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THE PRAGMATICS OF FIRST ORDER LANGUAGES. II

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The purpose of the present paper is to extend the results of [3], and to state and prove an assumption made tacitly therein. Section **6** of the present paper has the latter purpose, and sections **7** and **8** the former (the section-numbering of the present paper is consecutive with that of [3]). All terms and special symbols introduced without definition are intended in the sense of [3], and all theorems cited are those of [3] unless otherwise indicated.

Two expressions of L are defined to be pragmatically synonymous, relative to a polyadic interpretation Π, if they are interchangeable in all expressions of L, *salvo valore re* Π. Two formulas of $\boldsymbol{L}_\Pi$ are defined to be model synonymous if they have the same image in every semantic interpretation of $\boldsymbol{L}_\Pi$. Two predicates of $\boldsymbol{L}_\Pi$ are defined to be model synonymous if they have the same image under every interpretation of the theory expressed by $\boldsymbol{L}_\Pi$ in which the theory holds (in the customary model-theoretic sense). Model synonymy of individual constants is defined similarly. It is shown that if two formulas, or two predicates, of $\boldsymbol{L}_\Pi$ are pragmatically synonymous, then they are model synonymous. This result is suggested as an explication of Peirce's semiotic principle that if two signs have the same entire general intended interpretant, then they signify the same object, for the case of signs of the indicated type. But this result does not hold for individual constants, as is also shown.

It is shown, finally, that there is a one-one correspondence between the interpretations of $\boldsymbol{L}_\Pi$ onto Boolean models, and the interpretations of the theory expressed by $\boldsymbol{L}_\Pi$ in models of that theory, such that models under corresponding interpretations represent the same intuitive structure. This result justifies application of the term "model synonymous" to formulas, predicates, and individual constants of $\boldsymbol{L}_\Pi$ in the above three senses.

6 *Polyadic Interpretations and Their Cores* If Π is a polyadic interpretation, we shall refer to the polyadic algebra described in Theorem 1, as $\boldsymbol{L}_\Pi$. In virtue of Theorems 1-3, we shall refer to $\boldsymbol{L}_\Pi$ as a predicate calculus, and to $\mathfrak{G} = \langle \mathsf{P}, \mathsf{K}, \mathsf{I}, \{\&, \sim, \exists\}, \mathsf{S}\rangle$ as the (standard) syntax of $\boldsymbol{L}_\Pi$. Some

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conditions in the definition of a polyadic interpretation Π determine the syntax of L_Π, and the remaining conditions characterize the distribution of Π-values over **S**. In [3] it is tacitly assumed that the latter conditions on Π determine corresponding conditions on its core π, which are required to demonstrate the algebraic properties of L_Π. In this section we shall give an exact statement and proof of this assumption. Throughout this paper, we let Π (with or without subscripts) be a polyadic interpretation of the expressions **L** in the valuing dispositions **D**, unless the contrary is explicitly stated.

The conditions in the definition of Π which characterize the distribution of Π-values over the sentences of **S** are those of Definition 5, and conditions IV and V of Definition 1. We first consider Definition 1. Throughout this section, π is to be understood as the core of Π. The condition on π corresponding to D1 (V) is:

$$\text{If } \pi(s, c) = 1 = \pi(s', c), \text{ then } \pi(s \,\&\, s', c) = 1.$$

This proposition follows immediately from the definition of a core. We next show:

(6.1) *If $s, s' \in \mathbf{S}$, then for all $c \subset \mathbf{C}$ the values of $\pi(\sim s, c)$ and $\pi(s \,\&\, s', c)$ are fixed by the values of $\pi(s, c)$ and $\pi(s', c)$ according to the tables of* D1 (IV).

Proof: That (6.1) holds for the $\sim$-table follows immediately from (1.2). Let $\pi(s, c) = 0$. Then for all $s' \in \mathbf{S}$ and some $u \in \mathbf{U}$, $w \in \mathbf{W}$: $\Pi(s)(u, w, c) = 0 = \Pi(s \,\&\, s')(u, w, c)$. Then $\pi(s \,\&\, s', c) = 0$. In this way the desired entries in the &-table for $\pi(s, c) = 0$ or $\pi(s', c) = 0$ are obtained. If s and s' are both valued 1 under c by π, then so is $s \,\&\, s'$, by D1 (V). If $\pi(s, c) = 2 = \pi(s', c)$, then for all $u \in \mathbf{U}$, $w \in \mathbf{W}$, $\Pi(s)(u, w, c) = 2 = \Pi(s')(u, w, c)$, so that $\Pi(s \,\&\, s')(u, w, c) \neq 1 \neq \pi(s \,\&\, s', c)$.

It remains to consider the case in which s is valued 1 and s' is valued 2, by π under c (the situation is the same when s is valued 2 and s' is valued 1). We require the following lemma.

(6.2) *If* $\Pi(s)(u, w, c) = 1$ *for some* $u \in \mathbf{U}$, $w \in \mathbf{W}$, *and* $\Pi(s')(u, w, c) = 2$ *for all* $u \in \mathbf{U}$, $w \in \mathbf{W}$, *then* $\Pi(s \,\&\, s')(u, w, c) \neq 0$, *for all* $u \in \mathbf{U}$, $w \in \mathbf{W}$.

Let subscripts indicate elements fixed throughout the proof of (6.2). Let $\Pi(s)(u_1, w_1, c) = 1$ and $\Pi(s \,\&\, s')(u_0, w_0, c) = 0$. Then $\Pi(\sim(s \,\&\, s'))(u_0, w_0, c) = 1$, so that by D1 (V), $\Pi(s \,\&\, \sim(s \,\&\, s'))(u_2, w_2, c) = 1 = \Pi(s)(u_2, w_2, c) = \Pi(\sim(s \,\&\, s'))(u_2, w_2, c)$. Then $\Pi(s \,\&\, s')(u_2, w_2, c) = 0$, so that $\Pi(s')(u_2, w_2, c) = 0$. It follows from the above that, if $\Pi(s \,\&\, s')(u, w, c) = 0$ for some $u \in \mathbf{U}$, $w \in \mathbf{W}$, then either $\Pi(s)(u, w, c) = 1$ for all $u \in \mathbf{U}$, $w \in \mathbf{W}$, or $\Pi(s')(u, w, c) \neq 2$ for some $u \in \mathbf{U}$, $w \in \mathbf{W}$. Transposing this result gives (6.2).

Now let $\pi(s, c) = 1$ and $\pi(s', c) = 2$. Then by (6.2), $\Pi(s \,\&\, s')(u, w, c) \neq 0$, for all $u \in \mathbf{U}$, $w \in \mathbf{W}$. If $\Pi(s \,\&\, s')(u, w, c) = 1$, then $\Pi(s')(u, w, c) = 1$, against the hypothesis that $\pi(s', c) = 2$. Thus $\Pi(s \,\&\, s')(u, w, c) = 2$ for all $u \in \mathbf{U}$, $w \in \mathbf{W}$, and the desired &-table for π is thereby established. The proof of (6.1) is complete.

