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EXAMINATION OF THE AXIOMATIC FOUNDATIONS OF A THEORY OF CHANGE. III

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THIRD PART*

§§1-3

§1. Description of the deductive system. In the contemporary scientific literature we can find many variations of the first-order predicate calculus. It is then necessary to select one of these which shall serve to derive our propositions from our axioms. We shall use the first-order predicate calculus by H. Hermes in his "Einführung in die mathematische Logik" [13]. This calculus has the form of a calculus from assumptions. Hermes describes a proof in his calculus as follows: "Ein Beweis im Sinne eines Annahmenkalküls besteht . . . aus einer endlichen Folge von Zeilen, wobei jede Zeile aus endlich vielen Aussagen besteht, von denen die jeweils letzte als die Behauptung und die vorangehenden als die Annahmen dieser Zeile angesehen man von Ausgangszeilen (den Prämissen) übergehen kann zu einer weiteren Zeile (der Conclusio)", [13], pp. 34-35.

In the following presentation of the system of deductive rules, we shall use the symbols "ρ", "σ", "τ", and "a", "b", "c", and "Q", "R" as variables respectively for expressions, and individuals (momentaneous subjects or properties), and predicates.

(**A**) Rule of self-affirmation

$\rho\rho$

(Rule without premisses) The rule allows to write the sequence $\rho\rho$ for any expression ρ.

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(K_0) Rule for the introduction of the conjunctor

$$\frac{\begin{array}{r}\ldots\tau\\ ---\sigma\end{array}}{\ldots --- (\tau\wedge\sigma)}$$

Remark. The (eventually empty) beginning of sequences (or parts of their beginning) will be reproduced symbolically by . . . respectively by - - -. The order in which the beginnings of sequences forming the premisses of a rule appear in the conclusion, is arbitrary. Furthermore, it is allowed in the beginning of a sequence to repeat any number of times the expressions which already appear in it.

(K_1) First rule for the elimination of the conjunctor

$$\frac{\ldots(\tau\wedge\sigma)}{\ldots\tau}$$

(K_2) Second rule for the elimination of the conjunctor

$$\frac{\ldots(\tau\wedge\sigma)}{\ldots\sigma}$$

(E) Rule of exhaustion

$$\frac{\begin{array}{r}\ldots\tau\sigma\\ ---\neg\tau\sigma\end{array}}{\ldots ---\sigma}$$

(W) Rule of contradiction

$$\frac{\begin{array}{r}\ldots\tau\\ ---\neg\tau\end{array}}{\ldots ---\sigma}$$

(G) Rule for the elimination of the universal quantifier

$$\frac{\ldots\forall a\tau}{\ldots\tau}$$

(G_a) Rule for the introduction of the universal quantifier

$$\frac{\ldots\tau}{\ldots\forall a\tau},$$

provided a is not a free variable in . . . .

(S_b^a) Rule of substitution

$$\frac{\ldots\tau}{\ldots\sigma},$$

provided **Subst** . . . τba - - - σ.

(**Subst** . . . τba - - - σ shall mean that σ is the expression obtained from τ by substituting a for b provided b is free for a in τ, and that every part of . . . corresponds to its respective part of - - - through substitution of a for b.)

(**I**) First rule for identity $a = a$

($\mathbf{I}_b^a$) Second rule for identity

$$\frac{- - - \tau}{- - - b = a\ \sigma},$$

provided **Subst** $\tau ba\ \sigma$.

($\mathbf{A}_0$) Rule for the pre-introduction of the disjunctor

$$\frac{\begin{array}{r} . . .\ \sigma\rho \\ - - -\ \tau\rho \end{array}}{. . .\ - - -\ (\sigma \vee \tau)\rho}$$

($\mathbf{A}_1$) First rule of weakening of the disjunctor

$$\frac{- - -\ \sigma}{- - -\ (\sigma \vee \tau)}$$

($\mathbf{A}_2$) Second rule of weakening of the disjunctor

$$\frac{- - -\ \sigma}{- - -\ (\tau \vee \sigma)}$$

($\mathbf{I}_0$) Rule for the pre-introduction of the implicator

$$\frac{\begin{array}{r} . . .\ \neg\sigma\rho \\ - - -\ \tau\rho \end{array}}{. . .\ - - -\ (\sigma \rightarrow \tau)\rho}$$

($\mathbf{I}_1$) Rule for the introduction of the implicator

$$\frac{- - -\ \sigma\tau}{- - -\ (\sigma \rightarrow \tau)}$$

($\mathbf{P}_a'$) Rule for the introduction of the existential quantifier

$$\frac{- - -\ \sigma}{- - -\ \exists a\ \sigma}$$

($\mathbf{P}_a''$) Rule for the pre-introduction of the existential quantifier

$$\frac{- - -\ \sigma\tau}{- - -\ \exists a\sigma\tau},$$

provided a is neither free in τ nor in - - -.

In order to lighten the proofs we want to add here the following derived rules of inference.

(**SW**) Rule of self-refutation

$$\frac{- - -\ \sigma\neg\sigma}{- - -\ \neg\sigma}$$

(KS) Chain rule of the consequent

$$\frac{\begin{array}{r}\ldots\sigma\\ ---\sigma\tau\end{array}}{\ldots ---\tau}$$

(KP_1) First rule of contraposition

$$\frac{---\sigma\tau}{---\neg\tau\neg\sigma}$$

(EA) Rule for the introduction of an additional assumption

$\sigma\tau\sigma$

(ZA) Rule for the elimination of the conjunctor between parts of the ant cedent

$$\frac{---(\sigma\wedge\tau)\rho}{---\sigma\tau\rho}$$

(VA) Rule for the introduction of the conjunctor between parts of t antecedent

$$\frac{---\sigma\tau\rho}{---(\sigma\wedge\tau)\rho}$$

(SR) Rule of symmetry for identity

$a=b \quad b=a$

(TR) Rule of transitivity for identity

$a=b \quad b=c \quad a=c$

(ER) First rule of substitution for identity

$a_1=a_1' \ldots a_k=a_k' \quad Pa_1\ldots a_k \quad Pa_1'\ldots a_k'$

(BI) Rule for the elimination of the implicator

$$\frac{---(\sigma\rightarrow\tau)}{---\sigma\tau}$$

(MP) Rule of Modus Ponens

$$\frac{\begin{array}{r}\ldots\sigma\\ ---(\sigma\rightarrow\tau)\end{array}}{\ldots ---\tau}$$

§2. List of the primitive notions and of the definitions together with t axioms and propositions of the formalization.

Pn3.1. $x\sim y$: x and y are simultaneous.
A3.1.1 $x\sim x$
A3.1.2 $x\sim y\rightarrow y\sim x$
A3.1.3 $x\sim y\wedge y\sim z\rightarrow x\sim z$

Pn3.2. $x < y : x$ and y are genidentical and x is earlier in time than y.

A3.2.1 $\neg x < x$

A3.2.2 $x < y \rightarrow \neg y < x$

A3.2.3 $x < y \wedge y < z \rightarrow x < z$

A3.2.4 $x < z \rightarrow \exists y(x < y < z)$

A3.3 $x < y \rightarrow \neg x \sim y$

A3.4 $x_0 < x_1 < x_2 \wedge y_0 < y_2 \wedge x_0 \sim y_0 \wedge x_2 \sim y_2 \rightarrow \exists y_1(y_0 < y_1 < y_2 \wedge x_1 \sim y_1)$

D3.1 $x \leq y =_{Df} x < y \vee x = y$

D3.2 $\mathbf{G}xy =_{Df} x \leq y \vee y < x$

A3.5 $x_1 < y \wedge x_2 < y \rightarrow \mathbf{G}x_1x_2$

A3.6 $x < y_1 \wedge x < y_2 \rightarrow \mathbf{G}y_1y_2$

S3.1 $x < y \rightarrow \mathbf{G}xy$

S3.2.1 $\mathbf{G}xx$

S3.2.2 $\mathbf{G}xy \rightarrow \mathbf{G}yx$

S3.2.3 $\mathbf{G}xy \wedge \mathbf{G}yz \rightarrow \mathbf{G}xz$

S3.3 $\mathbf{G}xy \wedge x \sim y \rightarrow x = y$

S3.4 $x \sim y \wedge \neg x = y \rightarrow \neg \mathbf{G}xy$

S3.5 $\neg \mathbf{G}xy \wedge \mathbf{G}yz \rightarrow \neg \mathbf{G}xz$

S3.6 $x_1 < y \wedge x_2 < y \wedge x_1 \sim x_2 \rightarrow x_1 = x_2$

S3.7 $x < y_1 \wedge x < y_2 \wedge y_1 \sim y_2 \rightarrow y_1 = y_2$

Pn4.1. $\mathbf{A}x\alpha : x$ is actual with regard to α.

A4.1 $\exists \alpha \mathbf{A}x\alpha$

Pn5.1. $\mathbf{F}x\alpha : x$ is capable of α.

A5.1 $\mathbf{A}x\alpha \rightarrow \mathbf{F}x\alpha$

A5.2 $x_1 < x_2 \wedge \mathbf{F}x_2\alpha \rightarrow \mathbf{F}x_1\alpha$

D5.1 $\mathbf{P}x\alpha =_{Df} \mathbf{F}x\alpha \wedge \neg \mathbf{A}x\alpha$

S5.1 $\mathbf{G}x_1x_2 \wedge \mathbf{A}x_1\alpha \wedge \mathbf{P}x_2\alpha \rightarrow \neg x_1 \sim x_2$

S5.2 $\mathbf{A}x\alpha \wedge \mathbf{P}x\beta \rightarrow \neg \alpha = \beta$

S5.3 $\mathbf{P}x\alpha \rightarrow \exists \beta(\mathbf{A}x\beta \wedge \neg \alpha = \beta)$

S5.4 $x_1 < x_2 \wedge \neg \mathbf{A}x_1\alpha \wedge \mathbf{A}x_2\alpha \rightarrow \mathbf{P}x_1\alpha$

D5.2 $\mathbf{V}y_1y_2\alpha =_{Df} y_1 < y_2 \wedge \mathbf{A}y_2\alpha \wedge \forall y(y_1 \leq y < y_2 \rightarrow \neg \mathbf{A}y\alpha)$

S5.5 $\mathbf{V}y_1y_2\alpha \rightarrow \mathbf{G}y_1y_2$

S5.6 $\mathbf{V}y_0y_2\alpha \wedge \mathbf{V}y_1y_2\alpha \rightarrow \mathbf{G}y_0y_1$

S5.7 $\mathbf{V}y_0y_2\alpha \wedge y_0 < y_1 < y_2 \rightarrow \mathbf{V}y_1y_2\alpha$

S5.8 $\mathbf{V}y_1y_2\alpha \rightarrow \mathbf{P}y_1\alpha$

Pn6.1. $\mathbf{M}xy\alpha : x$ changes y to α.

