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# AN ECONOMY IN THE FORMATION RULES FOR QUANTIFICATION THEORY

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Church has devised[1] an elegant system of notation which appears to be so flexible (perhaps with minor extension[2]) that it may be used for the expression of any logistic system. The value of Church's approach lies in the uniformity, simplicity and transparency with which one may lay down the formation rules of a logistic system, and it has been widely used for this purpose. In Church's paper referred to, the system of notation was used in the formulation of a logistic system called by him "The simple theory of types": in this note, however, I am concerned with Church's notational system, not his logistic system, and in particular with the application of the notation to the formulation of quantification theory, and related systems. The purpose of this note is to point out that a simplification of the notation is possible in these applications, namely the "improper symbols" may be omitted. This represents an economy in the primitive machinery that must be presupposed in the formulation of such systems, and which must be allowed for in their analysis.

Church's notation is based upon a three-fold distinction amongst symbols, namely *constants*, *variables* and *improper*[3] *symbols* (the first two classes together comprise the *proper*[3] *symbols*). In practice the improper symbols are solely $\lambda$ and brackets: the brackets serve to mark the scope of each occurrence of $\lambda$. Church also used brackets to enclose all well-formed concatenates, but this is unnecessary, and I will here consider brackets to be used only in connection with $\lambda$. Each proper symbol is assigned a *type*. It is then very simple to lay down formation rules.

The improper symbols stand apart from the proper symbols in two respects:

i) improper symbols lack type, and the rules of concatenation where improper symbols are involved are different from those governing the concatenation of proper symbols;
ii) the concatenation rules for improper symbols require that improper symbols be used in conjunction with variables, in a specified manner.

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The utility of $\lambda$ is really seen in the context of "$\lambda$-conversion", and abstraction in general. When one turns to consider the operation of Church's notational system applied to quantification theory in particular,[4] one finds that the function of the special symbol $\lambda$ is really to collect in its single person all "impropriety" of the second kind; and to achieve this $\lambda$ has to be improper in the first respect also. The normal procedure for setting up quantification theory within Church's notational system is to have a symbol of type $(\circ(\alpha))$ for each type $\alpha$ which it is intended shall be quantified over; let the symbol $\pi^\alpha$ be used to express universal quantification over type $\alpha$. These are proper symbols. The notion "For all $\xi$, $\varphi$", where $\xi$ is a variable of type $\alpha$, is then represented in the system as $\pi^\alpha(\lambda\,\xi\varphi)$. There is no difficulty in avoiding "impropriety" in the first of the respects mentioned above: the symbol $\lambda$ is avoided if $\pi^\alpha$ is replaced by a symbol of type $((\circ\circ)\alpha)$, which is improper in the second only of the respects mentioned. The resultant notation is in effect the ordinary Polish system, but it does not fall within Church's framework, having many improper symbols. If "impropriety" in both respects is to be avoided, the systematic interpretability of the notation for quantification must depend only on the types of the symbols in use, and not at all on whether certain of them are variables.[5] This freedom may be achieved by expressing universal quantification by a symbol of type $((\circ\circ)\circ)$; let this be $\forall^\alpha$. The formula

$$\forall^\alpha\varphi\psi$$

is well-formed of type $\circ$ iff both $\varphi$ and $\psi$ are of type $\circ$: this follows from Church's formation rules. The formula is interpreted, whether $\varphi$ is a variable or not, as

$$(\xi_1)\,(\xi_2)\,\ldots\,(\xi_n)\cdot\varphi\supset\psi\quad,$$

where $\xi_1\ldots\xi_n$ is a complete list of all variables of type $\alpha$ which occur free in $\varphi$.[6] If no variable of type $\alpha$ appears free in $\varphi$, then $\forall^\alpha\varphi\psi$ is interpreted as $\varphi\supset\psi$. There is no occasion to make use of improper symbols at all. The existential quantifier may be taken as primitive, or defined for each type:

$$\exists^\alpha\varphi\psi\quad\text{for }N\forall^\alpha\varphi N\psi.$$

Thus $\exists^\alpha\varphi\psi$ is the translation into the present notation of $(\exists\,\xi_1)\ldots(\exists\,\xi_n)\cdot\varphi\ \&\ \psi$, where $\xi_1\ldots\xi_n$ is a complete list of the variables of type $\alpha$ which occur free in $\varphi$. Conversely, $(\xi)\varphi$ and $(\exists\,\xi)\varphi$ may be translated into the present notation as $\forall^\alpha\tau\varphi$ and $\exists^\alpha\tau\varphi$ respectively, where $\tau$ is any tautology in which $\xi$ occurs as only free variable of type $\alpha$.

