be true in the theory of Zadeh. From this point of view, the theory of Brown seems more natural or at least closer to the classical theory. However, although the theory of Brown is well-behaved with respect to complement, one should carefully observe the way that Brown's intersection words. Suppose, for example, that $A(x) = a$ and $B(x) = b$ with neither a nor b the 0 element of the lattice of values, but such that $a \wedge b = 0$. Then, although x is an element of A to some positive degree and element of B to some positive degree, x is an element of $A \cap B$ to the degree 0. This cannot happen in the theory of Zadeh, since all of the values are comparable so that x would belong to the intersection to the minimum of the two degrees a and b (and so to a non-zero degree). The corresponding situation for unions is not quite so disturbing, but still it should be noted. If, in the above example neither a nor b is 1, but $a \vee b = 1$, then $(A \cup B)(x) = 1$. This happens classically in the case that B is the complement of A, for example, (or when the complement of A is a subset of B), but the possibility of this sort of thing happening in other cases can seem mildly strange.

Having developed this elementary set theory, Brown goes on to discuss the general elementary properties of convexity; he does not prove most of the theorems that Zadeh does concerning convexity, since the theorem mentioned above on separation does not hold ([1], p. 33). He does observe, however ([1], pp. 38-39), that many properties of fuzzy sets can be defined in terms of the corresponding classical properties of the associated sets Γ_a. Among these properties are that of being simply connected, that of being star-shaped with respect to a point, and that of "having a hole" in an elementary sense. For the details, the reader is referred to the above-mentioned paper of Brown.

At this point, the reader should have a reasonable grasp of the intuitive idea of Zadeh's fuzzy set theory. In particular, he should note the difficulties that arose in connection with the definitions of complementation and intersection, and the complications that arose in the consideration of direct and inverse images of sets under functions. Further, it should be noted that there has been very little mention of the membership relation (ϵ). Zadeh ([3], pp. 341-342) observes that this relation "does not have the same role in the case of fuzzy sets" that it has in the classical theory. In the following sections, a Zermelo-Fraenkel-like theory based on a membership relation will be developed. This unified theory in which sets, function, etc. all are "fuzzy" helps to obviate some of the above difficulties and to clarify the nature of the others. Further, it eliminates the necessity of having a predetermined theory of ordinary sets on top of which the "fuzzy" sets are built as a superstructure by starting out axiomatically *ab initio*, as it were, assuming only elementary logic. Further, by developing the theory in a manner parallel to the usual development of other set theories, comparisons between this new theory and the more usual ones are facilitated.

2 The Axioms In this section we list all of the axioms (with the exception of the continuum hypotheses) for the various versions of Zadeh set theory. Certain basic definitions are also included. Those axioms whose numbers are preceded by an S are the ones to be added to the usual theory Za to obtain the stronger theory Za^+. For the convenience of the reader, the parallel axioms of classical Zermelo-Fraenkel theory, in the form that they are given in [2], will be listed in pairs with the new axioms where this listing is helpful for purposes of comparison.

Our theory will be formulated in ordinary first order logic with equality (=), the usual connectives $\wedge$ (and), $\vee$ (or), $\sim$ (not), $\supset$ (implies), $\equiv$ (if and only if), the usual quantifiers $\forall$ (for all), $\exists$ (there exists), lower case Latin letters with or without primes as variables (to be thought of as representing sets), the usual collection of Latin upper case letters, Greek letters, and combinations of various sorts of letters to be symbols for functions and relations. For certain sets, functions, and relations that have classically determined symbols associated with them, we shall use these symbols when it seems perspicacious to do so. We assume that there is in fact a denumerably infinite number of variables, but our only primitive relation will be the membership relation (ϵ). All other relations and functions will be defined. Constants will be considered as zero-ary relations.

The membership relation ϵ is a ternary relation; $\epsilon(x, y, z)$ should be intuitively interpreted as saying "x is an element of y with degree of membership at least z." Thus the third argument of ϵ corresponds to the value that the function $f_y(x)$ of [3] or $y(x)$ of [1] took. The correspondence is not exact since the membership relation in those papers could be viewed as a binary function whereas our ϵ is a ternary relation. It is helpful to view $\epsilon(x, y, z)$ as being true precisely in those cases in which $0 \leqq z \leqq f_y(x)$ in the sense of [3]. Thus ϵ accepts the degree of membership of x in y, but also accepts all "smaller" values. The utility of this device will become apparent as the axioms are stated. Note, however, that the actual values of the third argument are simply arbitrary sets, so that care must be taken in considering their ordering which certainly need not be linear.