A6.1 $\mathbf{M}xy\alpha \rightarrow \exists x_1(x_1 \sim y \wedge x < x_1)$

A6.2 $\mathbf{M}xy\alpha \rightarrow \exists y_0(x \sim y_0 \wedge y_0 < y)$

S6.1 $\mathbf{M}xy\alpha \rightarrow \neg x \sim y$

S6.2 $\mathbf{M}xy\alpha \wedge y_0 < y_1 < y \wedge y_0 \sim x \rightarrow \exists x_1(x_1 \sim y_1 \wedge x < x_1)$

S6.3 $\mathbf{M}xy\alpha \wedge x < x_1 < x_2 \wedge x_2 \sim y \rightarrow \exists y_1(x_1 \sim y_1 \wedge y_1 < y)$

A6.3 $\mathbf{M}xy\alpha \rightarrow \mathbf{A}y\alpha$

A6.4 $\mathbf{M}xy\alpha \wedge x \sim y_0 \leq y_1 < y \rightarrow \neg \mathbf{A}y_1\alpha$

A6.5 $\mathbf{M}x_1y\alpha \wedge \mathbf{M}x_2y\alpha \rightarrow \mathbf{G}x_1x_2$

S6.4 $\mathbf{M}xy\alpha \rightarrow \exists y_0(x \sim y_0 \wedge \mathbf{V}y_0y\alpha)$

S6.5 $\mathbf{M}xy_2\alpha \wedge x \sim y_1 < y_2 \rightarrow \mathbf{V}y_1y_2\alpha$

S6.6	$\mathbf{M}xy\alpha \wedge \mathbf{A}x\alpha \rightarrow \neg\mathbf{G}xy$
A6.6	$\mathbf{M}xy\alpha \wedge x < x_1 \sim y_1 < y \rightarrow \mathbf{M}x_1y\alpha$
S6.7	$\mathbf{M}xy_2\alpha \wedge x < x_1 \sim y_1 < y_2 \rightarrow \mathbf{V}y_1y_2\alpha$
A6.7	$\mathbf{V}y_1y_2\alpha \rightarrow \exists x \exists y\ (x \sim y \wedge y_1 \leq y < y_2 \wedge \mathbf{M}xy_2\alpha)$
S6.8	$\mathbf{V}y_0y_2\alpha \leftrightarrow \exists x_1 \exists y_1\ (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M}x_1y_2\alpha \wedge \forall y(y_0 \leq y < y_2 \rightarrow \neg\mathbf{A}y\alpha))$
Pn7.1.	$\mathbf{B}xy\alpha$: x has at most as large a share in α as y.
A7.1	$\mathbf{B}xx\alpha$
A7.2	$\mathbf{B}xy\alpha \wedge \mathbf{B}yz\alpha \rightarrow \mathbf{B}xz\alpha$
D7.1	$\mathbf{W}xy\alpha =_{Df} \mathbf{B}xy\alpha \wedge \neg\mathbf{B}yx\alpha$
D7.2	$\mathbf{I}xy\alpha =_{Df} \mathbf{B}xy\alpha \wedge \mathbf{B}yx\alpha$
S7.1	$\neg\mathbf{W}xx\alpha$
S7.2	$\mathbf{W}xy\alpha \rightarrow \neg\mathbf{W}yx\alpha$
S7.3	$\mathbf{W}xy\alpha \wedge \mathbf{W}yz\alpha \rightarrow \mathbf{W}xz\alpha$
A7.3	$\mathbf{P}x\alpha \wedge \mathbf{A}y\alpha \rightarrow \mathbf{W}xy\alpha$
A7.4	$\mathbf{A}x\alpha \wedge \mathbf{A}y\alpha \rightarrow \mathbf{I}xy\alpha$
A7.5	$\mathbf{M}xy\alpha \wedge \mathbf{A}z\alpha \rightarrow \mathbf{B}zx\alpha$
S7.4	$\mathbf{M}xy\alpha \wedge \mathbf{P}z\alpha \rightarrow \mathbf{W}zx\alpha$
S7.5	$\mathbf{M}xy\alpha \rightarrow \neg\mathbf{P}x\alpha$
S7.6	$\mathbf{M}xy\alpha \rightarrow \neg\mathbf{G}xy$
S7.7	$\mathbf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1 \rightarrow \neg\mathbf{G}xy$
S7.8	$\mathbf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1 \rightarrow \mathbf{W}yx\alpha$
S7.9	$\mathbf{V}y_1y_2\alpha \rightarrow \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathbf{M}xy_2\alpha \wedge \neg\mathbf{G}xy_2)$

§3. Proofs of the propositions. We want to point out that in writing the proofs we shall 1) treat the axioms, the definitions and the propositions as if they were schemata, 2) use the definitions as follows:

$$\frac{---\ \mathrm{D}_1}{---\ \mathrm{D}_2} \quad \text{or} \quad \frac{---\ \mathrm{D}_2}{---\ \mathrm{D}_1}$$

where $\mathrm{D}_1 =_{Df} \mathrm{D}_2$.

S3.1 $\quad x < y \rightarrow \mathbf{G}xy$

Proof

1.	$x < y$		$x < y$	$\langle \mathbf{A} \rangle$
2.	$x < y$		$x < y \vee x = y$	$\langle \mathbf{A}_1, 1 \rangle$
3.	$x < y$		$x \leq y$	$\langle$D3.1, 2$\rangle$
4.	$x < y$		$x \leq y \vee y < x$	$\langle \mathbf{A}_1, 3 \rangle$
5.	$x < y$		$\mathbf{G}xy$	$\langle$D3.2, 4$\rangle$
6.		$x < y \rightarrow \mathbf{G}xy$		$\langle \mathbf{I}_1, 5 \rangle$

S3.2.1 $\quad \mathbf{G}xx$

Proof

1.	$x = x$	$\langle \mathbf{I} \rangle$
2.	$x < x \vee x = x$	$\langle \mathbf{A}_2, 1 \rangle$
3.	$x \leq x$	$\langle$D3.1, 2$\rangle$

4.		$x \leq x \vee x < x$	$\langle \mathbf{A}_1, 3\rangle$
5.		$\mathbf{G}xx$	$\langle \text{D3.2}, 4\rangle$

S3.2.2 $\mathbf{G}xy \to \mathbf{G}yx$

Proof

1.	$\mathbf{G}xy$	$\mathbf{G}xy$	$\langle \mathbf{A}\rangle$
2.	$\mathbf{G}xy$	$x \leq y \vee y < x$	$\langle \text{D3.2}, 1\rangle$
3.	$x < y$	$x < y$	$\langle \mathbf{A}\rangle$
4.	$x < y$	$y \leq x \vee x < y$	$\langle \mathbf{A}_2, 3\rangle$
5.	$x = y$	$y = x$	$\langle \mathbf{SR}\rangle$
6.	$x = y$	$y < x \vee y = x$	$\langle \mathbf{A}_2, 5\rangle$
7.	$x = y$	$y \leq x$	$\langle \text{D3.1}, 6\rangle$
8.	$x = y$	$y \leq x \vee x < y$	$\langle \mathbf{A}_1, 7\rangle$
9.	$x < y \vee x = y$	$y \leq x \vee x < y$	$\langle \mathbf{A}_0, 4, 8\rangle$
10.	$x \leq y$	$x \leq y$	$\langle \mathbf{A}\rangle$
11.	$x \leq y$	$x < y \vee x = y$	$\langle \text{D3.1}, 10\rangle$
12.	$x \leq y$	$y \leq x \vee x < y$	$\langle \mathbf{KS}, 11, 9\rangle$
13.	$y < x$	$y < x$	$\langle \mathbf{A}\rangle$
14.	$y < x$	$y < x \vee y = x$	$\langle \mathbf{A}_1, 13\rangle$
15.	$y < x$	$y \leq x$	$\langle \text{D3.1}, 14\rangle$
16.	$y < x$	$y \leq x \vee x < y$	$\langle \mathbf{A}_1, 15\rangle$
17.	$x \leq y \vee y < x$	$y \leq x \vee x < y$	$\langle \mathbf{A}_0, 12, 16\rangle$
18.	$\mathbf{G}xy$	$y \leq x \vee x < y$	$\langle \mathbf{KS}, 2, 17\rangle$
19.	$\mathbf{G}xy$	$\mathbf{G}yx$	$\langle \text{D3.2}, 18\rangle$
20.		$\mathbf{G}xy \to \mathbf{G}yx$	$\langle \mathbf{I}_1, 19\rangle$

S3.2.3 $\mathbf{G}xy \wedge \mathbf{G}yx \to \mathbf{G}xz$

Proof

1.	$x < y$		$x < y$	$\langle \mathbf{A}\rangle$
2.	$y < z$		$y < z$	$\langle \mathbf{A}\rangle$
3.	$x < y$	$y < z$	$x < y \wedge y < z$	$\langle \mathbf{K}_0, 1, 2\rangle$
4.			$x < y \wedge y < z \to x < z$	$\langle \text{A3.2.3}\rangle$
5.	$x < y$	$y < z$	$x < z$	$\langle \mathbf{MP}, 3, 4\rangle$
6.	$y = z$	$x < y$	$x < z$	$\langle \mathbf{ER}_1\rangle$
7.	$x < y$	$y < z \vee y = z$	$x < z$	$\langle \mathbf{A}_0, 5, 6\rangle$
8.			$x < z \to \mathbf{G}xz$	$\langle \text{S3.1}\rangle$
9.	$x < y$	$y < z \vee y = z$	$\mathbf{G}xz$	$\langle \mathbf{MP}, 7, 8\rangle$
10.	$z < y$		$z < y$	$\langle \mathbf{A}\rangle$
11.	$x < y$	$z < y$	$x < y \wedge z < y$	$\langle \mathbf{K}_0, 1, 10\rangle$
12.			$x < y \wedge z < y \to \mathbf{G}xz$	$\langle \text{A3.5}\rangle$
13.	$x < y$	$z < y$	$\mathbf{G}xz$	$\langle \mathbf{MP}, 11, 12\rangle$
14.	$x < y$	$y < z \vee y = z \vee z < y$	$\mathbf{G}xz$	$\langle \mathbf{A}_0, 9, 13\rangle$
15.	$x = y$	$y < z$	$x < z$	$\langle \mathbf{ER}_1\rangle$
16.	$x = y$	$y < z$	$\mathbf{G}xz$	$\langle \mathbf{MP}, 15, 8\rangle$
17.	$x = y$	$y = z$	$x = z$	$\langle \mathbf{TR}\rangle$