It remains to show that the conditions of D5 on a polyadic interpretation Π have appropriate analogues which are satisfied by the core of Π.

(6.3) *If π is the core of a polyadic interpretation Π, then for all $p, q, r \in \mathbf{Q}$; $J \subset \mathbf{1}$; $e \in \mathbf{L}$; $c \subset \mathbf{C}$, π satisfies all the conditions obtained from those of* D5 *by replacing expressions of the form "$\Pi(e)(u, w, c) = 0$" with "$\pi(e, c) = 0$", by replacing expressions of the form "$\Pi(e) = \Pi(e')$" with "$\pi(e, c) = \pi(e', c)$", and by replacing expressions of the form "$\Pi(e) = d_0$" with "$\pi(e, c) = 0$".*

Proof: (6.3) holds for D5 (III) and (IX) by (1.3). By (1.4), (6.3) holds for D5 (IV)-(VI), (VIII), (X)-(XII). That (6.3) holds for D5 (I) follows immediately from (1.3). For D5 (II), let $\pi(\exists(p \,\&\, {\sim}q), c) = 0 = \pi(\exists(q \,\&\, {\sim}r), c)$. Then for some fixed u_1, w_1, $\Pi({\sim}\exists(p \,\&\, {\sim}q) \,\&\, {\sim}\exists(q \,\&\, {\sim}r))(u_1, w_1, c) = 1 = \pi({\sim}\exists(p \,\&\, {\sim}q))(u_1, w_1, c) = \Pi({\sim}\exists(q \,\&\, {\sim}r))(u_1, w_1, c)$, by D1 (IV), (V). Then $\Pi(\exists(p \,\&\, {\sim}q))(u_1, w_1, c) = 0 = \Pi(\exists(q \,\&\, {\sim}r))(u_1, w_1, c)$, so that by D5 (II), $\Pi(\exists(p \,\&\, {\sim}r))(u_1, w_1, c) = 0 = \pi(\exists(p \,\&\, {\sim}r), c)$. Thus (6.3) holds for D5 (II).

It remains to show that D5 (VII) satisfies (6.3), for which we require the lemma:

(6.4) *If $\Pi(\exists(p \,\&\, q))(u, w, c) = 1$, then $\Pi(\exists p)(u, w, c) = 1$.*

If $\Pi(\exists((p \,\&\, q) \,\&\, {\sim}p))(u, w, c) = 0$, then $\Pi(\exists(\exists(J)(p \,\&\, q) \,\&\, {\sim}\exists(J)p))(u, w, c) = 0$, by D5 (IX), where J is the set of free variables of $p \,\&\, q$, and hence by D5 (VII) and (IX), $\Pi(\exists(p \,\&\, q))(u, w, c) = \Pi(\exists(p \,\&\, q) \,\&\, \exists p)(u, w, c)$. By D5 (I), (VI), and (V), $\Pi(\exists((p \,\&\, q) \,\&\, {\sim}p))(u, w, c) = 0$ for all $u \in \mathbf{U}$, $w \in \mathbf{W}$. (6.4) is thereby established.

To establish the π-analogue of D5 (VII), let $\Pi(\exists(p \,\&\, {\sim}q))(u_0, w_0, c) = 0$, for fixed $u_0 \in \mathbf{U}$, $w_0 \in \mathbf{W}$. That D5 (VII) satisfies (6.3) is then evident, except when $\Pi(\exists p)(u_0, w_0, c) = 2 = \Pi(\exists(p \,\&\, q))(u_0, w_0, c)$. On this hypothesis we distinguish three cases.

Case 1. $\pi(\exists p, c) = 0$. Then $\pi(\exists(p \,\&\, q), c) = 0$, by (3.9).

Case 2. $\pi(\exists p, c) = 1$. Then $\Pi(\exists p)(u, w, c) = 1$ for some $u \in \mathbf{U}$, $w \in \mathbf{W}$. Then by D1 (V), $\Pi(\exists p \,\&\, {\sim}\exists(p \,\&\, {\sim}q))(u_1, w_1, c) = 1$, for fixed $u_1 \in \mathbf{U}$, $w_1 \in \mathbf{W}$, so that $\Pi(\exists(p \,\&\, {\sim}q))(u_1, w_1, c) = 0$. Then by D5 (VII), $1 = \Pi(\exists p)(u_1, w_1, c) = \Pi(\exists(p \,\&\, q))(u_1, w_1, c)$.

Case 3. $\Pi(\exists p, c) = 2$. Suppose $\Pi(\exists(p \,\&\, q))(u, w, c) \neq 2$, for some $u \in \mathbf{U}$, $w \in \mathbf{W}$. If $\Pi(\exists(p \,\&\, q))(u, w, c) = 0$ for some $u \in \mathbf{U}$, $w \in \mathbf{W}$, then by D1 (V) and (IV), $\Pi(\exists(p \,\&\, q))(u_1, w_1, c) = 0 = \Pi(\exists(p \,\&\, {\sim}q))(u_1, w_1, c)$, for fixed $u_1 \in \mathbf{U}$, $w_1 \in \mathbf{W}$, so that $\Pi(\exists p)(u_1, w_1, c) = \Pi(\exists(p \,\&\, q))(u_1, w_1, c) = 0$, against the hypothesis of Case 3. On the other hand, if $\Pi(\exists(p \,\&\, q))(u, w, c) = 1$, for fixed $u \in \mathbf{U}$, $w \in \mathbf{W}$, then by (6.4), $\Pi(\exists p)(u, w, c) = 1$, against the hypothesis of Case 3. We have therefore shown that D5 (VII) satisfies (6.3). The proof of (6.3) is complete.

We conclude this section with the observation that the conditions of D2 on a sentential interpretation have analogues in the sense of (6.3). For

D2 (I) has the form of D5 (II); D2 (II) has the form of D5 (III); and D2 (III)-(VI) have the form of D5 (IV). This fact about sentential interpretations is required in the proof of (2.5).

7 *Some Foundations of Semiotic Theory* The intuitive content of (5.1) is that formulas of a predicate calculus L_Π which are interchangeable in all their occurrences in expressions of L_Π, preserving the valuations of those expressions by the users of L_Π, signify the same object, for Π as interpretant (in the sense of Peirce). As suggested in [3], the specification of some distinguished semantic interpretation of L_Π which could be said to contain the objects signified by the formulas of L_Π, for Π as interpretant, is the fundamental problem of first order semiotic theory. In this section we shall investigate various analogues of (5.1), which throw some light upon the above problem. We shall also study more deeply the relation between polyadic interpretations Π and predicate calculi L_Π.

We first define a relation of pragmatic synonymy on the expressions of L. Let Π be a pragmatic interpretation of L (in D) and let e and e' be expressions of L. We define $\mathsf{E}_\Pi(e, e')$ iff $\Pi(e''(e)) = \Pi(e''(e'))$, for all expressions $e''(e)$ and $e''(e')$ of L in the substitution notation (2.2). (We continue to employ the convention of [3] that in an expression represented as $e(e')$ for (2.2), e' occurs at least once.) $\mathsf{E}_\Pi(e, e')$ asserts, in an obvious sense, that e and e' are pragmatically synonymous, relative to Π: e and e' are interchangeable in expressions of L, preserving the Π-valuations of those expressions.