The Axiom of Extensionality The classical formulation of this axiom states:

$$(\forall x)(\forall y)\left[(\forall z)(z \epsilon x \equiv z \epsilon y) \supset x = y\right] \qquad ([2], \text{p. } 51)$$

Thus two sets are to be considered equal classically if they have precisely the same elements. Hence the determination of the suitable parallel axioms for Za-set theory is not difficult:

Ax.1: $(\forall x)(\forall y)[(\forall z)(\forall w)(\epsilon(z, x, w) \equiv \epsilon(z, y, w)) \supset x = y]$.

Two sets are equal if every set belongs to the first with precisely the same degree(s) of membership as it does to the second. At this point it is convenient to make the following definition.

Def.1: $x \subseteq y \equiv (\forall z)(\forall w)(\epsilon(z, x, w) \supset \epsilon(z, y, w))$. (Subset)

So, using the suggested interpretation, to say that x is a subset of y means that if any z belongs to x at least to the degree w, then z also belongs to y at least to that degree. It is here that the non-functional nature of ϵ first proves itself useful. We are certainly in no position to define a suitable ordering relation and require that x be a subset of y if any arbitrary z belongs to x to a lesser or equal degree than it does to y, but we can state this axiom about the relation ϵ, an axiom that seems to have the correct intuitive interpretation and which is at the same time one half of the hypothesis of the Axiom of Extensionality, as is true in the classical case. Hence we obtain the usual classical theorem:

Thm.1: $(\forall x)(\forall y)[(x \subseteq y \wedge y \subseteq x) \supset (x = y)]$.

The Relational Axiom This axiom has no classical counterpart. Its purpose is to indicate that the relation ϵ does behave in the manner suggested in the first paragraph before the beginning of the axioms.

Ax.2: $(\forall x)(\forall y)(\forall z)(\forall w)[(\epsilon(x, y, z) \wedge w \subseteq z) \supset \epsilon(x, y, w)]$.

If x is an element of y at least to the degree z and w is "smaller" than z (with respect to $\subseteq$), then x is also an element of y at least to the degree w.

The Axiom of the Null Set Classically this axiom simply says

$$(\exists x)(\forall y)[\sim(y \epsilon x)] \qquad ([2], \text{p. } 51)$$

There exists a set with no elements. It would be convenient to say that there exists a set to which all elements belong only to the degree zero. But we have no zero (or null set) yet. So we say instead

Ax.3: $(\exists x)[(\forall y)(\forall z)(\epsilon(y, x, z) \supset x = z) \wedge (\forall v)(\forall w)(\epsilon(v, w, x))]$.

Think of the x whose existence this theorem insures as the null set and as the "minimal" degree to which any set may belong to any other set. Then the axiom says that if y belongs to the empty set at least to the degree z, then that degree is the minimal degree (the null set) and further that any set v belongs to any set w at least to the minimal degree (the null set).

Thm.2: *The set x whose existence is asserted by* Axiom 3 *is unique.*

Proof: Consider any two sets satisfying the axiom, x and x'. By the second part of the axiom applied to x', we obtain, in particular, $\epsilon(x, x, x')$. But from this, together with the first part of the axiom applied to x, we obtain $x = x'$. QED

Def.2: $\emptyset$ is the x whose existence and uniqueness are given by Axiom 3 and Theorem 2. (Null Set)

Note that the Axiom of Extensionality was not used in the proof of the uniqueness of the null set, but rather just the Axiom of the Null Set itself.

The Axiom of Non-Triviality At this stage of the development, it is conceivable that $\emptyset$ is the only set accepted as a third argument of ϵ.

Def.3: $\mathrm{D}(x) \equiv (\exists y)(\exists z)(\epsilon(y, z, x))$. (Degree)

$\mathrm{D}(x)$ holds (read "x is a degree") if and only if x is the third element of some triple in ϵ.

Ax.4: $(\exists x)[\mathrm{D}(x) \wedge \sim(x = \emptyset)]$.

This axiom stating that $\emptyset$ is not the only possible degree has, of course, no classical counterpart.

The Axiom of Pairs The classical version of this axiom insures us that if x and y are sets, then there is a set z whose elements are precisely x and y.

$$(\forall x)(\forall y)(\exists z)(\forall w)[w \in z \equiv (w = x \vee w = y)] \qquad ([2], \text{ p. } 51)$$

Our situation is somewhat more complicated, since we must also decide to what degree x and y are to belong to z. The situation is further complicated by the fact that, although we now have a minimal degree ($\emptyset$) of sorts, there is no particular "maximal" degree. For this reason, we introduce two axioms about pairs.