18.	$x = y \quad y = z$	$x < z \vee x = z$	$\langle \mathsf{A}_2, 17\rangle$
19.	$x = y \quad y = z$	$x \leq z$	$\langle \mathrm{D}3.1, 18\rangle$
20.	$x = y \quad y = z$	$x \leq z \vee z < x$	$\langle \mathsf{A}_1, 19\rangle$
21.	$x = y \quad y = z$	$\mathbf{G}xz$	$\langle \mathrm{D}3.2, 20\rangle$
22.	$x = y \quad z < y$	$z < x$	$\langle \mathsf{ER}_1\rangle$
23.		$z < x \rightarrow \mathbf{G}zx$	$\langle \mathrm{S}3.1\rangle$
24.	$x = y \quad z < y$	$\mathbf{G}zx$	$\langle \mathsf{MP}, 22, 23\rangle$
25.		$\mathbf{G}zx \rightarrow \mathbf{G}xz$	$\langle \mathrm{S}3.2.2\rangle$
26.	$x = y \quad z < y$	$\mathbf{G}xz$	$\langle \mathsf{MP}, 24, 25\rangle$
27.	$x = y \quad y < z \vee y = z$	$\mathbf{G}xz$	$\langle \mathsf{A}_0, 16, 21\rangle$
28.	$x = y \quad y < z \vee y = z \vee z < y$	$\mathbf{G}xz$	$\langle \mathsf{A}_0, 27, 26\rangle$
29.	$y < x$	$y < x$	$\langle \mathsf{A}\rangle$
30.	$y < x \quad y < z$	$y < x \wedge y < z$	$\langle \mathsf{K}_0, 29, 2\rangle$
31.		$y < x \wedge y < z \rightarrow \mathbf{G}xz$	$\langle \mathrm{A}3.6\rangle$
32.	$y < x \quad y < z$	$\mathbf{G}xz$	$\langle \mathsf{MP}, 30, 31\rangle$
33.	$y = z \quad y < x$	$z < x$	$\langle \mathsf{ER}_1\rangle$
34.	$y < x \quad z < y$	$z < y \wedge y < x$	$\langle \mathsf{K}_0, 29, 10\rangle$
35.		$z < y \wedge y < x \rightarrow z < x$	$\langle \mathrm{A}3.2.3\rangle$
36.	$y < x \quad z < y$	$z < x$	$\langle \mathsf{MP}, 34, 35\rangle$
37.	$y < x \quad y = z \vee z < y$	$z < x$	$\langle \mathsf{A}_0, 33, 36\rangle$
38.	$y < x \quad y = z \vee z < y$	$\mathbf{G}zx$	$\langle \mathsf{MP}, 37, 23\rangle$
39.	$y < x \quad y = z \vee z < y$	$\mathbf{G}xz$	$\langle \mathsf{MP}, 38, 25\rangle$
40.	$y < x \quad y < z \vee y = z \vee z < y$	$\mathbf{G}xz$	$\langle \mathsf{A}_0, 32, 39\rangle$
41.	$x < y \vee x = y \quad y < z \vee y = z \vee z < y$	$\mathbf{G}xz$	$\langle \mathsf{A}_0, 14, 28\rangle$
42.	$x < y \vee x = y \vee y < x \quad y < z \vee y = z \vee z < y$	$\mathbf{G}xz$	$\langle \mathsf{A}_0, 41, 40\rangle$
43.	$\mathbf{G}xy$	$\mathbf{G}xy$	$\langle \mathsf{A}\rangle$
44.	$\mathbf{G}xy$	$x \leq y \vee y < x$	$\langle \mathrm{D}3.2, 43\rangle$
45.	$x \leq y$	$x \leq y$	$\langle \mathsf{A}\rangle$
46.	$x \leq y$	$x < y \vee x = y$	$\langle \mathrm{D}3.1, 45\rangle$
47.	$x \leq y$	$x < y \vee x = y \vee y < x$	$\langle \mathsf{A}_1, 46\rangle$
48.	$y < x$	$y < x$	$\langle \mathsf{A}\rangle$
49.	$y < x$	$x < y \vee x = y \vee y < x$	$\langle \mathsf{A}_2, 48\rangle$
50.	$x \leq y \vee y < x$	$x < y \vee x = y \vee y < x$	$\langle \mathsf{A}_0, 47, 49\rangle$
51.	$\mathbf{G}xy$	$x < y \vee x = y \vee y < x$	$\langle \mathsf{KS}, 44, 50\rangle$
52.	$\mathbf{G}xy \quad y < z \vee y = z \vee z < y$	$\mathbf{G}xz$	$\langle \mathsf{KS}, 51, 42\rangle$
53.	$\mathbf{G}yz$	$\mathbf{G}yz$	$\langle \mathsf{A}\rangle$
54.	$\mathbf{G}yz$	$y \leq z \vee z < y$	$\langle \mathrm{D}3.2, 53\rangle$
55.	$y \leq z$	$y \leq z$	$\langle \mathsf{A}\rangle$
56.	$y \leq z$	$y < z \vee y = z$	$\langle \mathrm{D}3.1, 55\rangle$
57.	$y \leq z$	$y < z \vee y = z \vee z < y$	$\langle \mathsf{A}_1, 56\rangle$

58.	$z < y$	$z < y$	⟨**A**⟩
59.	$z < y$	$y < z \vee y = z \vee z < y$	⟨$\mathbf{A}_2$, 58⟩
60.	$y \leq z \vee z < y$	$y < z \vee y = z \vee z < y$	⟨$\mathbf{A}_0$, 57, 59⟩
61.	$\mathbf{G}yz$	$y < z \vee y = z \vee z < y$	⟨**KS**, 54, 60⟩
62.	$\mathbf{G}xy \quad \mathbf{G}yz$	$\mathbf{G}xz$	⟨**KS**, 52, 61⟩
63.	$\mathbf{G}xy \wedge \mathbf{G}yz$	$\mathbf{G}xz$	⟨**VA**, 62⟩
64.	$\mathbf{G}xy \wedge \mathbf{G}yz \to \mathbf{G}xz$		⟨$\mathbf{I}_1$, 63⟩

S3.3 $\quad \mathbf{G}xy \wedge x \sim y \to x = y$

Proof

1.	$x < y \to \neg x \sim y$		⟨A3.3⟩
2.	$x < y$	$\neg x \sim y$	⟨**BI**, 1⟩
3.	$x \sim y$	$x \sim y$	⟨**A**⟩
4.	$x < y \quad x \sim y$	$x = y$	⟨**W**, 2, 3⟩
5.	$y < x \to \neg y \sim x$		⟨A3.3⟩
6.	$y < x$	$\neg y \sim x$	⟨**BI**, 5⟩
7.	$x \sim y \to y \sim x$		⟨A3.1.2⟩
8.	$x \sim y$	$y \sim x$	⟨**BI**, 7⟩
9.	$y < x \quad x \sim y$	$x = y$	⟨**W**, 6, 8⟩
10.	$x = y$	$x = y$	⟨**A**⟩
11.	$x = y \quad x \sim y$	$x = y$	⟨**EA**, 10⟩
12.	$x < y \vee x = y \quad x \sim y$	$x = y$	⟨$\mathbf{A}_0$, 4, 11⟩
13.	$x < y \vee x = y \vee y < x \quad x \sim y$	$x = y$	⟨$\mathbf{A}_0$, 12, 9⟩
14.	$\mathbf{G}xy$	$\mathbf{G}xy$	⟨**A**⟩
15.	$\mathbf{G}xy$	$x \leq y \vee y < x$	⟨D3.2, 14⟩
16.	$x \leq y$	$x \leq y$	⟨**A**⟩
17.	$x \leq y$	$x < y \vee x = y$	⟨D3.1, 16⟩
18.	$x \leq y$	$x < y \vee x = y \vee y < x$	⟨$\mathbf{A}_1$, 17⟩
19.	$y < x$	$y < x$	⟨**A**⟩
20.	$y < x$	$x < y \vee x = y \vee y < x$	⟨$\mathbf{A}_2$, 19⟩
21.	$x \leq y \vee y < x$	$x < y \vee x = y \vee y < x$	⟨$\mathbf{A}_0$, 18, 20⟩
22.	$\mathbf{G}xy$	$x < y \vee x = y \vee y < x$	⟨**KS**, 15, 21⟩
23.	$\mathbf{G}xy \quad x \sim y$	$x = y$	⟨**KS**, 22, 13⟩
24.	$\mathbf{G}xy \wedge x \sim y$	$x = y$	⟨**VA**, 23⟩
25.	$\mathbf{G}xy \wedge x \sim y \to x = y$		⟨$\mathbf{I}_1$, 24⟩

S3.4 $\quad x \sim y \wedge \neg x = y \to \neg \mathbf{G}xy$

Proof

1.	$\mathbf{G}xy \wedge x \sim y \to x = y$		⟨S3.3⟩
2.	$\mathbf{G}xy \wedge x \sim y$	$x = y$	⟨**BI**, 1⟩
3.	$x \sim y \quad \mathbf{G}xy$	$x = y$	⟨**ZA**, 2⟩
4.	$x \sim y \quad \neg x = y$	$\neg \mathbf{G}xy$	⟨$\mathbf{KP}_1$, 3⟩
5.	$x \sim y \wedge \neg x = y$	$\neg \mathbf{G}xy$	⟨**VA**, 4⟩
6.	$x \sim y \wedge \neg x = y \to \neg \mathbf{G}xy$		⟨$\mathbf{I}_1$, 5⟩