Let π be the core of a polyadic interpretation Π. We then define, for predicates F and G of L_Π, $\mathsf{E}_\pi(F, G)$ iff F and G are of the same degree and all formulas $p(F)$ and $p(G)$ of L_Π related by substitution according to (2.2) are E_π-congruent in the sense of (4.1). Analogously we define, for individual constants a and b of L_Π, $\mathsf{E}_\pi(a, b)$ iff all formulas $p(a)$ and $p(b)$ related by (2.2) are E_π-congruent. Finally we observe that, for formulas p and q of L_Π, $\mathsf{E}_\pi(p, q)$ iff $\mathsf{E}_\pi(r(p), r(q))$ for all formulas $r(p)$ and $r(q)$ related by (2.2). If predicates, individual constants, or formulas are in the appropriate E_π-relation, we shall say that they are congruent.

From the above definitions it follows immediately that pragmatic synonymy entails congruence, for predicates F and G, individual constants a and b, and formulas p and q:

(7.1) *If* $\mathsf{E}_\Pi(F, G)$, *then* $\mathsf{E}_\pi(F, G)$.
If $\mathsf{E}_\Pi(a, b)$, *then* $\mathsf{E}_\pi(a, b)$.
If $\mathsf{E}_\Pi(p, q)$, then $\mathsf{E}_\pi(p, q)$.

The following propositions are concerned with the relation between the syntactic and algebraic structures of L_Π. Let the predicate expressions F and G be represented, respectively, by the algebraic predicates F and G.

(7.2) $\mathsf{E}_\pi(F,G)$ *iff* $\mathsf{F} = \mathsf{G}$.

Proof: Let F and G be of degree n, and let $\{i_1, \ldots, i_n\} = \mathsf{J} \subset \mathsf{I}$ and $\{t_1, \ldots, t_n\} \subset \mathsf{J} \cup \mathsf{K}$. If $\mathsf{E}_\pi(Fi_1 \ldots i_n, Gi_1 \ldots i_n)$, then $\mathsf{E}_\pi(Ft_1 \ldots t_n, Gt_1 \ldots t_n)$, so that $\mathsf{E}_\pi(p(Ft_1 \ldots t_n), p(Gt_1 \ldots t_n))$ for all formulas $p(Ft_1 \ldots t_n)$, $p(Gt_1 \ldots t_n)$ related by (2.2). Thus if $\mathsf{E}_\pi(Fi_1 \ldots i_n, Gi_1 \ldots i_n)$ holds for all $i_1, \ldots, i_n \in \mathsf{I}$, i.e., if $\mathsf{F} = \mathsf{G}$, then $\mathsf{E}_\pi(F, G)$. The converse is obvious.

(7.2) Corollary *The representation of predicate expressions by algebraic predicates, according to* Theorem 2, *is one-one iff* L_Π *contains no distinct congruent predicate expressions.*

Let the individual constants a and b be represented, respectively, by the algebraic constants a and b.

(7.3) $\mathsf{E}_\pi(a, b)$ *iff* $\mathsf{a} = \mathsf{b}$.

Proof: If a and b are not congruent, then there are formulas $p(a)$ and $p(b)$ such that it is not the case that $\mathsf{E}_\pi(p(a), p(b))$, where $p(a)$ is of the form $p(a_\mathsf{J})$, for some variables J and formula p, from which $p(a_\mathsf{J})$ is got by putting a for free $i \in \mathsf{J}$ in p, and from which $p(b) = p(b_\mathsf{J})$ is got by putting b for free $i \in \mathsf{J}$ in p (as in Definition (3.16)); then $p(a) = \mathsf{a}(\mathsf{J})p$ and $p(b) = \mathsf{b}(\mathsf{J})p$ are not congruent, so that $\mathsf{a} \neq \mathsf{b}$. To show the converse, let $\mathsf{a} \neq \mathsf{b}$. Then it is not the case that for all $\mathsf{J} \subset \mathsf{I}$, $p \in \mathsf{Q}$, $\mathsf{E}_\pi(\mathsf{a}(\mathsf{J})p, \mathsf{b}(\mathsf{J})p)$; and hence for some $p(a) = p(a_\mathsf{J})$ and $p(b) = p(b_\mathsf{J})$, it is not the case that $\mathsf{E}_\pi(p(a), p(b))$.

(7.3) Corollary *The representation of individual constants by algebraic constants, according to* Theorem 3, *is one-one iff* L_Π *contains no distinct congruent individual constants.*

If two polyadic interpretations have the same logical constants and individual variables we shall say that they are *similar* interpretations. We define $|\Pi| = \mathsf{P} \cup \mathsf{K}$ to be the *parameters* of Π (or of L_Π). If two similar interpretations Π_1 and Π_2 have the same parameters, then they have the same syntax. For if s is a sentence of Π_1, then by (2.1) s has one of the forms of D4 (V). Then by hypothesis and D4 (I)-(IV), $\Pi_2(s \,\&\, {\sim}s) = d_0$, so that s is a sentence of Π_2.

Π_1 is defined to be a *subinterpretation* of Π_2 (abbreviated $\Pi_1 < \Pi_2$) iff Π_1 and Π_2 are similar and for all $e \in \mathsf{L}$, $u \in \mathsf{U}$, $w \in \mathsf{W}$, $c \subset \mathsf{C}$: $\Pi_1(e)(u, w, c) = \Pi_2(e)(u, w, c)$ whenever $\Pi_1(e)(u, w, c) \neq 2$. The subinterpretation relation is a partial ordering of the polyadic interpretations of L in D. The following propositions follow immediately from the relevant definitions. Let Q_1 and Q_2 be the formulas of Π_1 and Π_2, respectively.

(7.4) *If* $\Pi_1 < \Pi_2$, *then* $|\Pi_1| \subset |\Pi_2|$ *and* $\mathsf{Q}_1 \subset \mathsf{Q}_2$.

We define the *theory* of a polyadic interpretation Π (or of L_Π) to be the set $\mathsf{T}_\mathsf{S} = \mathsf{T} \cap \mathsf{S}$, where T is defined by (4.5) and S by (2.1). Let T_1 and T_2 be theories of Π_1 and Π_2, respectively.

(7.5) *If* $\Pi_1 < \Pi_2$, *then* $\mathsf{T}_1 \subset \mathsf{T}_2$.

(7.6) *If* $\Pi_1 < \Pi_2$ *and* Π_1 *agrees with* Π_2 *on all sentences of* Π_1, *then* L_{Π_1} *is a polyadic subalgebra of* L_{Π_2}.