Ax.5: $(\forall x)(\forall y)(\exists z)[(\forall v)(\forall w)[(\epsilon(w, z, v) \wedge \sim(v = \emptyset)) \supset (w = x .\vee. w = y)]$
$\wedge (\forall v')[\mathrm{D}(v') \supset (\epsilon(x, z, v') \wedge \epsilon(y, z, v'))]]$.

Thus, for every x and y, there is a z whose only elements to a degree different from $\emptyset$ are x and y. Further, if v' is a degree, then both x and y belong to z at least to the degree v'. This is a way of creating a pair containing x and y to the "maximal" degree, at least in the sense they belong to the pair to every available degree.

Ax.6: $(\forall x)(\forall y)(\forall v')(\forall w')[(\mathrm{D}(v') \wedge \mathrm{D}(w')) \supset ((\exists z)(\forall v)(\forall w)[(\epsilon(w, z, v)$
$\wedge \sim(v = \emptyset)) \supset (w = x .\vee. w = y)] \wedge [\epsilon(x, z, v') \wedge \epsilon(y, z, w')]$
$\wedge [(\exists v'')(\exists w'')(\epsilon(x, z, v'') \wedge \epsilon(y, z, w'')) \supset (v'' \subset v' .\wedge. w'' \subset w')])]$.

Thus, if x and y are any two sets and v' and w' any two degrees, there is a set z satisfying the following three properties:

a) x and y are the only elements of z on non-$\emptyset$ degree;
b) x belongs to z at least to the degree v' and y belongs to z at least to the degree w';
c) if also x belongs to z at least tò the degree v'' and y belongs to z at least to the degree w'', then v'' is "smaller" than or equal to v' ($v'' \subseteq v'$) and w'' is "smaller" than or equal to w' ($w'' \subseteq w'$).

Hence, for every pair of sets and every pair of degrees, we get a new set that contains each of the original sets as an element "precisely to the corresponding degree."

Thm.3: *For each x and y (respectively x, y, v', w') the set whose existence is assured by* Axiom 5 *(respectively* Axiom 6) *is unique.*

Proof: This follows immediately from the Axiom of Extensionality, using the appropriate axiom (5 or 6) together with the Relational Axiom and the second part of the Axiom of the Null Set. QED

Def.4: The sets whose existence and uniqueness are given by Axiom 5 and Theorem 3 are designated by $\{x, y\}$. (Strong Pair)

Def.5: The sets whose existence and uniqueness are given by Axiom 6 and Theorem 3 are designated by $\{x, y\}_{v'w'}$. (Weak Pair)

Def.6: $\{x\}$ is $\{x, x\}$. (Strong Unit Set of x)

Def.7: $\{x\}_{v'}$ is $\{x, x\}_{v'v'}$. (Weak Unit Set of x)

Thus the presence of both Axiom 5 and Axiom 6 gives us pairs whose elements belong with quite varying degrees of membership. The necessity of both axioms, rather than just one or the other becomes apparent if the degrees are imagined to correspond to the natural numbers with their natural ordering.

Def.8: $\langle x, y\rangle$ is $\{\{x\}, \{x, y\}\}$. (Ordered Pair)

One could also define Weak Ordered Pairs of various sorts, but such pairs do not seem to be of immediate use. Functions will consist of sets of ordered pairs, but their "fuzzyness" will consist in the degree to which the pairs belong to the function, not in the degree to which the pairs are "fuzzy." Because the sets mentioned in the definition of $\langle x, y\rangle$ are strong, it is easy to use the Axiom of Extensionality together with Definition 8 and its predecessors to prove (in the usual manner)

Thm.4: $\langle x, y\rangle = \langle z, w\rangle$ *if and only if* $x = z$ *and* $y = w$. QED

Def.9: $\mathrm{Funct}(x) \equiv [(\forall y)(\forall z)[(\epsilon(y, x, z) \wedge \sim(z = \emptyset)) \supset (\exists u')(\exists v')(y = \langle u', v'\rangle)]$
$\wedge (\forall u)(\forall v)(\forall v')(\forall w)(\forall w')[(\epsilon(\langle u, v\rangle, x, w) \wedge \epsilon(\langle u, v'\rangle, x, w') \wedge \sim(w = \emptyset)$
$\wedge \sim(w' = \emptyset)) \supset v = v']]$. (Function)

Thus, the only elements of a function to non-$\emptyset$ degrees are ordered pairs and any two ordered pairs to belong to non-$\emptyset$ degrees and have the same first elements have the same second elements. This sort of definition corresponds to the situation in which you have a function whose domain and range are unclear, but for any element that might be in the domain, there is no confusion as to what element in the range (if any) it would correspond to. To handle the situation in which for each element of the domain, there are several possible elements of the range that might correspond, perhaps with different degrees of certitude, the most satisfactory solution is to require that the function be redefined so that its range consists of sets of the previous range values, the elements of the new range set that corresponds to any x in the domain being the values that x might take, each belonging to the set "$f(x)$" to the appropriate degree.