S3.5 $\neg \mathsf{G}xy \wedge \mathsf{G}yz \rightarrow \neg \mathsf{G}xz$

Proof

1.		$\mathsf{G}xz \rightarrow \mathsf{G}zx$	$\langle$S3.2.2$\rangle$
2.	$\mathsf{G}xz$	$\mathsf{G}zx$	$\langle \mathsf{BI}, 1\rangle$
3.		$\mathsf{G}yz \wedge \mathsf{G}zx \rightarrow \mathsf{G}yx$	$\langle$S3.2.3$\rangle$
4.	$\mathsf{G}yz \wedge \mathsf{G}zx$	$\mathsf{G}yx$	$\langle \mathsf{BI}, 3\rangle$
5.	$\mathsf{G}yz \quad \mathsf{G}zx$	$\mathsf{G}yx$	$\langle \mathsf{ZA}, 4\rangle$
6.		$\mathsf{G}yx \rightarrow \mathsf{G}xy$	$\langle$S3.2.2$\rangle$
7.	$\mathsf{G}yz \quad \mathsf{G}xz$	$\mathsf{G}yx$	$\langle \mathsf{KS}, 2, 5\rangle$
8.	$\mathsf{G}yz \quad \mathsf{G}xz$	$\mathsf{G}xy$	$\langle \mathsf{MP}, 7, 6\rangle$
9.	$\mathsf{G}yz \quad \neg \mathsf{G}xy$	$\neg \mathsf{G}xz$	$\langle \mathsf{KP}_1, 8\rangle$
10.	$\neg \mathsf{G}xy \wedge \mathsf{G}yz$	$\neg \mathsf{G}xz$	$\langle \mathsf{VA}, 9\rangle$
11.		$\neg \mathsf{G}xy \wedge \mathsf{G}yz \rightarrow \neg \mathsf{G}xz$	$\langle \mathsf{I}_1, 10\rangle$

S3.6 $x_1 < y \wedge x_2 < y \wedge x_1 \sim x_2 \rightarrow x_1 = x_2$

Proof

1.		$x_1 < y \wedge x_2 < y \rightarrow \mathsf{G}x_1x_2$	$\langle$A3.5$\rangle$
2.	$x_1 < y \wedge x_2 < y$	$\mathsf{G}x_1x_2$	$\langle \mathsf{BI}, 1\rangle$
3.	$x_1 < y \quad x_2 < y$	$\mathsf{G}x_1x_2$	$\langle \mathsf{ZA}, 2\rangle$
4.	$x_1 \sim x_2$	$x_1 \sim x_2$	$\langle \mathsf{A}\rangle$
5.	$x_1 < y \quad x_2 < y \quad x_1 \sim x_2$	$\mathsf{G}x_1x_2 \wedge x_1 \sim x_2$	$\langle \mathsf{K}_0, 3, 4\rangle$
6.		$\mathsf{G}x_1x_2 \wedge x_1 \sim x_2 \rightarrow x_1 = x_2$	$\langle$S3.3$\rangle$
7.	$x_1 < y \quad x_2 < y \quad x_1 \sim x_2$	$x_1 = x_2$	$\langle \mathsf{MP}, 5, 6\rangle$
8.	$x_1 < y \wedge x_2 < y \wedge x_1 \sim x_2$	$x_1 = x_2$	$\langle \mathsf{VA}, 7\rangle$
9.		$x_1 < y \wedge x_2 < y \wedge x_1 \sim x_2 \rightarrow x_1 = x_2$	$\langle \mathsf{I}_1, 8\rangle$

S3.7 $x < y_1 \wedge x < y_2 \wedge y_1 \sim y_2 \rightarrow y_1 = y_2$

Proof

1.		$x < y_1 \wedge x < y_2 \rightarrow \mathsf{G}y_1y_2$	$\langle$A3.6$\rangle$
2.	$x < y_1 \wedge x < y_2$	$\mathsf{G}y_1y_2$	$\langle \mathsf{BI}, 1\rangle$
3.	$x < y_1 \quad x < y_2$	$\mathsf{G}y_1y_2$	$\langle \mathsf{ZA}, 2\rangle$
4.	$y_1 \sim y_2$	$y_1 \sim y_2$	$\langle \mathsf{A}\rangle$
5.	$x < y_1 \quad x < y_2 \quad y_1 \sim y_2$	$\mathsf{G}y_1y_2 \wedge y_1 \sim y_2$	$\langle \mathsf{K}_0, 3, 4\rangle$
6.		$\mathsf{G}y_1y_2 \wedge y_1 \sim y_2 \rightarrow y_1 = y_2$	$\langle$S3.3$\rangle$
7.	$x < y_1 \quad x < y_2 \quad y_1 \sim y_2$	$y_1 = y_2$	$\langle \mathsf{MP}, 5, 6\rangle$
8.	$x < y_1 \wedge x < y_2 \wedge y_1 \sim y_2$	$y_1 = y_2$	$\langle \mathsf{VA}, 7\rangle$
9.		$x < y_1 \wedge x < y_2 \wedge y_1 \sim y_2 \rightarrow y_1 = y_2$	$\langle \mathsf{I}_1, 8\rangle$

S5.1 $\mathsf{G}x_1x_2 \wedge \mathsf{A}x_1\alpha \wedge \mathsf{P}x_2\alpha \rightarrow \neg x_1 \sim x_2$

Proof

1.		$\mathsf{G}x_1x_2 \wedge x_1 \sim x_2 \rightarrow x_1 = x_2$	$\langle$S3.3$\rangle$

2.	$\mathbf{G}x_1x_2 \wedge x_1 \sim x_2$	$x_1 = x_2$	$\langle$**BI**, 1$\rangle$
3.	$\mathbf{G}x_1x_2 \quad x_1 \sim x_2$	$x_1 = x_2$	$\langle$**ZA**, 2$\rangle$
4.	$x_1 = x_2 \quad \mathbf{A}x_1\alpha$	$\mathbf{A}x_2\alpha$	$\langle \mathbf{ER}_1 \rangle$
5.	$\mathbf{G}x_1x_2 \quad \mathbf{A}x_1\alpha \quad x_1 \sim x_2$	$\mathbf{A}x_2\alpha$	$\langle$**KS**, 3, 4$\rangle$
6.	$\mathbf{P}x_2\alpha$	$\mathbf{P}x_2\alpha$	$\langle \mathbf{A} \rangle$
7.	$\mathbf{P}x_2\alpha$	$\mathbf{F}x_2\alpha \wedge \neg \mathbf{A}x_2\alpha$	$\langle$D5.1, 6$\rangle$
8.	$\mathbf{P}x_2\alpha$	$\neg \mathbf{A}x_2\alpha$	$\langle \mathbf{K}_2$, 7$\rangle$
9.	$\mathbf{G}x_1x_2 \quad \mathbf{A}x_1\alpha \quad \mathbf{P}x_2\alpha \quad x_1 \sim x_2$	$\neg x_1 \sim x_2$	$\langle$**W**, 5, 8$\rangle$
10.	$\mathbf{G}x_1x_2 \quad \mathbf{A}x_1\alpha \quad \mathbf{P}x_2\alpha$	$\neg x_1 \sim x_2$	$\langle$**SW**, 9$\rangle$
11.	$\mathbf{G}x_1x_2 \wedge \mathbf{A}x_1\alpha \wedge \mathbf{P}x_2\alpha$	$\neg x_1 \sim x_2$	$\langle$**VA**, 10$\rangle$
12.		$\mathbf{G}x_1x_2 \wedge \mathbf{A}x_1\alpha \wedge \mathbf{P}x_2\alpha \rightarrow \neg x_1 \sim x_2$	$\langle \mathbf{I}_1$, 11$\rangle$

S5.2 $\qquad \mathbf{A}x\alpha \wedge \mathbf{P}x\beta \rightarrow \neg \alpha = \beta$

Proof

1.	$\mathbf{A}x\alpha$	$\mathbf{A}x\alpha$	$\langle \mathbf{A} \rangle$
2.	$\mathbf{P}x\beta$	$\mathbf{P}x\beta$	$\langle \mathbf{A} \rangle$
3.	$\mathbf{P}x\beta$	$\mathbf{F}x\beta \wedge \neg \mathbf{A}x\beta$	$\langle$D5.1, 2$\rangle$
4.	$\mathbf{P}x\beta$	$\neg \mathbf{A}x\beta$	$\langle \mathbf{K}_2$, 3$\rangle$
5.	$\alpha = \beta \quad \mathbf{A}x\alpha$	$\mathbf{A}x\beta$	$\langle \mathbf{ER}_1 \rangle$
6.	$\mathbf{A}x\alpha \quad \mathbf{P}x\beta \quad \alpha = \beta$	$\neg \alpha = \beta$	$\langle$**W**, 4, 5$\rangle$
7.	$\mathbf{A}x\alpha \quad \mathbf{P}x\beta$	$\neg \alpha = \beta$	$\langle$**SW**, 6$\rangle$
8.	$\mathbf{A}x\alpha \wedge \mathbf{P}x\beta$	$\neg \alpha = \beta$	$\langle$**VA**, 7$\rangle$
9.		$\mathbf{A}x\alpha \wedge \mathbf{P}x\beta \rightarrow \neg \alpha = \beta$	$\langle \mathbf{I}_1$, 8$\rangle$

S5.3 $\qquad \mathbf{P}x\alpha \rightarrow \exists\beta(\mathbf{A}x\beta \wedge \neg \alpha = \beta)$