Proof: By (7.4) and the hypothesis of (7.6), the domain Q_1 of L_{Π_1} is included in the domain $\mathbf{Q}_2$ of L_{Π_2}. Now let $\mathbf{E}_{\pi_2}(p, q)$, where $p, q \in \mathbf{Q}_1$ and π_2 is the core of Π_2. Then $\pi_2(\forall(p \leftrightarrow q), \mathbf{C}) = 1 = \pi_1(\forall(p \leftrightarrow q), \mathbf{C})$, where π_1 is the core of Π_1, since $\forall(p \leftrightarrow q)$ is a sentence of L_{Π_1}. Then $\mathbf{E}_{\pi_1}(p, q)$. Clearly $\mathbf{E}_{\pi_1} \subset \mathbf{E}_{\pi_2}$, so that $\mathbf{E}_{\pi_1}$ is the restriction to $\mathbf{Q}_1$ of $\mathbf{E}_{\pi_2}$. Since Π_1 and Π_2 are similar, the operations of L_{Π_1} on $\mathbf{Q}_1$ are the restrictions to $\mathbf{Q}_1$ of the operations of L_{Π_2} on $\mathbf{Q}_2$. (7.6) is thereby established.

If Π_1 and Π_2 satisfy the antecedent of (7.6), then every semantic interpretation of L_{Π_2} is an extension of a semantic interpretation of L_{Π_1}. This fact is of semiotic importance, but we shall not in this paper pursue its consequences.

We define $\mathbf{T}$ to be a *subtheory* of the theory of Π iff there is a non-empty subset $\mathbf{A}$ of the theory of Π such that $\mathbf{T}$ is the set of sentences of Π over the parameters of $\mathbf{A}$ which are logical consequences of $\mathbf{A}$.

(7.7) *If* $\mathbf{T}$ *is a subtheory of* Π, *then* $\mathbf{T}$ *is the theory of a subinterpretation of* Π.

Proof: Let $\boldsymbol{P}_{\mathbf{T}}$ be the set of all subinterpretations of Π which agree with Π on $\mathbf{T}$. $\boldsymbol{P}_{\mathbf{T}}$ is not empty, since $\Pi \in \boldsymbol{P}_{\mathbf{T}}$. In terms of $\boldsymbol{P}_{\mathbf{T}}$ we define the mapping Π_0 from $\mathbf{L}$ into $\mathbf{D}$:

$$\Pi_0(e)(u, w, c) = \begin{cases} \Pi(e)(u, w, c), \text{ if } \Pi(e)(u, w, c) = \Pi_1(e)(u, w, c) \text{ for all } \Pi_1 \in \boldsymbol{P}_{\mathbf{T}}. \\ 2, \text{ otherwise.} \end{cases}$$

It is straightforward to verify that Π_0 is a polyadic interpretation similar to Π. Now let $\Pi_0(e)(u, w, c) \neq 2$. Then $\Pi_0(e)(u, w, c) = \Pi(e)(u, w, c)$, so that $\Pi_0 < \Pi$. Finally, let $e \in \mathbf{T}$. Then for some $u \in \mathbf{U}$, $w \in \mathbf{W}$, and for all $\Pi_1 \in \boldsymbol{P}_{\mathbf{T}}$, $\Pi_1(e)(u, w, \mathbf{C}) = 1 = \Pi(e)(u, w, \mathbf{C})$, so that e is in the theory of Π_0. Conversely, if $\Pi_0(e)(u, w, \mathbf{C}) = 1 = \Pi(e)(u, w, \mathbf{C})$ for some $u \in \mathbf{U}$, $w \in \mathbf{W}$, then $e \in \mathbf{T}$. Thus $\mathbf{T}$ is the theory of $\Pi_0 < \Pi$. (7.7) is thereby established.

Let $\boldsymbol{P}_{\mathfrak{G}}$ be the set of all polyadic interpretations of $\mathbf{L}$ in $\mathbf{D}$ with syntax $\mathfrak{G}$. The set of all polyadic interpretations of $\mathbf{L}$ in $\mathbf{D}$ is partitioned into sets of the form $\boldsymbol{P}_{\mathfrak{G}}$. Let $\mathbf{E}$ be the intersection of all congruences $\mathbf{E}_\pi$, defined by (4.1), where π is the core of some $\Pi \in \boldsymbol{P}_{\mathfrak{G}}$. The least polyadic interpretation in $\boldsymbol{P}_{\mathfrak{G}}$ (with respect to $<$) determines a predicate calculus with respect to $\mathbf{E}$, which is free in the family of predicate calculi L_Π, for $\Pi \in \boldsymbol{P}_{\mathfrak{G}}$, and may be regarded as the algebraic representation of the pure predicate calculus for the standard syntax $\mathfrak{G}$.

For $\mathfrak{G}$ and $\mathbf{E}$ as above, let $\Pi \in \boldsymbol{P}_{\mathfrak{G}}$, and let $t = \sim(s \mathbin{\&} \sim s)$, for some sentence s of $\mathfrak{G}$. It follows from the definition of a polyadic interpretation that $\mathbf{E}(s, t)$ if $\Pi(s) = d_1$. The $\mathbf{E}$-congruence class $\mathbf{T}_{\mathfrak{G}}$ of t may then be regarded as the set of logical truths of predicate calculi with syntax $\mathfrak{G}$: $s \in \mathbf{T}_{\mathfrak{G}}$ iff s is mapped on the unit element by every semantic interpretation of L_Π, for every $\Pi \in \boldsymbol{P}_{\mathfrak{G}}$.

Now let $\mathbf{M} = \{s \in \mathbf{S}: \Pi(s) = d_1\}$. Then $\mathbf{T}_{\mathfrak{G}} \subset \mathbf{M} \subset \mathbf{T}_{\mathbf{S}}$, where $\mathbf{T}_{\mathbf{S}}$ is the theory of Π. In the non-trivial case of proper inclusion, this result may be understood to mean that (in the familiar terminology of Quine) the theory $\mathbf{M}$ is more remote from the evidence $\mathbf{C}$ than $\mathbf{T}_{\mathbf{S}}$, but not as remote as $\mathbf{T}_{\mathfrak{G}}$. The status of $\mathbf{M}$ may perhaps be accorded to (first order) analytic theories:

M may be invoked for establishing some particular theory $\mathsf{T_S}$ (under the evidence C, relative to the rules of acceptance represented by Π), but not necessarily for establishing any theory whatever (in the syntax $\mathfrak{S}$ of M), as is the set of logical truths $\mathsf{T}_{\mathfrak{S}}$.

If Π is a polyadic interpretation of L in D, then the interpreting dispositions in D are clearly an idealization of the actual verbal behavior of even the most careful scientific users of L. For example, as we have seen above, every logical truth of L_Π is accepted by every user at every time under every condition. The condition V of D1 is also a manifest idealization of actual verbal behavior. D1 (V) may be weakened to hold only for the total evidence C, in which case (6.1) and (6.3) hold only for C; and this is sufficient for the existence of the languages L_Π. Moreover, if D1 (V) holds only for C, then the above consequence about logical truths is not forthcoming; but it reappears for condition C itself. We shall not pursue such possibilities for diminishing the idealization of actual verbal behavior represented by polyadic interpretations; we rather contemplate their application to actual behavior by suitable approximation.