Note that, using our relatively strong definition of function, if f is a function and $\langle x, y\rangle \epsilon f$, it makes sense to say such things as $y = f(x)$ which would not have been meaningful had we allowed f to be multivalued even in the weak sense that it was allowed to contain pairs of the form $\langle x, y\rangle$, $\langle x, y'\rangle$, with y not equal to y', provided the degree of membership of these pairs was different.

Def.10: $\mathrm{Dom}(x,f) \equiv [\mathrm{Funct}(f) \wedge (\forall z)(\forall w) \sim (w = \emptyset)[\epsilon(z, x, w) \equiv (\exists v)(\epsilon(\langle z, v\rangle, f, w))]]$. (Domain)

The domain is the usual set of first elements of pairs from a function; each such element belongs to the domain of f precisely to the degree that the corresponding pair belongs to f.

Def.11: $\mathrm{Ran}(x,f) \equiv [\mathrm{Funct}(f) \wedge (\forall z)(\forall w) \sim (w = \emptyset)[\epsilon(z, x, w) \equiv (\exists v)(\epsilon(\langle v, z\rangle, f, w))]]$. (Range)

Although Definitions 10 and 11 seem rather similar, note the difference in results: the range is the collection of second members of pairs from the function f; each such z belongs to the range to all the degrees that any pair with second element z belongs to f. Thus with our interpretation, x belongs to the range with the "maximal" degree among the degrees of the corresponding pairs in f. The following definitions are standard.

Def.12: A function f maps into a set x means that $\mathrm{Ran}(f) \subseteq x$. (Into)

Def.13: A function f maps onto a set x means that $\mathrm{Ran}(f) = x$. (Onto)

The Axiom of Unions A common classical form of this axiom is

$$(\forall x)(\exists y)(\forall z)[z \epsilon y \equiv (\exists t)(z \epsilon t \wedge t \epsilon x)] \qquad \text{([2], p. 52)}$$

Here y is the union of all of the sets in x, i.e., the set of the elements of the elements of x. Our situation here is slightly complicated by the possibility that an element z of x might belong to x to one degree and, at the same time, z might itself have elements that belonged to z to other degrees.

Def.14: $\mathrm{Std}(x) \equiv (\forall y)[(\exists z)(\epsilon(z, x, w) \wedge \sim(\emptyset = w)) \supset (\forall z')(\mathrm{D}(z') \supset \epsilon(z, x, z'))]$. (Standard)

A standard set x is one having the property that if any z belongs to x at least to a non-$\emptyset$ degree, then z belongs to x to every degree ("maximally"). For standard sets we could state the following axiom of unions.

$$(\forall x)[\mathrm{Std}(x) \supset (\exists y)(\forall z)(\forall w)(\epsilon(z, y, w) \equiv (\exists t)(\exists t')(\epsilon(z, t, w) \wedge \epsilon(t, x, t') \wedge \sim(t' = \emptyset)))]$$

For standard x, a y satisfying this axiom would contain the elements of the elements of x, each to all degrees (thus "maximally") that it was an element of some set y which was an element of x at least to some non-$\emptyset$ degree t'. Since all elements of x that are elements of some non-$\emptyset$ degree at least are elements of x of every degree, we can safely ignore t'. The usual extensionality proof shows that for each standard x, the y postulated above is unique.

Def.15: The set y of the above axiom will be designated as $\bigcup_s x$. (Standard Union)

Ax.7: $(\forall x)(\exists y)(\forall z)(\forall w)[\epsilon(z, y, w) \equiv (\exists t)(\exists t')(\exists t'')(\epsilon(z, t, t') \wedge \epsilon(t, x, t'') \wedge w \subseteq t' \wedge w \subseteq t'')]$.

Thus if z belongs to t at least to the degree t' and t belongs to x at least to the degree t'', then z belongs to the union of x (y) to every degree "smaller" than both t' and t''. Thus we might say that the degree of membership of z in y is the "smaller" of the degrees of membership of z in t and t in x. However, observe that if z in several different t which are in z, the degrees assigned to z in the union of x are all of those assigned by the above process for the different t. Thus we take "the maximum of the minimal degrees" in defining the union. Again the set y postulated to exist by Axiom 7 is unique by extensionality.