Proof

1.		$\mathbf{A}x\beta \wedge \mathbf{P}x\alpha \rightarrow \neg \alpha = \beta$	$\langle$S5.2$\rangle$
2.	$\mathbf{A}x\beta \wedge \mathbf{P}x\alpha$	$\neg \alpha = \beta$	$\langle$**BI**, 1$\rangle$
3.	$\mathbf{A}x\beta \quad \mathbf{P}x\alpha$	$\neg \alpha = \beta$	$\langle$**ZA**, 2$\rangle$
4.	$\mathbf{A}x\beta$	$\mathbf{A}x\beta$	$\langle \mathbf{A} \rangle$
5.	$\mathbf{P}x\alpha \quad \mathbf{A}x\beta$	$\mathbf{A}x\beta \wedge \neg \alpha = \beta$	$\langle \mathbf{K}_0$, 4, 3$\rangle$
6.	$\mathbf{P}x\alpha \quad \mathbf{A}x\beta$	$\exists\beta(\mathbf{A}x\beta \wedge \neg \alpha = \beta)$	$\langle \mathbf{P}'_\beta$, 5$\rangle$
7.	$\mathbf{P}x\alpha \quad \exists\beta\, \mathbf{A}x\beta$	$\exists\beta(\mathbf{A}x\beta \wedge \neg \alpha = \beta)$	$\langle \mathbf{P}''_\beta$, 6$\rangle$
8.		$\exists\beta\, \mathbf{A}x\beta$	$\langle$A4.1$\rangle$
9.	$\mathbf{P}x\alpha$	$\exists\beta(\mathbf{A}x\beta \wedge \neg \alpha = \beta)$	$\langle$**KS**, 8, 7$\rangle$
10.		$\mathbf{P}x\alpha \rightarrow \exists\beta(\mathbf{A}x\beta \wedge \neg \alpha = \beta)$	$\langle \mathbf{I}_1$, 9$\rangle$

S5.4 $\qquad x_1 < x_2 \wedge \neg \mathbf{A}x_1\alpha \wedge \mathbf{A}x_2\alpha \rightarrow \mathbf{P}x_1\alpha$

Proof

1.	$x_1 < x_2$	$x_1 < x_2$	$\langle \mathbf{A} \rangle$
2.		$\mathbf{A}x_2\alpha \rightarrow \mathbf{F}x_2\alpha$	$\langle$A5.1$\rangle$
3.	$\mathbf{A}x_2\alpha$	$\mathbf{F}x_2\alpha$	$\langle$**BI**, 2$\rangle$
4.	$x_1 < x_2 \quad \mathbf{A}x_2\alpha$	$x_1 < x_2 \wedge \mathbf{F}x_2\alpha$	$\langle \mathbf{K}_0$, 1, 3$\rangle$
5.		$x_1 < x_2 \wedge \mathbf{F}x_2\alpha \rightarrow \mathbf{F}x_1\alpha$	$\langle$A5.2$\rangle$

6.	$x_1 < x_2 \quad \mathsf{A}\, x_2\alpha$	$\mathsf{F}\, x_1\alpha$	$\langle \mathsf{MP}, 4, 5\rangle$
7.	$\neg \mathsf{A}\, x_1\alpha$	$\neg \mathsf{A}\, x_1\alpha$	$\langle \mathsf{A}\rangle$
8.	$x_1 < x_2 \quad \neg \mathsf{A}\, x_1\alpha \quad \mathsf{A}\, x_2\alpha$	$\mathsf{F}\, x_1\alpha \wedge \neg \mathsf{A}\, x_1\alpha$	$\langle \mathsf{K}_0, 6, 7\rangle$
9.	$x_1 < x_2 \quad \neg \mathsf{A}\, x_1\alpha \quad \mathsf{A}\, x_2\alpha$	$\mathsf{P}\, x_1\alpha$	$\langle \mathrm{D}5.1, 8\rangle$
10.	$x_1 < x_2 \wedge \neg \mathsf{A}\, x_1\alpha \wedge \mathsf{A}\, x_2\alpha$	$\mathsf{P}\, x_1\alpha$	$\langle \mathsf{VA}, 9\rangle$
11.	$x_1 < x_2 \wedge \neg \mathsf{A}\, x_1\alpha \wedge \mathsf{A}\, x_2\alpha \rightarrow \mathsf{P}\, x_1\alpha$		$\langle \mathsf{I}_1, 10\rangle$

S5.5 $\quad \mathsf{V}\, y_1 y_2\alpha \rightarrow \mathsf{G}\, y_1 y_2$

Proof

1.	$\mathsf{V}\, y_1 y_2\alpha$	$\mathsf{V}\, y_1 y_2\alpha$	$\langle \mathsf{A}\rangle$
2.	$\mathsf{V}\, y_1 y_2\alpha$	$y_1 < y_2 \wedge \mathsf{A}\, y_2\alpha \wedge \forall y(y_1 \leq y < y_2 \rightarrow \neg \mathsf{A}\, y\alpha)$	$\langle \mathrm{D}5.2, 1\rangle$
3.	$\mathsf{V}\, y_1 y_2\alpha$	$y_1 < y_2$	$\langle \mathsf{K}_1, 1\rangle$
4.	$y_1 < y_2 \rightarrow \mathsf{G}\, y_1 y_2$		$\langle \mathrm{S}3.1\rangle$
5.	$\mathsf{V}\, y_1 y_2\alpha$	$\mathsf{G}\, y_1 y_2$	$\langle \mathsf{MP}, 3, 4\rangle$
6.	$\mathsf{V}\, y_1 y_2\alpha \rightarrow \mathsf{G}\, y_1 y_2$		$\langle \mathsf{I}_1, 5\rangle$

S5.6 $\quad \mathsf{V}\, y_0 y_2\alpha \wedge \mathsf{V}\, y_1 y_2\alpha \rightarrow \mathsf{G}\, y_0 y_1$

Proof

1.	$\mathsf{V}\, y_0 y_2\alpha$	$\mathsf{V}\, y_0 y_2\alpha$	$\langle \mathsf{A}\rangle$
2.	$\mathsf{V}\, y_0 y_2\alpha \rightarrow \mathsf{G}\, y_0 y_2$		$\langle \mathrm{S}5.5\rangle$
3.	$\mathsf{V}\, y_0 y_2\alpha$	$\mathsf{G}\, y_0 y_2$	$\langle \mathsf{MP}, 1, 2\rangle$
4.	$\mathsf{V}\, y_1 y_2\alpha$	$\mathsf{V}\, y_1 y_2\alpha$	$\langle \mathsf{A}\rangle$
5.	$\mathsf{V}\, y_1 y_2\alpha \rightarrow \mathsf{G}\, y_1 y_2$		$\langle \mathrm{S}5.5\rangle$
6.	$\mathsf{V}\, y_1 y_2\alpha$	$\mathsf{G}\, y_1 y_2$	$\langle \mathsf{MP}, 4, 5\rangle$
7.	$\mathsf{G}\, y_1 y_2 \rightarrow \mathsf{G}\, y_2 y_1$		$\langle \mathrm{S}3.2.2\rangle$
8.	$\mathsf{V}\, y_1 y_2\alpha$	$\mathsf{G}\, y_2 y_1$	$\langle \mathsf{MP}, 6, 7\rangle$
9.	$\mathsf{V}\, y_0 y_2\alpha \quad \mathsf{V}\, y_1 y_2\alpha$	$\mathsf{G}\, y_0 y_2 \wedge \mathsf{G}\, y_2 y_1$	$\langle \mathsf{K}_0, 3, 8\rangle$
10.	$\mathsf{G}\, y_0 y_2 \wedge \mathsf{G}\, y_2 y_1 \rightarrow \mathsf{G}\, y_0 y_1$		$\langle \mathrm{S}3.2.3\rangle$
11.	$\mathsf{V}\, y_0 y_2\alpha \quad \mathsf{V}\, y_1 y_2\alpha$	$\mathsf{G}\, y_0 y_1$	$\langle \mathsf{MP}, 9, 10\rangle$
12.	$\mathsf{V}\, y_0 y_2\alpha \wedge \mathsf{V}\, y_1 y_2\alpha$	$\mathsf{G}\, y_0 y_1$	$\langle \mathsf{VA}, 11\rangle$
13.	$\mathsf{V}\, y_0 y_2\alpha \wedge \mathsf{V}\, y_1 y_2\alpha \rightarrow \mathsf{G}\, y_0 y_1$		$\langle \mathsf{I}_1, 2\rangle$

S5.7 $\quad \mathsf{V}\, y_0 y_2\alpha \wedge y_0 < y_1 < y_2 \rightarrow \mathsf{V}\, y_1 y_2\alpha$

Proof

1.	$\mathsf{V}\, y_0 y_2\alpha$	$\mathsf{V}\, y_0 y_2\alpha$	$\langle \mathsf{A}\rangle$
2.	$\mathsf{V}\, y_0 y_2\alpha$	$y_0 < y_2 \wedge \mathsf{A}\, y_2\alpha \wedge \forall y_1(y_0 \leq y_1 < y_2 \rightarrow \neg \mathsf{A}\, y_1\alpha)$	$\langle \mathrm{D}5.2, 1\rangle$
3.	$\mathsf{V}\, y_0 y_2\alpha$	$\forall y_1(y_0 \leq y_1 < y_2 \rightarrow \neg \mathsf{A}\, y_1\alpha)$	$\langle \mathsf{K}_2, 2\rangle$
4.	$\mathsf{V}\, y_0 y_2\alpha$	$y_0 \leq y_1 < y_2 \rightarrow \neg \mathsf{A}\, y_1\alpha$	$\langle \mathsf{G}, 3\rangle$
5.	$\mathsf{V}\, y_0 y_2\alpha \quad y_0 \leq y_1 < y_2$	$\neg \mathsf{A}\, y_1\alpha$	$\langle \mathsf{BI}, 4\rangle$
6.	$y_1 = y' \quad y_0 < y_1$	$y_0 < y'$	$\langle \mathsf{ER}_1\rangle$
7.	$y_1 = y' \quad y_0 < y_1$	$y_0 < y' \vee y_0 = y'$	$\langle \mathsf{A}_1, 6\rangle$
8.	$y_0 < y_1 \wedge y_1 < y' \rightarrow y_0 < y'$		$\langle \mathrm{A}3.2.3\rangle$
9.	$y_0 < y_1 \wedge y_1 < y'$	$y_0 < y'$	$\langle \mathsf{BI}, 8\rangle$
10.	$y_0 < y_1 \quad y_1 < y'$	$y_0 < y'$	$\langle \mathsf{ZA}, 9\rangle$