We shall conclude this section by considering analogues of (5.1), for the case of interpretations of the theory $\mathsf{T_S}$ of L_Π, in models which are relational structures. For this purpose we are led to understand the concept of a (relational) model of a set of sentences of L_Π in the following way. By a *relation* of degree n on a set X we understand a mapping from X^n into the (domain of the) simple Boolean algebra. By a *relational structure* we understand a pair $\mathfrak{X} = \langle \mathsf{X}, \mathsf{R}\rangle$, where R is a non-empty set of relations on the non-empty set X. Let A be a non-empty set of sentences of L_Π and let $\mathfrak{X} = \langle \mathsf{X}; \mathsf{R}\rangle$ be a relational structure. Let μ be a mapping from predicates of (sentences of) A to relations of like degree in R, and from individual constants of A to elements of X. μ may be called a semantic interpretation of A in $\mathfrak{X}$.

Let p be any formula of L_Π over the parameters of A (i.e., each predicate and individual constant of p is in some sentence of A). p may then be said to be defined in $\mathfrak{X}$ under μ. Let X^I be the set of all functions from the set I of variables of L_Π into the domain X of $\mathfrak{X}$. Let $x \in \mathsf{X}^\mathsf{I}$. If $t \in \mathsf{I} \cup \mathsf{K}$, we define $x_t = x(t)$ if $t \in \mathsf{I}$, and $x_t = \mu t$ if $t \in \mathsf{K}$. We then define x *satisfies* p *under* μ iff one of the following four conditions holds:

(1) $p = Ft_1 \ldots t_n$, where $F \in \boldsymbol{P}^n$ and $t_1, \ldots, t_n \in \mathsf{I} \cup \mathsf{K}$, and $\mu F(x_{t_1}, \ldots, x_{t_n}) = 1$.
(2) $p = q \;\&\; r$, where q and r are formulas of L_Π, and x satisfies both q and r under μ.
(3) $p = {\sim} q$, where q is a formula of L_Π, and x does not satisfy q under μ.
(4) $p = \exists iq$, where q is a formula of L_Π, $i \in \mathsf{I}$, and for some $y \in \mathsf{X}^\mathsf{I}$ which differs from x at most at x_i, y satisfies q under μ.

Now we may define, for any sentence s of A: s *holds in* $\mathfrak{X}$ *under* μ iff all $x \in \mathsf{X}^\mathsf{I}$ satisfy s under μ. Finally, $\mathfrak{X}$ is defined to be a *model of* A *under* μ iff (every sentence of) A holds in $\mathfrak{X}$ under μ, and μ is onto the set of relations of $\mathfrak{X}$.

The above definitions do not require that all elements of the domain of $\mathfrak{X}$ be named by individual constants of A under μ, nor that L_Π be a sublanguage of a language with sufficient constants for this property to hold (as in Robinson's [2]). If, however, the latter condition holds, the above definition of *x satisfies p under* μ may be restricted to one defining *s holds in* $\mathfrak{X}$ *under* μ, for closed formulas s of L_Π. We do not require that interpretations be one-one from predicates and individual constants of $\mathbf{A}$ to relations and individuals of $\mathfrak{X}$, since we wish to investigate analogues of (5.1) for predicates and individual constants. If, however, interpretations μ are required to be one-one in this sense, the definition of *s holds in* $\mathfrak{X}$ *under* μ obtainable by restricting the above definition of satisfaction to closed sentences s, is essentially the same as Definition 1.4 of Robinson's [2], modified so that relations are regarded as mappings in the above sense, and iterated quantification of formulas is allowed. We shall at the end of this section consider the conditions under which an interpretation of $\mathbf{T_S}$ is one-one on $\mathbf{P}$ and $\mathbf{K}$.

For the case of interpretations of $\mathbf{T_S}$ in relational models, we have the following analogue of (5.1). Let $\mathfrak{X} = \langle \mathbf{X}, \mathbf{R} \rangle$ be a model of $\mathbf{T_S}$ under μ. Let $x \in \mathbf{X}^{\mathbf{I}}$. Then for all formulas p and q of L_Π:

(7.8) *If* $\mathsf{E}_\Pi(p, q)$, *then x satisfies p under* μ *iff x satisfies q under* μ.

Proof: p and q are over the parameters of $\mathbf{T_S}$, which is $\mathbf{P} \cup \mathbf{K}$, by (3.8). By (7.1) and the hypothesis of (7.8), $\mathsf{E}_\pi(p, q)$, so that $\forall(p \leftrightarrow q) \in \mathbf{T_S}$. Thus $\forall(p \leftrightarrow q)$ holds in $\mathfrak{X}$ under μ; i.e., all $x \in \mathbf{X}^{\mathbf{I}}$ satisfy $\forall(p \leftrightarrow q)$ under μ. The consequent of (7.8) follows by expanding $\forall(p \leftrightarrow q)$ in terms of $\exists$, $\sim$, and &.

We now introduce a semantical synonymy relation on the predicates, individual constants, and formulas of L_Π. As observed in the proof of (7.8), the set of parameters of $\mathbf{T_S}$ is $\mathbf{P} \cup \mathbf{K}$. For predicates F and G of L_Π we define $\mathsf{E_M}(F, G)$ iff $\mu F = \mu G$ for all interpretations μ of $\mathbf{T_S}$ under which $\mathbf{T_S}$ holds. For individual constants a and b of L_Π we define $\mathsf{E_M}(a, b)$ iff $\mu a = \mu b$ for all interpretations μ of $\mathbf{T_S}$ under which $\mathbf{T_S}$ holds. For formulas p and q of L_Π we define $\mathsf{E_M}(p, q)$ iff $\mu p = \mu q$ for all semantic interpretations μ of L_Π (i.e., polyadic homomorphisms μ of L_Π in a Boolean model, in the sense of [1]).

$\mathsf{E_M}(p, q)$ may just as well be defined in terms of relational models, as the consequent of (7.8) for all interpretations of $\mathbf{T_S}$ under which $\mathbf{T_S}$ holds. For if $\mu p = \mu q$, where μ is a (semantic) interpretation of L_Π, then $\mu(p \leftrightarrow q) = 1$, so that if $\mathsf{E_M}(p, q)$ then $p \leftrightarrow q \in \mathbf{T}$, which is the set of formulas mapped on 1 by every interpretation of L_Π. Thus if $\mathsf{E_M}(p, q)$ then $\forall(p \leftrightarrow q) \in \mathbf{T_S}$, and the consequent of (7.8) follows as in the proof of (7.8). Conversely, assume the consequent of (7.8) for all interpretations μ of $\mathbf{T_S}$ under which $\mathbf{T_S}$ holds. Then for every such interpretation of $\mathbf{T_S}$, all $x \in \mathbf{X}^{\mathbf{I}}$ satisfy $p \leftrightarrow q$, and hence satisfy $\forall(p \leftrightarrow q)$, so that $\forall(p \leftrightarrow q)$ holds. It follows that $\forall(p \leftrightarrow q) \in \mathbf{T_S}$ (*cf.*, Robinson 8.1.3). Thus for all interpretations μ of L_Π, $\mu p = \mu q$; i.e., $\mathsf{E_M}(p, q)$.