Def.16: The set y of Axiom 7 will be designated $\bigcup x$. (Union)

If x is standard, in Axiom 7, if there is one non-$\emptyset$ t'' for a particular t, then all degrees are available to serve as t'', so that Axiom 7 can easily be reduced to the axiom above asserting the existence of the standard union of a standard set. Hence we have the following theorem.

Thm.5: $\mathrm{Std}(x) \supset \left(\bigcup_s x = \bigcup x\right)$. QED

The following is the usual definition.

Def.17: $x \cup y$ is $\bigcup\{x, y\}$.

The Axiom of Infinity The purpose of this axiom is to insure that we have an infinite number of sets and possibly a collection of sets looking rather like the natural numbers. A classical formulation is as follows.

$$(\exists x)[\emptyset \epsilon x \wedge (\forall y)(y \epsilon x \supset y \cup \{y\} \epsilon x)] \qquad \text{([2], p. 52)}$$

Using the fact that null set $\emptyset$ is standard and that $x \cup y$ is defined as the union of a standard set $\{x, y\}$, one may apply the Axiom of Non-Triviality to verify, for example, that $\emptyset$ and $\emptyset \cup \{\emptyset\}$ are distinct. Note that the converse of the Axiom of Extensionality follows from our first order logic with equality.

Ax.S8: $(\exists x)[\mathrm{Std}(x) \wedge (\exists t)(\sim(t = \emptyset) \wedge \epsilon(\emptyset, x, t)) \wedge (\forall y)(\epsilon(y, x, t) \supset \epsilon(y \cup \{y\}, x, t))]$.

This strong axiom simply gives us a standard infinite set of the usual kind. The following axiom is weaker, since although it gives us an infinite set, that set may not be standard.

Ax.8: $(\exists x)[(\exists t)(\epsilon(\emptyset, x, t) \wedge \sim(t = \emptyset)) \wedge (\forall y)(\forall t')[(\epsilon(y, x, t') \wedge \sim(t' = \emptyset)) \supset (\exists t'')(\epsilon(y \cup \{y\}, x, t'') \wedge \sim(t'' = \emptyset))]]$.

Using this axiom, the different members of the infinite set x may all belong to x to different degrees. Thus, although we are assured that the elements that are supposed to belong to x do belong to some non-$\emptyset$ degrees, it is conceivable, for example, that as we go out further in the sequence, the degrees "decrease" toward $\emptyset$. Axiom 8 seems to be the weakest useful version of the Axiom of Infinity for our system. Axioms of the type below, asserting the existence of an infinite set whose elements all belong at least (or precisely) to some fixed non-$\emptyset$ degree, could be added to strengthen the theory Za, if this were desired. Note that even the third and fourth of these axioms which state that the first (respectively, second) of these possible additions is available for each degree t, need not imply the truth of Axiom S8. Again, if one imagines the degrees as natural numbers with their usual ordering, this becomes clear.

(1) $(\exists x)(\exists t)[\epsilon(\emptyset, x, t) \wedge \sim(t = \emptyset) \wedge (\forall y)(\epsilon(y, x, t) \supset \epsilon(y \cup \{y\}, x, t))]$

(2) $(\exists x)(\exists t)[\epsilon(\emptyset, x, t) \wedge \sim(t = \emptyset) \wedge (\forall y)(\epsilon(y, x, t) \supset \epsilon(y \cup \{y\}, x, t))$
$\wedge (\forall z)(\forall t')(\epsilon(z, x, t') \supset t' \subseteq t)]$

(3) $(\forall t)[D(t) \supset (\exists x)[\epsilon(\emptyset, x, t) \wedge (\forall y)(\epsilon(y, x, t) \supset \epsilon(y \cup \{y\}, x, t))]]$

(4) $(\forall t)[D(t) \supset (\exists x)[\epsilon(\emptyset, x, t) \wedge (\forall y)(\epsilon(y, x, t) \supset \epsilon(y \cup \{y\}, x, t))$
$\wedge (\forall z)(\forall t')(\epsilon(z, x, t') \supset t' \subseteq t)]]$

Note that, in (3) and (4), it is not necessary to assume that $\sim(t = \emptyset)$: the case that $t = \emptyset$ is true by the Axiom of the Null Set; we are sure that there are other values of t by the Axiom of Non-Triviality.