11.	$y_0 < y_1 \quad y_1 < y'$	$y_0 < y' \vee y_0 = y'$	$\langle \mathbf{A}_1, 10\rangle$
12.	$y_0 < y_1 \quad y_1 < y' \vee y_1 = y'$	$y_0 < y' \vee y_0 = y'$	$\langle \mathbf{A}_0, 7, 11\rangle$
13.	$y_0 < y_1 \quad y_1 < y' \vee y_1 = y'$	$y_0 \leq y'$	$\langle \mathrm{D3.1}, 12\rangle$
14.	$y_1 \leq y'$	$y_1 \leq y'$	$\langle \mathbf{A}\rangle$
15.	$y_1 \leq y'$	$y_1 < y' \vee y_1 = y'$	$\langle \mathrm{D3.1}, 14\rangle$
16.	$y_0 < y_1 \quad y_1 \leq y'$	$y_0 \leq y'$	$\langle \mathbf{KS}, 15, 13\rangle$
17.	$y' < y_2$	$y' < y_2$	$\langle \mathbf{A}\rangle$
18.	$y_0 < y_1 \quad y_1 \leq y' \quad y' < y_2$	$y_0 \leq y' \wedge y' < y_2$	$\langle \mathbf{K}_0, 16, 17\rangle$
19.	$\mathbf{V} y_0 y_2 \alpha \quad y_0 \leq y' < y_2$	$\neg \mathbf{A} y' \alpha$	$\langle \mathbf{S}^{y'}_{y_1}, 5\rangle$
20.	$\mathbf{V} y_0 y_2 \alpha \quad y_0 < y_1 \quad y_1 \leq y' \quad y' < y_2$	$\neg \mathbf{A} y' \alpha$	$\langle \mathbf{KS}, 18, 19\rangle$
21.	$\mathbf{V} y_0 y_2 \alpha \quad y_0 < y_1 \quad y_1 \leq y' < y_2$	$\neg \mathbf{A} y' \alpha$	$\langle \mathbf{VA}, 20\rangle$
22.	$\mathbf{V} y_0 y_2 \alpha \quad y_0 < y_1$	$y_1 \leq y' < y_2 \rightarrow \neg \mathbf{A} y' \alpha$	$\langle \mathbf{I}_1, 21\rangle$
23.	$\mathbf{V} y_0 y_2 \alpha \quad y_0 < y_1$	$\forall y'(y_1 \leq y' < y_2 \rightarrow \neg \mathbf{A} y' \alpha)$	$\langle \mathbf{G}_{y'}, 22\rangle$
24.	$\mathbf{V} y_0 y_2 \alpha$	$y_0 < y_2 \wedge \mathbf{A} y_2 \alpha$	$\langle \mathbf{K}_1, 2\rangle$
25.	$\mathbf{V} y_0 y_2 \alpha$	$\mathbf{A} y_2 \alpha$	$\langle \mathbf{K}_2, 24\rangle$
26.	$y_1 < y_2$	$y_1 < y_2$	$\langle \mathbf{A}\rangle$
27.	$\mathbf{V} y_0 y_2 \alpha \quad y_1 < y_2$	$y_1 < y_2 \wedge \mathbf{A} y_2 \alpha$	$\langle \mathbf{K}_0, 26, 25\rangle$
28.	$\mathbf{V} y_0 y_2 \alpha \quad y_0 < y_1 \quad y_1 < y_2$	$y_1 < y_2 \wedge \mathbf{A} y_2 \alpha \wedge \forall y'(y_1 \leq y' < y_2 \rightarrow \neg \mathbf{A} y' \alpha)$	$\langle \mathbf{K}_0, 27, 23\rangle$
29.	$\mathbf{V} y_0 y_2 \alpha \quad y_0 < y_1 \quad y_1 < y_2$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathrm{D3.2}, 28\rangle$
30.	$\mathbf{V} y_0 y_2 \alpha \wedge y_0 < y_1 < y_2$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{VA}, 29\rangle$
31.		$\mathbf{V} y_0 y_2 \alpha \wedge y_0 < y_1 < y_2 \rightarrow \mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{I}_1, 30\rangle$

S5.8 $\quad \mathbf{V} y_1 y_2 \alpha \rightarrow \mathbf{P} y_1 \alpha$

Proof

1.	$\mathbf{V} y_1 y_2 \alpha$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{A}\rangle$
2.	$\mathbf{V} y_1 y_2 \alpha$	$y_1 < y_2 \wedge \mathbf{A} y_2 \alpha \wedge \forall y(y_1 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha)$	$\langle \mathrm{D5.2}, 1\rangle$
3.	$\mathbf{V} y_1 y_2 \alpha$	$\forall y(y_1 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha)$	$\langle \mathbf{K}_2, 2\rangle$
4.	$\mathbf{V} y_1 y_2 \alpha$	$y_1 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha$	$\langle \mathbf{G}, 3\rangle$
5.	$\mathbf{V} y_1 y_2 \alpha \quad y_1 \leq y < y_2$	$\neg \mathbf{A} y \alpha$	$\langle \mathbf{BI}, 4\rangle$
6.	$\mathbf{V} y_1 y_2 \alpha \quad y_1 \leq y_1 < y_2$	$\neg \mathbf{A} y_1 \alpha$	$\langle \mathbf{S}^{y_1}_{y}, 5\rangle$
7.	$\mathbf{V} y_1 y_2 \alpha \quad y_1 \leq y_1 \quad y_1 < y_2$	$\neg \mathbf{A} y_1 \alpha$	$\langle \mathbf{ZA}, 6\rangle$
8.		$y_1 = y_1$	$\langle \mathbf{I}\rangle$
9.		$y_1 < y_1 \vee y_1 = y_1$	$\langle \mathbf{A}_2, 8\rangle$
10.		$y_1 \leq y_1$	$\langle \mathrm{D3.1}, 9\rangle$
11.	$\mathbf{V} y_1 y_2 \alpha \quad y_1 < y_2$	$\neg \mathbf{A} y_1 \alpha$	$\langle \mathbf{KS}, 10, 7\rangle$
12.	$\mathbf{V} y_1 y_2 \alpha$	$y_1 < y_2 \wedge \mathbf{A} y_2 \alpha$	$\langle \mathbf{K}_1, 2\rangle$
13.	$\mathbf{V} y_1 y_2 \alpha$	$\mathbf{A} y_2 \alpha$	$\langle \mathbf{K}_2, 12\rangle$
14.	$\mathbf{V} y_1 y_2 \alpha$	$y_1 < y_2$	$\langle \mathbf{K}_1, 12\rangle$
15.		$\mathbf{A} y_2 \alpha \rightarrow \mathbf{F} y_2 \alpha$	$\langle \mathrm{A5.1}\rangle$

16.	$\mathbf{V} y_1 y_2 \alpha$	$\mathbf{F} y_2 \alpha$	$\langle \mathbf{MP}, 13, 15 \rangle$
17.		$y_1 < y_2 \wedge \mathbf{F} y_2 \alpha \to \mathbf{F} y_1 \alpha$	$\langle \mathrm{A5.2} \rangle$
18.	$\mathbf{V} y_1 y_2 \alpha$	$y_1 < y_2 \wedge \mathbf{F} y_2 \alpha$	$\langle \mathbf{K}_0, 14, 16 \rangle$
19.	$\mathbf{V} y_1 y_2 \alpha$	$\mathbf{F} y_1 \alpha$	$\langle \mathbf{MP}, 18, 17 \rangle$
20.	$\mathbf{V} y_1 y_2 \alpha$	$\neg \mathbf{A} y_1 \alpha$	$\langle \mathbf{KS}, 14, 11 \rangle$
21.	$\mathbf{V} y_1 y_2 \alpha$	$\mathbf{F} y_1 \alpha \wedge \neg \mathbf{A} y_1 \alpha$	$\langle \mathbf{K}_0, 19, 20 \rangle$
22.	$\mathbf{V} y_1 y_2 \alpha$	$\mathbf{P} y_1 \alpha$	$\langle \mathrm{D5.1}, 21 \rangle$
23.		$\mathbf{V} y_1 y_2 \alpha \to \mathbf{P} y_1 \alpha$	$\langle \mathbf{I}_1, 22 \rangle$

S6.1 $\quad \mathbf{M} x y \alpha \to \neg x \sim y$

Proof

1.		$x < x_1 \to \neg x \sim x_1$	$\langle \mathrm{A3.3} \rangle$
2.	$x < x_1$	$\neg x \sim x_1$	$\langle \mathbf{BI}, 1 \rangle$
3.	$x_1 \sim y$	$x_1 \sim y$	$\langle \mathbf{A} \rangle$
4.		$x \sim y \to y \sim x$	$\langle \mathrm{A3.1.2} \rangle$
5.	$x \sim y$	$y \sim x$	$\langle \mathbf{BI}, 4 \rangle$
6.	$x_1 \sim y \quad x \sim y$	$x_1 \sim y \wedge y \sim x$	$\langle \mathbf{K}_0, 3, 5 \rangle$
7.		$x_1 \sim y \wedge y \sim x \to x_1 \sim x$	$\langle \mathrm{A3.1.3} \rangle$
8.	$x_1 \sim y \quad x \sim y$	$x_1 \sim x$	$\langle \mathbf{MP}, 6, 7 \rangle$
9.		$x_1 \sim x \to x \sim x_1$	$\langle \mathrm{A3.1.2} \rangle$
10.	$x_1 \sim y \quad x \sim y$	$x \sim x_1$	$\langle \mathbf{MP}, 8, 9 \rangle$
11.	$x_1 \sim y \quad x < x_1 \quad x \sim y$	$\neg x \sim y$	$\langle \mathbf{W}, 2, 10 \rangle$
12.	$x_1 \sim y \quad x < x_1$	$\neg x \sim y$	$\langle \mathbf{SW}, 11 \rangle$
13.	$x_1 \sim y \wedge x < x_1$	$\neg x \sim y$	$\langle \mathbf{VA}, 12 \rangle$
14.	$\exists x_1 (x_1 \sim y \wedge x < x_1)$	$\neg x \sim y$	$\langle \mathbf{P}''_{x_1}, 13 \rangle$
15.		$\mathbf{M} x y \alpha \to \exists x_1 (x_1 \sim y \wedge x < x_1)$	$\langle \mathrm{A6.1} \rangle$
16.	$\mathbf{M} x y \alpha$	$\exists x_1 (x_1 \sim y \wedge x < x_1)$	$\langle \mathbf{BI}, 15 \rangle$
17.	$\mathbf{M} x y \alpha$	$\neg x \sim y$	$\langle \mathbf{KS}, 16, 14 \rangle$
18.		$\mathbf{M} x y \alpha \to \neg x \sim y$	$\langle \mathbf{I}_1, 17 \rangle$