The foregoing intended interpretations may be secured by appropriate axiomatisation. This may be formed by suitably rewriting any known axiomatisation according to the principles of translation just indicated; but in practice simplification is possible. I will show, as a sufficient illustration, an axiomatisation of $F^1$ following Łukasiewicz.[7] Add to a convenient formulation of Propositional Logic the following:

*Vocabulary:*

$\forall^{\iota}$-type ((oo)o).

Variables and constants of type $\iota$, $o\iota$, $o\iota\iota$ etc. *ad lib*. These symbols are proper.

*Formation rules*: According to Church's notational system.

*Special metatheoretic notions*: Assuming the notion of "occurrence" is well defined for elementary expressions, and for expressions involving only propositional connectives, the following extends the definition:

$\xi$ (a variable of type $\iota$) *occurs freely* in $\forall^{\iota}\varphi\psi$ iff $\xi$ occurs freely in $\psi$ and does not occur freely in $\varphi$.

The principal clause of a definition of the relation between variables "is substitutable for", intended as the converse of the relation "is free for", may be given:

$\eta$ *is substitutable for* $\xi$ (variables of type $\iota$) in $\forall^{\iota}\varphi\psi$ iff $\eta$ is substitutable for $\xi$ in $\psi$ and either $\xi$ does not occur freely in $\psi$ or $\eta$ does not occur freely in $\varphi$.

*Rules of inference for quantification:*

From $CC\varphi\psi\chi$ to infer $C\forall^{\iota}\varphi\psi\chi$.

From $C\varphi C\psi\chi$ to infer $C\varphi\forall^{\iota}\psi\chi$, where no $\iota$-variable occurs freely in both $\varphi$ and $\psi$.

From $\varphi$ to infer the result of substituting $\nu$ for all free occurrences of $\xi$ in $\varphi$: where $\xi$ is an $\iota$-variable; and $\nu$ is either an $\iota$-constant, or an $\iota$-variable which is substitutable for $\xi$ in $\varphi$.

The first point of the notation presented is the absence of improper symbols, an economy in the basic machinery which must be presupposed in a discussion of quantification theory. But there are two further features. The notation has a practical advantage which is in a sense converse to an advantage of the normal notation: normally one states explicitly the variables which are bound, by specifying each separately; in the present notation, on the other hand, one handles classes of variables. Thus $\forall^{\iota}\varphi\psi$ binds just those variables which are free in $\varphi$, whatever they may be, leaving free any other variables which may occur in $\psi$. Hence, for example, the notation offers the facility of expressing the closure of a wff without the introduction of special notation or extraneous verbiage. The universal $\iota$-closure of $\varphi$ may be expressed $\forall^{\iota}N\varphi\varphi$, the existential $\iota$-closure being $\exists^{\iota}\varphi\varphi$. Secondly, the two-part structure of the basic quantification expresses the subject-predicate analysis of natural language and classical formal logic. The expressions $\forall^{\iota}\varphi\psi$ and $\exists^{\iota}\varphi\psi$ may be read "All men are mortal" and "Some men are mortal" respectively, $\varphi$ representing the subject and $\psi$ the predicate. The forms in natural language correspond, and so do the forms in this notation. Associated with these two points, however, are two limitations. An expression substituted into the "subject" position of a quantified statement must contain exactly the same free variables; and $\forall`\varphi\psi$ is not equivalent to $\forall`N\psi N\varphi$ unless the same free variables occur free in $\varphi$ and $\psi$.

## NOTES

1. Alonzo Church, "A formulation of the simple theory of types," *The Journal of Symbolic Logic*, vol. 5 (1940), pp. 56-68.
2. See for example, J. G. Kemeny, "Models of Logical Systems," *The Journal of Symbolic Logic*, vol. 13 (1948), p. 18.
3. In this note the words 'proper' and 'improper' are to be understood solely in relation to this system of notation of Church. They carry no evaluative overtones.
4. I discuss quantification, but my remarks apply to other operators, as least number, description, etc.
5. In addition, of course, reference to variables and their binding is necessary for the specification of the axioms and/or rules of inference of quantification theory.
6. Where $\alpha$ is $o$, $\varphi$ may be simply a propositional variable.
7. Prior, A. N., "Formal Logic": Oxford (U.P.) 1955, 1962, p. 304. Substitution is added as a third rule to show its formulation in this notation.

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