We shall refer to the relations $\mathsf{E_M}$ as *model synonymies*. From (5.1) it follows that if two formulas are pragmatically synonymous they are model synonymous. The same is true for predicates F and G.

(7.9) *If* $\mathsf{E}_{\Pi}(F, G)$, *then* $\mathsf{E}_{\mathsf{M}}(F, G)$.

Proof: By hypothesis and (7.1), for all $i_1, \ldots, i_n \in \mathsf{I}$, $\mathsf{E}_{\pi}(Fi_1 \ldots i_n, Gi_1 \ldots i_n)$. Then by (7.8), for all interpretations μ of $\mathsf{T_S}$, and all $x \in \mathsf{X}^{\mathsf{I}}$ relative to μ, x satisfies $Fi_1 \ldots i_n$ under μ iff x satisfies $Gi_1 \ldots i_n$ under μ, so that $\mu F(x_{i_1}, \ldots, x_{i_n}) = \mu G(x_{i_1}, \ldots, x_{i_n})$. Now for each n-tuple $(x_1, \ldots, x_n)$ of X^n, there are $x \in \mathsf{X}^{\mathsf{I}}$ and distinct $i_1, \ldots, i_n \in \mathsf{I}$ such that $x_1 = x_{i_1}, \ldots, x_n = x_{i_n}$, since X^{I} contains all functions from I into X. Then for each such $(x_1, \ldots, x_n)$, by choosing appropriate elements of X and I, we have $\mu F(x_1, \ldots, x_n) = \mu G(x_1, \ldots, x_n)$, for all interpretations μ of $\mathsf{T_S}$; i.e., $\mathsf{E}_{\mathsf{M}}(F, G)$.

There is no analogue of (7.9) for individual constants. For if $\mathsf{E}_{\pi}(a, b)$ then $\mathsf{T_S}$ asserts only that μa and μb are indistinguishable by the predicates of $\mathsf{T_S}$.

We have shown that for predicates, individual constants, and formulas of $\boldsymbol{L}_{\Pi}$, pragmatic synonymy implies congruence; and for predicates and formulas of $\boldsymbol{L}_{\Pi}$, congruence implies model synonymy. It is also the case that, for formulas, predicates, and individual constants, model synonymy implies congruence. We conclude this section by observing that in order for an interpretation of $\mathsf{T_S}$ to be one-one on the predicates P, it is sufficient that $\mathsf{T_S}$ be maximal and P have no distinct congruent predicates. Conversely, if an interpretation of $\mathsf{T_S}$ is one-one on P, then P has no distinct congruent predicates. Finally, in order for an interpretation of $\mathsf{T_S}$ to be one-one on the individual constants K, it is sufficient that $\mathsf{T_S}$ be maximal and K have no distinct congruent constants. But if an interpretation of $\mathsf{T_S}$ is one-one on K, it does not follow that K has no distinct congruent constants.

8 *Boolean and Relational Models* Every interpretation of $\boldsymbol{L}_{\Pi}$ into a Boolean model is an interpretation of $\boldsymbol{L}_{\Pi}$ onto its range (*cf.*, [1], p. 130). (5.1) is stated for onto interpretations (though of course it holds in general) since we assume on intuitive semiotic grounds that a calculus $\boldsymbol{L}_{\Pi}$, regarded as a sign, has only its onto interpretations as possible objects. On the same intuitive grounds, we assume that the possible objects of the theory $\mathsf{T_S}$ of $\boldsymbol{L}_{\Pi}$, regarded as a sign, are the interpretations of $\mathsf{T_S}$ in relational models of $\mathsf{T_S}$. Thus (7.8) is the semiotic equivalent of (5.1), and (7.9) is a semiotic analogue of (5.1), provided that:

(8.1) *There is a one-one correspondence between the interpretations* μ *of* $\boldsymbol{L}_{\Pi}$ *onto Boolean models and the interpretations* μ^* *of* $\mathsf{T_S}$ *in relational models of* $\mathsf{T_S}$, *such that models under corresponding interpretations have the same domain, and for all sequences of values* $x \in \mathsf{X}^{\mathsf{I}}$ *and formulas* p *of* $\boldsymbol{L}_{\Pi}$:

$$\mu p(x) = 1 \textit{ iff } x \textit{ satisfies } p \textit{ under } \mu^*$$

where X *is the common domain of models under* μ *and* μ^*, *and* I *is the set of variables of* $\boldsymbol{L}_{\Pi}$.

We shall give a detailed proof of (8.1), reference to which will be useful in subsequent investigations of the semiotics of first order languages. We first show:

(A) *For every interpretation* μ *of* $\boldsymbol{L}_{\Pi}$ *onto a Boolean model, there is an interpretation* μ^* *of* $\mathbf{T_S}$ *such that* μ *and* μ^* *are related according to* (8.1).

Proof: Let the I-algebra $\mathfrak{B}$ over X be a model of $\boldsymbol{L}_{\Pi}$ under μ. Let F be a predicate of degree n of $\boldsymbol{L}_{\Pi}$. Let $\mathsf{H} = \{i_1, \ldots, i_n\} \subset \mathsf{I}$. Let $x, y \in \mathsf{X}^{\mathsf{I}}$. We first establish the proposition:

(8.2) *If* $x(\mathsf{I} - \mathsf{H})_* y$, *then* $\mu F i_1 \ldots i_n(x) = \mu F i_1 \ldots i_n(y)$.

The antecedent of (8.2) is defined to mean that $x_i = y_i$ if $i \notin \mathsf{I} - \mathsf{H}$. (8.2) asserts that $\mu F i_1 \ldots i_n$ is independent of $\mathsf{I} - \mathsf{H}$, a (necessary and) sufficient condition for which is that $\mu F i_1 \ldots i_n = \exists(\mathsf{I} - \mathsf{H})\mu F i_1 \ldots i_n$ (*cf.*, [1], p. 114). By Definition (3.6),

$$\exists(\mathsf{I} - \mathsf{H})F i_1 \ldots i_n = \exists(\Lambda)F i_1 \ldots i_n = F i_1 \ldots i_n.$$

Then since μ is a polyadic homomorphism, $\mu F i_1 \ldots i_n = \exists(\mathsf{I} - \mathsf{H})\mu F i_1 \ldots i_n$; (8.2) is thereby established.

With F, x, y, μ as above, let $\mathsf{H} = \{i_1, \ldots, i_n\}$, $\mathsf{J} = \{j_1, \ldots, j_n\} \subset \mathsf{I}$, where the variables in H and in J are distinct. We then have a lemma for (A).

(8.3) *If* $x_{i_1} = y_{j_1}, \ldots, x_{i_n} = y_{j_n}$, *then* $\mu F i_1 \ldots i_n(x) = \mu F j_1 \ldots j_n(y)$.