S6.2 $\quad \mathbf{M} x y \alpha \wedge y_0 < y_1 < y \wedge y_0 \sim x \to \exists x_1 (x_1 \sim y_1 \wedge x < x_1)$

Proof

1.		$x_2 \sim y \to y \sim x_2$	$\langle \mathrm{A3.1.2} \rangle$
2.	$x_2 \sim y$	$y \sim x_2$	$\langle \mathbf{BI}, 1 \rangle$
3.	$x < x_2$	$x < x_2$	$\langle \mathbf{A} \rangle$
4.	$x_2 \sim y \quad x < x_2$	$y \sim x_2 \wedge x < x_2$	$\langle \mathbf{K}_0, 2, 3 \rangle$
5.	$y_0 < y_1 < y \wedge y_0 \sim x$	$y_0 < y_1 < y \wedge y_0 \sim x$	$\langle \mathbf{A} \rangle$
6.		$y_0 < y_1 < y \wedge x < x_2 \wedge y_0 \sim x \wedge y \sim x_2 \to \exists x_1 (x < x_1 < x_2 \wedge y_1 \sim x_1)$	$\langle \mathrm{A3.4} \rangle$
7.	$y_0 < y_1 < y \wedge y_0 \sim x \quad x_2 \sim y \quad x < x_2$	$y_0 < y_1 < y \wedge y_0 \sim x \wedge y \sim x_2 \wedge x < x_2$	$\langle \mathbf{K}_0, 5, 4 \rangle$
8.	$y_0 < y_1 < y \wedge y_0 \sim x \quad x_2 \sim y \quad x < x_2$	$\exists x_1 (x < x_1 < x_2 \wedge y_1 \sim x_1)$	$\langle \mathbf{MP}, 7, 6 \rangle$
9.	$x < x_1 < x_2 \wedge y_1 \sim x_1$	$x < x_1 < x_2 \wedge y_1 \sim x_1$	$\langle \mathbf{A} \rangle$
10.	$x < x_1 < x_2 \wedge y_1 \sim x_1$	$x < x_1 < x_2$	$\langle \mathbf{K}_1, 9 \rangle$

11.	$x < x_1 < x_2 \wedge y_1 \sim x_1$	$x < x_1$	$\langle K_1, 10\rangle$
12.	$x < x_1 < x_2 \wedge y_1 \sim x_1$	$y_1 \sim x_1$	$\langle K_2, 9\rangle$
13.		$y_1 \sim x_1 \rightarrow x_1 \sim y_1$	$\langle A3.1.2\rangle$
14.	$x < x_1 < x_2 \wedge y_1 \sim x_1$	$x_1 \sim y_1$	$\langle MP, 12, 13\rangle$
15.	$x < x_1 < x_2 \wedge y_1 \sim x_1$	$x_1 \sim y_1 \wedge x < x_1$	$\langle K_0, 14, 11\rangle$
16.	$x < x_1 < x_2 \wedge y_1 \sim x_1$	$\exists x_1(x_1 \sim y_1 \wedge x < x_1)$	$\langle P'_{x_1}, 15\rangle$
17.	$\exists x_1(x < x_1 < x_2 \wedge y_1 \sim x_1)$	$\exists x_1(x_1 \sim y_1 \wedge x < x_1)$	$\langle P''_{x_1}, 16\rangle$
18.	$y_0 < y_1 < y \wedge y_0 \sim x \quad x_2 \sim y \quad x < x_2$	$\exists x_1(x_1 \sim y_1 \wedge x < x_1)$	$\langle KS, 8, 17\rangle$
19.	$y_0 < y_1 < y \wedge y_0 \sim x \quad x_2 \sim y \wedge x < x_2$	$\exists x_1(x_1 \sim y_1 \wedge x < x_1)$	$\langle VA, 18\rangle$
20.	$y_0 < y_1 < y \wedge y_0 \sim x \quad \exists x_2(x_2 \sim y \wedge x < x_2)$	$\exists x_1(x_1 \sim y_1 \wedge x < x_1)$	$\langle P''_{x_2}, 19\rangle$
21.		$\mathbf{M}xy\alpha \rightarrow \exists x_2(x_2 \sim y \wedge x < x_2)$	$\langle A6.1\rangle$
22.	$\mathbf{M}xy\alpha$	$\exists x_2(x_2 \sim y \wedge x < x_2)$	$\langle BI, 21\rangle$
23.	$\mathbf{M}xy\alpha \quad y_0 < y_1 < y \wedge y_0 \sim x$	$\exists x_1(x_1 \sim y_1 \wedge x < x_1)$	$\langle KS, 22, 20\rangle$
24.	$\mathbf{M}xy\alpha \wedge y_0 < y_1 < y \wedge y_0 \sim x$	$\exists x_1(x_1 \sim y_1 \wedge x < x_1)$	$\langle VA, 23\rangle$
25.	$\mathbf{M}xy\alpha \wedge y_0 < y_1 < y \wedge y_0 \sim x \rightarrow$	$\exists x_1(x_1 \sim y_1 \wedge x < x_1)$	$\langle I_1, 24\rangle$

S6.3 $\quad \mathbf{M}xy\alpha \wedge x < x_1 < x_2 \wedge x_2 \sim y \rightarrow \exists y_1(x_1 \sim y_1 \wedge y_1 < y)$

Proof

1.	$x \sim y_0$	$x \sim y_0$	$\langle A\rangle$
2.	$y_0 < y$	$y_0 < y$	$\langle A\rangle$
3.	$x \sim y_0 \quad y_0 < y$	$x \sim y_0 \wedge y_0 < y$	$\langle K_0, 1, 2\rangle$
4.	$x < x_1 < x_2 \wedge x_2 \sim y$	$x < x_1 < x_2 \wedge x_2 \sim y$	$\langle A\rangle$
5.	$x < x_1 < x_2 \wedge x_2 \sim y \quad x \sim y_0 \quad y_0 < y$	$x < x_1 < x_2 \wedge x_2 \sim y \wedge x \sim y_0 \wedge y_0 < y$	$\langle K_0, 4, 3\rangle$
6.	$x < x_1 < x_2 \wedge y_0 < y \wedge x \sim y_0 \wedge x_2 \sim y \rightarrow$	$\exists y_1(y_0 < y_1 < y \wedge x_1 \sim y_1)$	$\langle A3.4\rangle$
7.	$x < x_1 < x_2 \wedge x_2 \sim y \quad x \sim y_0 \quad y_0 < y$	$\exists y_1(y_0 < y_1 < y \wedge x_1 \sim y_1)$	$\langle MP, 5, 6\rangle$
8.	$y_0 < y_1 < y \wedge x_1 \sim y_1$	$y_0 < y_1 < y \wedge x_1 \sim y_1$	$\langle A\rangle$
9.	$y_0 < y_1 < y \wedge x_1 \sim y_1$	$y_0 < y_1 < y$	$\langle K_1, 8\rangle$
10.	$y_0 < y_1 < y \wedge x_1 \sim y_1$	$y_1 < y$	$\langle K_2, 9\rangle$
11.	$y_0 < y_1 < y \wedge x_1 \sim y_1$	$x_1 \sim y_1$	$\langle K_2, 8\rangle$
12.	$y_0 < y_1 < y \wedge x_1 \sim y_1$	$x_1 \sim y_1 \wedge y_1 < y$	$\langle K_0, 11, 10\rangle$
13.	$y_0 < y_1 < y \wedge x_1 \sim y_1$	$\exists y_1(x_1 \sim y_1 \wedge y_1 < y)$	$\langle P'_{y_1}, 12\rangle$
14.	$\exists y_1(y_0 < y_1 < y \wedge x_1 \sim y_1)$	$\exists y_1(x_1 \sim y_1 \wedge y_1 < y)$	$\langle P''_{y_1}, 13\rangle$
15.	$x < x_1 < x_2 \wedge x_2 \sim y \quad x \sim y_0 \quad y_0 < y$	$\exists y_1(x_1 \sim y_1 \wedge y_1 < y)$	$\langle KS, 7, 14\rangle$

16.	$x < x_1 < x_2 \wedge x_2 \sim y \quad x \sim y_0 \wedge y_0 < y$	$\exists y_1(x_1 \sim y_1 \wedge y_1 < y)$	$\langle$**VA**, 15$\rangle$
17.	$x < x_1 < x_2 \wedge x_2 \sim y \quad \exists y_0(x \sim y_0 \wedge y_0 < y)$	$\exists y_1(x_1 \sim y_1 \wedge y_1 < y)$	$\langle \mathbf{P}''_{y_0}$, 16$\rangle$
18.		$\mathbf{M}xy\alpha \to \exists y_0(x \sim y_0 \wedge y_0 < y)$	$\langle$A6.2$\rangle$
19.	$\mathbf{M}xy\alpha$	$\exists y_0(x \sim y_0 \wedge y_0 < y)$	$\langle$**BI**, 18$\rangle$
20.	$\mathbf{M}xy\alpha \quad x < x_1 < x_2 \wedge x_2 \sim y$	$\exists y_1(x_1 \sim y_1 \wedge y_1 < y)$	$\langle$**KS**, 19, 17$\rangle$
21.	$\mathbf{M}xy\alpha \wedge x < x_1 < x_2 \wedge x_2 \sim y$	$\exists y_1(x_1 \sim y_1 \wedge y_1 < y)$	$\langle$**VA**, 20$\rangle$
22.		$\mathbf{M}xy\alpha \wedge x < x_1 < x_2 \wedge x_2 \sim y \to \exists y_1(x_1 \sim y_1 \wedge y_1 < y)$	$\langle \mathbf{I}_1$, 21$\rangle$

S6.4 $\quad \mathbf{M}xy\alpha \to \exists y_0(x \sim y_0 \wedge \mathbf{V}y_0y\alpha)$