The Axiom of Replacement To state this axiom, we must first enumerate the countably many formulae of our language which have at least two free variables. Call the n'th of these $A_n(x, y, t_1, t_2, \ldots, t_k)$ (where k may depend on n). Then the classical "Axiom" of Replacement consists of the following infinite collection as sentences.

$$(\forall t)(\forall t_2) \ldots (\forall t_k)[(\forall x)(\exists! y)\, A_n(x, y; t_1, t_2, \ldots, t_k) \supset (\forall u)(\exists v)(\forall r)[r \epsilon v \equiv (\exists s)(s \epsilon u \wedge A_n(s, r; t_1, t_2, \ldots, t_k))]] \qquad \text{([2], p. 52)}$$

If, for fixed values of the t's, A_n defines y as a function F of x, then if u is a set, the range of the function obtained by restricting the domain of F to u is also a set. (As usual, $\exists! y$ means "there exists a unique y," i.e., there is a y and any two are identical.)

Ax.9: $(\forall t_1)(\forall t_2) \ldots (\forall t_k)[(\forall x)(\exists! y)\, A_n(x, y; t_1, t_2 \ldots, t_k) \supset (\forall u)(\exists v)[(\forall w)$
$\forall r\, [\sim(w = \emptyset) \supset [\epsilon(r, v, w) \equiv (\exists s)(\epsilon(s, u, w) \wedge A_n(s, r; t_1, t_2, \ldots, t_k))]]]]$.

This axiom closely parallels the classical one above, giving the same range set v corresponding to the domain set u. However, we also had to specify the degree that any element r belongs to v: if $A_n(s, r; t_1, t_2, \ldots t_k)$ holds true for s in u, then we put r into all of those degrees that s was in u. Thus, if r is the image of several different s, we put r in the range set v to all of the appropriate degrees (to the "maximal" degree). This axiom takes into account the "fuzzyness" of the domain set u and transfers it to the corresponding range set v, but it ignores the "fuzzyness" of the sets that may be involved in the construction of A_n. Hence we add the following axiom.

Ax.10: $(\forall f)\,[\mathsf{Funct}(f) \supset (\forall u)(\exists v)(\forall r)(\forall w)\,[\sim(w = \emptyset) \supset \epsilon(r, v, w)$
$\equiv (\exists s)(\exists w')(\exists w'')(\epsilon(s, u, w') \wedge \epsilon(\langle s, r\rangle, f, w'') \wedge w \subseteq w' \wedge w \subseteq w'')]]$.

Thus, if f is a function and we restrict its domain to u, the set v is the range of f. If s belongs to u at least to the degree w' and $\langle s, r\rangle$ belongs to the function f at least to the degree w'', then we put r into the range set v with all degrees "smaller" than both w' and w''. Since, if r occurs as the second element of more than one appropriate pair, we put it into the range with all of the suitable degrees, Axiom 10 may be viewed as constructing the range with degrees arranged as "maxima of minima." It takes into account both the "fuzzyness" of the domain set u and of the function f.

A common weakening of the classical Axiom of Replacement is the Axiom of Separation (Aussonderungsaxiom). For this axiom, we need a listing of all of the formulae of our language with at least one free variable. Let the n'th such formula be $B_n(x; t_1, t_2 \ldots t_k)$ where again k may depend on n.

$(\forall t_1)(\forall t_2) \ldots (\forall t_k)(\forall x)(\exists y)(\forall z)\,[z \epsilon y \equiv (z \epsilon x \wedge B_n(z; t_1, t_2 \ldots, t_k))]$. ([2], p. 55)

A suitable parallel collection of axioms for our theory would be the following:

$$(\forall t_1)(\forall t_2) \ldots (\forall t_k)(\forall x)(\exists y)(\forall z)(\forall w)([\epsilon(z, y, w) \equiv (\epsilon(z, x, w) \wedge B_n(z; t_1, t_2, \ldots t_k))] \vee w = \emptyset). \qquad \text{(Separation)}$$

This axiom is a consequence of the Axioms of Replacement in the usual way, by defining the proper function (*cf.*, [2], p. 55).

The Axiom of the Power Set The classical form of this axiom is very simple.

$$(\forall x)(\exists y)(\forall z)(z \epsilon y \equiv z \subseteq x). \qquad \text{([2], p. 53)}$$

This tells us that y, the power set of x is the set of all subsets of x. The question that arises in creating a power set in Za-set theory is that of the degrees of membership of the various elements of the power set.

Ax.S11: $(\forall x)(\exists y)(\forall z)(\forall w)(\epsilon(z, y, w) \equiv (\mathsf{D}(w) \wedge z \subseteq x) \vee w = \emptyset)$.

This very strong power set has all of the subsets of x as elements to every degree. Proving its uniqueness (for fixed x) by extensionality, we may give the following definition.