For proof of (8.3) we observe that there exists a transformation τ on I such that $\tau j_1 = i_1, \ldots, \tau j_n = i_n$. For such τ, $F i_1 \ldots i_n = \mathsf{S}(\tau)F j_1 \ldots j_n$, by Definition (3.2). Then since μ is a polyadic homomorphism:

(8.3)′ $\mu F i_1 \ldots i_n(x) = \mu\mathsf{S}(\tau)F j_1 \ldots j_n(x) = \mathsf{S}(\tau)\mu F j_1 \ldots j_n(x) = \mu F j_1 \ldots j_n(\tau_* x)$, *where* $(\tau_* x)_i = x_{\tau_i}$, *for* $i \in \mathsf{I}$.

By hypothesis, $(\tau_* x)_{j_1} = x_{\tau_{j_1}} = x_{i_1} = y_{j_1}, \ldots, (\tau_* x)_{j_n} = y_{j_n}$; i.e., $\tau_* x\ (\mathsf{I} - \mathsf{J})_* y$. Then by (8.2), $\mu F j_1 \ldots j_n(\tau_* x) = \mu F j_1 \ldots j_n(y)$. (8.3) then follows by (8.3)′.

With each predicate F of degree n of $\boldsymbol{L}_{\Pi}$ we may now associate a relation f on X as follows. For all $(x_1, \ldots, x_n) \in \mathsf{X}^n$, for any distinct variables $i_1, \ldots, i_n \in \mathsf{I}$, and for all $x \in \mathsf{X}^{\mathsf{I}}$ such that $x_{i_1} = x_1, \ldots, x_{i_n} = x_n$:

(8.4) $f(x_1, \ldots, x_n) = \mu F i_1 \ldots i_n(x)$.

For each $(x_1, \ldots, x_n) \in \mathsf{X}^n$, such $x, i_1, \ldots, i_n$ will always exist, since X^{I} contains all mappings from I into X. And by (8.3), $f(x_1, \ldots, x_n)$ is uniquely defined.

Now with the model $\mathfrak{B}$ of $\boldsymbol{L}_{\Pi}$ under μ we associate the structure $\mathfrak{X}_{\mathfrak{B}} = \langle\mathsf{X}, \mathsf{R}\rangle$, where R is the set of relations f on X defined by (8.4). We wish to show that $\mathfrak{X}_{\mathfrak{B}}$ is a model of $\mathbf{T_S}$ under an appropriate interpretation μ^*, which may be defined as follows. If F is a predicate of $\mathbf{T_S}$ we let $\mu^* F = f \in \mathsf{R}$, defined by (8.4). In order to define $\mu^* a$, where a is an individual constant of $\mathbf{T_S}$, we reason as follows. Every constant b of $\boldsymbol{L}_\Pi$ corresponds to a unique element x_b of $\mathbf{X}$ such that, for all $\mathbf{J} \subset \mathbf{I}$, $x \in \mathbf{X}^\mathbf{I}$, and functions α of $\mathfrak{B}$:

(8.5) $\mathsf{b}(\mathbf{J})\alpha(x) = \alpha(x^\mathbf{J})$,

where $x_i^\mathbf{J} = x_\mathsf{b}$ if $i \in \mathbf{J}$, and $x_i^\mathbf{J} = x_i$ otherwise. The correspondence $b \to x_\mathsf{b}$ is one-one. Moreover, constants of $\boldsymbol{L}_\Pi$ are preserved under the (onto) homomorphism μ, in the sense that if a is a constant of $\boldsymbol{L}_\Pi$ then there is a unique constant b of $\mathfrak{B}$ such that:

(8.6) $\mu\mathsf{a}(\mathbf{J})p = \mathsf{b}(\mathbf{J})\,\mu p$

for all formulas p and sets $\mathbf{J}$ of variables of $\boldsymbol{L}_\Pi$ (*cf.*, [1]: p. 155, Lemma (15.3)). Then we define $\mu^*a = x_\mathsf{b}$, where b is the constant of $\mathfrak{B}$ which corresponds, according to (8.6), with the constant a of $\boldsymbol{L}_\Pi$ which represents a, and x_b is in turn defined by (8.5).

In order to show that $\mathbf{T_S}$ holds in $\mathfrak{X}_\mathfrak{B}$ under μ^*, we shall require the following lemmas (8.7) and (8.8). Let $F \in \mathbf{P}^n$; $t_1, \ldots, t_n \in \mathbf{I} \cup \mathbf{K}$. Then for all $x \in \mathbf{X}^\mathbf{I}$:

(8.7) $\mu F t_1 \ldots t_n(x) = f(x_{t_1}, \ldots, x_{t_n})$,

where $f = \mu^*F$, and (for $m = 1, \ldots, n$) $x_{t_m} = x(t_m)$ if $t_m \in \mathbf{I}$, and $x_{t_m} = \mu^*t_m$ if $t_m \in \mathbf{K}$. We shall prove (8.7) for the case that only one t_m is in $\mathbf{K}$; the general case follows by induction. Let $t_m = a \in \mathbf{K}$. Then for some $\mathbf{J} = \{i_1, \ldots, i_n\} \subset \mathbf{I}$ and for some $\mathbf{H} \subset \mathbf{J}$, $Ft_1 \ldots t_n = Fi_1 \ldots i_n(a_\mathbf{H})$, the formula got from $Fi_1 \ldots i_n$ by putting a for $i \in \mathbf{H}$ (*cf.*, (3.16)). Let a be represented by the algebraic constant a, so that $Fi_1 \ldots i(a_\mathbf{H}) = \mathsf{a}(\mathbf{H})Fi_1 \ldots i_n$. Then for all $x \in \mathbf{X}^\mathbf{I}$:

$$\begin{aligned}
\mu F t_1 \ldots t_n(x) &= \mu\mathsf{a}(\mathbf{H})Fi_1 \ldots i_n(x) & \\
&= \mathsf{b}(\mathbf{H})\,\mu Fi_1 \ldots i_n(x) & \text{(by (8.6))}\\
&= \mu Fi_1 \ldots i_n(x^\mathbf{H}) & \text{(by (8.5))}\\
&= f(x^\mathbf{H}_{i_1}, \ldots, x^\mathbf{H}_{i_n}) & \text{(by (8.4))}\\
&= f(x_1, \ldots, x_n) &
\end{aligned}$$

where (for $m = 1, \ldots, n$)$x_m = x_{i_m}$ if $i_m \notin \mathbf{H}$, and $x_m = x_\mathsf{b}$ if $i_m \in \mathbf{H}$. Now if $t_m \in \mathbf{I}$, then $t_m \notin \mathbf{H}$, so that $x_m = x^\mathbf{H}_{i_m} = x_{i_m} = x_{t_m}$. And if $t_m \notin \mathbf{I}$, then $i_m \notin \mathbf{H}$, so that $x_m = x_\mathsf{b} = \mu^*t_m = x_{t_m}$. (8.7) is thereby proved.

Let p be a formula of $\boldsymbol{L}_\Pi$, with μ, μ^*, and x as above.