Def.18: The set y whose existence is assured by Axiom S11 will be denoted $\mathcal{P}_s(x)$. (Strong Power Set)

The following weaker axiom simply guarantees the existence of a power set, i.e., a set containing as elements the subsets of x, each to some non-$\emptyset$ degree.

Ax.11: $(\forall x)(\exists y)(\forall z)\,[(\exists w)(\epsilon(z, y, w) \wedge \sim(w = \emptyset)) \equiv z \subseteq x]$.

In the strong system Za$^+$, which includes Axiom S11, Axiom 11 becomes redundant, being a consequence of Axiom S11 and the Axiom of Non-

Triviality. A set whose existence in guaranteed by Axiom 11 will be called *a power set of* x; we cannot conveniently assign it a symbol for regular use, since there is no guarantee that a power set is unique.

The Axiom of Products Classically, no axiom is needed to assert the existence of Cartesian Products since, for Zermelo-Fraenkel-like set theories, the Cartesian Product of x and y is a subset of $\mathcal{P}\,\mathcal{P}(x \cup y)$ and there is a sentence of the language that allows us to specify precisely just which such subset we want by the Axiom of Replacement. This is not the case in our theory Za. Here, since the pairs involved in the definition of the ordered pair $\langle x', y' \rangle$ are all standard, we cannot expect that the Cartesian Product be a subset of $\mathcal{P}\mathcal{P}(x \cup y)$ (or even of $\mathcal{P}_s\,\mathcal{P}_s(x \cup y)$). The possibility of defining weaker ordered pairs in a manner that the pairs in question would form a Cartesian Product that is a subset of $\mathcal{P}\mathcal{P}(x \cup y)$ seems promising at first glance, but then the statement A_n necessary to apply the Axiom of Separation becomes the difficulty: in most cases, there are several collections of "weak" ordered pairs which could serve as a "weak" Cartesian Product of x and y, and quite often, many of these collections are subsets of $\mathcal{P}\mathcal{P}(x \cup y)$. The obvious device that comes to mind is the application of the Axiom of Choice, but this depends on the construction of a function the proof of whose existence seems possible to prove only if we can already construct Cartesian Products.

Ax.S12: $(\forall x)(\forall y)(\exists z)(\forall w)(\forall w')[(\epsilon(w, z, w') \wedge \sim(w' = \emptyset)) \equiv (\exists x')(\exists y')(\exists v')$
$(\exists v'')(\epsilon(x', x, v') \wedge \epsilon(y', y, v'') \wedge \sim(v' = \emptyset) \wedge \sim(v' = \emptyset) \wedge w$
$= \langle x', y' \rangle \wedge \mathrm{D}(w') \wedge \sim(w' = \emptyset))]$.

This strong axiom gives us a sort of standard Cartesian Product, each of the suitable ordered pairs belonging to the product to all degrees. The uniqueness of the set z whose existence the axiom asserts is assured by extensionality.

Def.19: The set z of Axiom S12 will be denoted $x \times_s y$.
(Strong Cartesian Product)

Ax.12: $(\forall x)(\forall y)(\exists z)(\forall w)[(\exists w')(\epsilon(w, z, w') \wedge \sim(w' = \emptyset)) \equiv (\exists x')(\exists y')(\exists v')$
$(\exists v'')[\epsilon(x', x, v') \wedge \epsilon(y', y, v'') \wedge \sim(v' = \emptyset) \wedge \sim(v'' = \emptyset) \wedge w = \langle x', y' \rangle]]$.

This weaker sort of Cartesian Product (which need not be unique) contains each of the ordered pairs in question to some non-$\emptyset$ degree, but not necessarily to all degrees.

Def.20: Any set z asserted to exist by Axiom 12 will be denoted by $x \times y$.
(Cartesian Product)

Note carefully that notations such as $x \times y$ cannot be used in quite the same manner as $x \times_s y$, since the set that the former denotes is not unique.

The Axiom of Regularity This axiom appears in Zermelo-Fraenkel set theory chiefly because of its convenience in writing proofs. Unlike the other axioms, it is sometimes used to exclude certain types and classes of sets.

$$(\forall x)(\exists y)\,[x = \emptyset \vee (y \in x \wedge \forall z(z \in x \supset \sim(z \in y)))]. \qquad ([2], \text{ p. } 53)$$

If x is not $\emptyset$, then y is an element of x which is disjoint from x, an element of x which is minimal with respect to ϵ.