(8.8) *x satisfies p under μ^* iff $\mu p(x) = 1$.*

Proof: For atomic formulas $Ft_1 \ldots t_n$ of $\boldsymbol{L}_\Pi$, by (8.7) and condition (1) in the definition of satisfaction in section 7:

(I) x satisfies $Ft_1 \ldots t_n$ under μ^* iff $\mu Ft_1 \ldots t_n(x) = 1$.

Since μ is a polyadic homomorphism, for all formulas q and r of $\boldsymbol{L}_\Pi$:

(II) $\mu(q \;\&\; r)(x) = 1$ iff $\mu q(x) = 1 = \mu r(x)$
(III) $\mu(\sim q)(x) = 1$ iff $\mu q(x) = 0$

(IV) $\mu(\exists iq)(x) = 1$ iff $\exists\{i\}\ \mu q(x) = 1$
iff $\vee\{\mu q(y):\ x\{i\}_* y\} = 1$
iff $\mu q(y) = 1$ for some y which differs from x at most at x_i.

Thus by (2)-(4) in the definition of satisfaction, and by the above properties (II)-(IV) of the homomorphism μ, valuation of $x \in \mathsf{X}^{\mathsf{I}}$ as 1 by μp is fixed exactly as is satisfaction of p by x under μ^*. This proves (8.8).

From (8.8) it follows that, since all sentences of $\mathsf{T_S}$ are mapped by μ to the unit element of $\mathfrak{B}$ (the model of $\boldsymbol{L}_\Pi$ under μ), all sentences of $\mathsf{T_S}$ are satisfied by all $x \in \mathsf{X}^{\mathsf{I}}$ under μ^*. Then $\mathsf{T_S}$ holds in $\mathfrak{X}_{\mathfrak{B}}$ under μ^*, and (A) is thereby proved.

(B) *The mapping* $\mu \to \mu^*$ *defined in* (A) *is an onto mapping.*

Proof: Let $\mathfrak{X} = \langle \mathsf{X}, \mathsf{R} \rangle$ be a model of $\mathsf{T_S}$ under an interpretation μ^*. Then each predicate F of $\boldsymbol{L}_\Pi$ is associated with a relation $\mu^* F = f \in \mathsf{R}$, and each name a of $\boldsymbol{L}_\Pi$ is associated with an element $\mu^* a$ of X. Let B be the set of all functions from X^{I} into the simple Boolean algebra. We then define a mapping μ from the formulas Q of $\boldsymbol{L}_\Pi$ into B, such that for each atomic formula $Ft_1 \ldots t_n$ of Q and for each $x \in \mathsf{X}^{\mathsf{I}}$:

(8.9) $\mu F t_1 \ldots t_n(x) = f(x_{t_1}, \ldots, x_{t_n})$,

where $f = \mu^* F$ and (for $m = 1, \ldots, n$) $x_{t_m} = x(t_m)$ if $t_m \in \mathsf{I}$, and $x_{t_m} = \mu^* t_m$ if $t_m \in \mathsf{K}$. The mapping μ may then be extended over Q, and polyadic operations defined on the range $\mu\mathsf{Q}$ of μ, for all subsets J and transformations τ of I, in the following natural way. For all $p, q \in \mathsf{Q}$:

$$
(8.10)\quad
\begin{aligned}
&\mu(p \,\&\, q)(x) = \mu p(x) \wedge \mu q(x) = (\mu p \wedge \mu q)(x).\\
&\mu(\sim p)(x) = \mu p(x)' = (\mu p)'(x).\\
&\mu\exists(\mathsf{J})p(x) = \vee\{\mu p(y):\ x \mathsf{J}_* y\} = \exists(\mathsf{J})\,\mu p(x).\\
&\mu\mathsf{S}(\tau)p(x) = \mu p(\tau_* x) = \mathsf{S}(\tau)\,\mu p(x).
\end{aligned}
$$

Since for all $\mu p \in \mu\mathsf{Q}$, $\exists(\mathsf{J})\mu p$ exists and belongs to $\mu\mathsf{Q}$, and $\mathsf{S}(\tau)\mu p$ belongs to $\mu\mathsf{Q}$, $\mathfrak{B} = \langle \mu\mathsf{Q}, \mathsf{I}, \mathsf{S}, \exists \rangle$ is a model of $\boldsymbol{L}_\Pi$ under the polyadic homomorphism μ defined by (8.9) and (8.10). Moreover, by (8.9) (since μ^* is onto R) each relation $f = \mu^* F$ of R satisfies (8.4), so that $\mathfrak{X}$ is of the form $\mathfrak{X}_{\mathfrak{B}}$ in the proof of (A). In this case, (8.8) holds, and (B) is thereby established. It remains to show:

(C) *The mapping defined in* (A) *is one-one.*

Proof: Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be models of $\boldsymbol{L}_\Pi$, under distinct interpretations μ_1 and μ_2, respectively, and let $\mathfrak{X}_1 = \mathfrak{X}_{\mathfrak{B}_1}$ and $\mathfrak{X}_2 = \mathfrak{X}_{\mathfrak{B}_2}$ be the associated models of $\mathsf{T_S}$. If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have distinct domains, (C) is evident. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have domain X. It is required to find some $f_1 = \mu_1^* F$ of $\mathfrak{X}_1$ distinct from $f_2 = \mu_2^* F$ of $\mathfrak{X}_2$. If the polyadic homomorphisms μ_1 and μ_2 agree on atomic formulas of $\boldsymbol{L}_\Pi$, then they agree on Q, which is constructed from atomic formulas by operations of the polyadic algebra $\boldsymbol{L}_\Pi$. In this case $\mathfrak{B}_1 = \mathfrak{B}_2$, against the hypothesis of (C). Then there must be an atomic formula

$p = Ft_1 \ldots t_n$ of $\mathbf{Q}$ such that $\mu_1 Ft_1 \ldots t_n \neq \mu_2 Ft_1 \ldots t_n$. If all terms of p are in $\mathbf{I}$, then by (8.4), $f_1 \neq f_2$, so that $\mathfrak{X}_1 \neq \mathfrak{X}_2$. If some term of p is in $\mathbf{K}$, then this case reduces to the previous one as follows.

We consider the case in which one term of p is in $\mathbf{K}$; the general case follows by induction. As shown in the proof of (8.7), $\mu_1 Ft_1 \ldots t_n(x) = \mu_1 Fi_1 \ldots i_n(x^{\mathbf{H}})$, for all $x \in \mathbf{X}^{\mathbf{I}}$, for some $\mathbf{J} = \{i_1, \ldots, i_n\} \subset \mathbf{I}$, and for some $\mathbf{H} \subset \mathbf{J}$; similarly for μ_2. By hypothesis $\mu_1 Ft_1 \ldots t_n(x) \neq \mu_2 Ft_1 \ldots t_n(x)$, for some $x \in \mathbf{X}^{\mathbf{I}}$; then for such x, $\mu_1 Fi_1 \ldots i_n(x^{\mathbf{H}}) \neq \mu_2 Fi \ldots i_n(x^{\mathbf{H}})$, which is the case already considered. (C) is thereby established. (8.1) follows from (A), (B), and (C).

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