Ax.13: $(\forall x)(\exists y)\,[x = \emptyset \vee (\exists w)(\sim(w = \emptyset) \wedge \epsilon(y, x, w) \wedge (\forall z)(\forall w')\,[(\epsilon(z, x, w') \wedge \sim(w' = \emptyset)) \supset (\forall w'')(\epsilon(z, y, w'') \supset w'' = \emptyset)])]$.

The reader will recognize this axiom as a direct translation of the corresponding classical axiom. The usual consideration of $\{x\}$ assures us, by means of Axiom 13, that x is an element of itself only to the degree $\emptyset$. The other usual results on descending epsilon chains, epsilon loops, etc., also hold.

The Axiom of Choice Classically, this axiom takes many forms, not all equivalent. As usual we follow the example of [2] for the traditional statement.

If $a \to A(a) \neq \emptyset$ is a function defined for all $a \in x$, then there exists another function $f(a)$ for $a \in x$, and $f(a) \in A(a)$. ([2], p. 53)

The function f is the choice function for the collection of sets $A(a)$. It is rather tempting to declare the existence of a strong choice function in the following manner.

$(\forall x)(\forall f)\,[[\mathrm{Funct}(f) \wedge x \subseteq \mathrm{Dom}(f) \wedge (\forall y)(\forall z)(\forall w)(\forall w')(\epsilon(y, x, z) \wedge \sim(z = \emptyset) \wedge \epsilon(\langle y, w\rangle, f, w') \wedge \sim(w' = \emptyset)) \supset \sim(w = \emptyset)] \supset (\exists g)\,[\mathrm{Funct}(g) \wedge x \subseteq \mathrm{Dom}(g) \wedge (\forall x')(\forall x'')(\forall v)\,[(\epsilon(x', x, v) \wedge \sim(v = \emptyset)) \supset (\mathrm{D}(x'') \supset \epsilon(g(x), f(x), x''))]]]$.

The choice function g in this proposed axiom selects a member $g(x)$ of each set $f(x)$ and requires that $g(x)$ belongs to $f(x)$ to every degree. This would be a pleasant ideal situation, but unfortunately, even though we know that $f(x)$ is not $\emptyset$, we do not know that it has any member belonging to it of every degree. Even modifications of this idea requiring that the element picked be of as "high" a degree as possible are not particularly satisfactory, because of their quasi-constructive nature: in some sense all such axioms do not pick an arbitrary point from each of the non-empty sets, but rather one that is already designated to some extent. Hence we adopt the following rather weak axiom which gives us an element of each non-empty set in question, but an element about whose membership we can only say that it is of some degree other than $\emptyset$, perhaps a different degree for each set in question.

Ax.14: $(\forall x)(\forall f)\,[[\mathrm{Funct}(f) \wedge x \subseteq \mathrm{Dom}(f) \wedge (\forall y)(\forall z)(\forall w)(\forall w)\,[\epsilon(y, x, z) \wedge \sim(z = \emptyset) \wedge \epsilon(\langle y, w\rangle, f, w') \wedge \sim(w' = \emptyset)] \supset \sim(w = \emptyset)] \supset [(\exists g)(\mathrm{Funct}(g) \wedge x \subseteq \mathrm{Dom}(g) \wedge (\forall x')(\forall x'')(\forall v')(\forall v'')(\forall t)(\forall t')[(\epsilon(x', x, v') \wedge \sim(v' = \emptyset)) \wedge \epsilon(\langle x', x''\rangle, f, v'') \wedge \epsilon(\langle x', t\rangle, g, t') \wedge \sim(v'' = \emptyset) \wedge \sim(t = \emptyset)] \supset (\exists t'')(\sim(t'' = \emptyset) \wedge \epsilon(t, x'', t'')))]]$.

Here, g is the choice function: assuming that f is a function with x as a subset of its domain and such that the range of f restricted to u consists of non-empty sets w, then g is a function with domain including x, such that if

x' is an element of x, then the g-image of x' (t) is an element of the f-image of x' (x'') to some non-$\emptyset$ degree (t'').

Def.21: Za is the first order theory with equality whose only non-logical axioms are Axioms 1 through 13. (Za)

Def.22: Za^+ is the extension of Za by the addition of the Axioms S8, S11, and S12.

Def.23: ZaCh is the extension of Za by the addition of the single Axiom 14.
Def.24: $\mathrm{Za}^+\mathrm{Ch}$ is the extension of Za^+ by the addition of single Axiom 14.

In the next section, the above theories are developed to the point (natural numbers, ordinals, cardinals, etc.) that it becomes clear that classical mathematics can be done on the basis of such a theory. The differences as well as similarities with the usual Zermelo-Fraenkel theory will be indicated.

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To be continued.

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