Proof

1.		$\mathbf{M}xy\alpha \wedge x \sim y_0 \leq y_1 < y \to \neg \mathbf{A}y_1\alpha$	$\langle$A6.4$\rangle$
2.	$\mathbf{M}xy\alpha \wedge x \sim y_0 \leq y_1 < y$	$\neg \mathbf{A}y_1\alpha$	$\langle$**BI**, 1$\rangle$
3.	$\mathbf{M}xy\alpha \quad x \sim y_0 \quad y_0 \leq y_1 < y$	$\neg \mathbf{A}y_1\alpha$	$\langle$**ZA**, 2$\rangle$
4.	$\mathbf{M}xy\alpha \quad x \sim y_0$	$y_0 \leq y_1 < y \to \neg \mathbf{A}y_1\alpha$	$\langle \mathbf{I}_1$, 3$\rangle$
5.	$\mathbf{M}xy\alpha \quad x \sim y_0$	$\forall y_1(y_0 \leq y_1 < y \to \neg \mathbf{A}y_1\alpha)$	$\langle \mathbf{G}_{y_1}$, 4$\rangle$
6.		$\mathbf{M}xy\alpha \to \mathbf{A}y\alpha$	$\langle$A6.3$\rangle$
7.	$\mathbf{M}xy\alpha$	$\mathbf{A}y\alpha$	$\langle$**BI**, 6$\rangle$
8.	$\mathbf{M}xy\alpha \quad x \sim y_0$	$\mathbf{A}y\alpha \wedge \forall y_1(y_0 \leq y_1 < y \to \neg \mathbf{A}y_1\alpha)$	$\langle \mathbf{K}_0$, 7, 5$\rangle$
9.	$y_0 < y$	$y_0 < y$	$\langle$**A**$\rangle$
10.	$\mathbf{M}xy\alpha \quad x \sim y_0 \quad y_0 < y$	$y_0 < y \wedge \mathbf{A}y\alpha \wedge \forall y_1(y_0 \leq y_1 < y \to \neg \mathbf{A}y_1\alpha)$	$\langle \mathbf{K}_0$, 9, 8$\rangle$
11.	$\mathbf{M}xy\alpha \quad x \sim y_0 \quad y_0 < y$	$\mathbf{V}y_0y\alpha$	$\langle$D5.2, 10$\rangle$
12.	$x \sim y_0$	$x \sim y_0$	$\langle$**A**$\rangle$
13.	$\mathbf{M}xy\alpha \quad x \sim y_0 \quad y_0 < y$	$x \sim y_0 \wedge \mathbf{V}y_0y\alpha$	$\langle \mathbf{K}_0$, 11, 12$\rangle$
14.	$\mathbf{M}xy\alpha \quad x \sim y_0 \quad y_0 < y$	$\exists y_0(x \sim y_0 \wedge \mathbf{V}y_0y\alpha)$	$\langle \mathbf{P}'_{y_0}$, 13$\rangle$
15.	$\mathbf{M}xy\alpha \quad x \sim y_0 \wedge y_0 < y$	$\exists y_0(x \sim y_0 \wedge \mathbf{V}y_0y\alpha)$	$\langle$**VA**, 14$\rangle$
16.	$\mathbf{M}xy\alpha \quad \exists y_0(x \sim y_0 \wedge y_0 < y)$	$\exists y_0(x \sim y_0 \wedge \mathbf{V}y_0y\alpha)$	$\langle \mathbf{P}''_{y_0}$, 15$\rangle$
17.		$\mathbf{M}xy\alpha \to \exists y_0(x \sim y_0 \wedge y_0 < y)$	$\langle$A6.2$\rangle$
18.	$\mathbf{M}xy\alpha$	$\exists y_0(x \sim y_0 \wedge y_0 < y)$	$\langle$**BI**, 17$\rangle$
19.	$\mathbf{M}xy\alpha$	$\exists y_0(x \sim y_0 \wedge \mathbf{V}y_0y\alpha)$	$\langle$**KS**, 18, 16$\rangle$
20.		$\mathbf{M}xy\alpha \to \exists y_0(x \sim y_0 \wedge \mathbf{V}y_0y\alpha)$	$\langle \mathbf{I}_1$, 19$\rangle$

S6.5 $\quad \mathbf{M}xy_2\alpha \wedge x \sim y_1 < y_2 \to \mathbf{V}y_1y_2\alpha$

Proof

1.	$\mathbf{V}y_0y_2\alpha$	$\mathbf{V}y_0y_2\alpha$	$\langle$**A**$\rangle$
2.	$\mathbf{V}y_0y_2\alpha$	$y_0 < y_2 \wedge \mathbf{A}y_2\alpha \wedge \forall y(y_0 \leq y < y_2 \to \neg \mathbf{A}y\alpha)$	$\langle$D5.2, 1$\rangle$

3.	$x \sim y_1$	$x \sim y_1$	$\langle \mathbf{A} \rangle$
4.	$y_1 < y_2$	$y_1 < y_2$	$\langle \mathbf{A} \rangle$
5.		$x \sim y_0 \rightarrow y_0 \sim x$	$\langle$A3.1.2$\rangle$
6.	$x \sim y_0$	$y_0 \sim x$	$\langle \mathbf{BI}, 5 \rangle$
7.	$x \sim y_0 \quad x \sim y_1$	$y_0 \sim x \wedge x \sim y_1$	$\langle \mathbf{K}_0, 6, 3 \rangle$
8.		$y_0 \sim x \wedge x \sim y_1 \rightarrow y_0 \sim y_1$	$\langle$A3.1.3$\rangle$
9.	$x \sim y_0 \quad x \sim y_1$	$y_0 \sim y_1$	$\langle \mathbf{MP}, 7, 8 \rangle$
10.	$\mathbf{V} y_0 y_2 \alpha$	$y_0 < y_2$	$\langle \mathbf{K}_1, 2 \rangle$
11.	$\mathbf{V} y_0 y_2 \alpha \quad y_1 < y_2$	$y_0 < y_2 \wedge y_1 < y_2$	$\langle \mathbf{K}_0, 10, 4 \rangle$
12.	$x \sim y_1 \quad y_1 < y_2 \quad x \sim y_0 \quad \mathbf{V} y_0 y_2 \alpha$	$y_0 < y_2 \wedge y_1 < y_2 \wedge y_0 \sim y_1$	$\langle \mathbf{K}_0, 11, 9 \rangle$
13.		$y_0 < y_2 \wedge y_1 < y_2 \wedge y_0 \sim y_1 \rightarrow y_0 = y_1$	$\langle$S3.6$\rangle$
14.	$x \sim y_1 \quad y_1 < y_2 \quad x \sim y_0 \quad \mathbf{V} y_0 y_2 \alpha$	$y_0 = y_1$	$\langle \mathbf{MP}, 12, 13 \rangle$
15.	$\mathbf{V} y_0 y_2 \alpha \quad y_0 = y_1$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{I}^{y_1}_{y_0}, 1 \rangle$
16.	$x \sim y_1 \quad y_1 < y_2 \quad x \sim y_0 \quad \mathbf{V} y_0 y_2 \alpha$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{KS}, 14, 15 \rangle$
17.	$x \sim y_1 \quad y_1 < y_2 \quad x \sim y_0 \wedge \mathbf{V} y_0 y_2 \alpha$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{VA}, 16 \rangle$
18.	$x \sim y_1 \quad y_1 < y_2 \quad \exists y_0 (x \sim y_0 \wedge \mathbf{V} y_0 y_2 \alpha)$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{P}''_{y_0}, 17 \rangle$
19.		$\mathbf{M} x y_2 \alpha \rightarrow \exists y_0 (x \sim y_0 \wedge \mathbf{V} y_0 y_2 \alpha)$	$\langle$S6.4$\rangle$
20.	$\mathbf{M} x y_2 \alpha$	$\exists y_0 (x \sim y_0 \wedge \mathbf{V} y_0 y_2 \alpha)$	$\langle \mathbf{BI}, 19 \rangle$
21.	$\mathbf{M} x y_2 \alpha \quad x \sim y_1 \quad y_1 < y_2$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{KS}, 20, 18 \rangle$
22.	$\mathbf{M} x y_2 \alpha \wedge x \sim y_1 < y_2$	$\mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{VA}, 21 \rangle$
23.		$\mathbf{M} x y_2 \alpha \wedge x \sim y_1 < y_2 \rightarrow \mathbf{V} y_1 y_2 \alpha$	$\langle \mathbf{I}_1, 22 \rangle$

S6.6 $\quad \mathbf{M} x y \alpha \wedge \mathbf{A} x \alpha \rightarrow \neg \mathbf{G} x y$

Proof

1.		$\mathbf{V} y_0 y \alpha \rightarrow \mathbf{P} y_0 \alpha$	$\langle$S3.8$\rangle$
2.	$\mathbf{V} y_0 y \alpha$	$\mathbf{P} y_0 \alpha$	$\langle \mathbf{BI}, 1 \rangle$
3.	$\mathbf{V} y_0 y \alpha$	$\mathbf{F} y_0 \alpha \wedge \neg \mathbf{A} y_0 \alpha$	$\langle$D5.1, 2$\rangle$
4.	$\mathbf{V} y_0 y \alpha$	$\neg \mathbf{A} y_0 \alpha$	$\langle \mathbf{K}_2, 3 \rangle$
5.		$\mathbf{V} y_0 y \alpha \rightarrow \mathbf{G} y_0 y$	$\langle$S5.5$\rangle$
6.	$\mathbf{V} y_0 y \alpha$	$\mathbf{G} y_0 y$	$\langle \mathbf{BI}, 5 \rangle$
7.		$\mathbf{G} y_0 y \rightarrow \mathbf{G} y y_0$	$\langle$S3.2.2$\rangle$
8.	$\mathbf{V} y_0 y \alpha$	$\mathbf{G} y y_0$	$\langle \mathbf{MP}, 6, 7 \rangle$
9.	$\mathbf{G} x y$	$\mathbf{G} x y$	$\langle \mathbf{A} \rangle$
10.	$\mathbf{V} y_0 y \alpha \quad \mathbf{G} x y$	$\mathbf{G} x y \wedge \mathbf{G} y y_0$	$\langle \mathbf{K}_0, 8, 9 \rangle$
11.		$\mathbf{G} x y \wedge \mathbf{G} y y_0 \rightarrow \mathbf{G} x y_0$	$\langle$S3.2.3$\rangle$
12.	$\mathbf{V} y_0 y \alpha \quad \mathbf{G} x y$	$\mathbf{G} x y_0$	$\langle \mathbf{MP}, 10, 11 \rangle$
13.	$x \sim y_0$	$x \sim y_0$	$\langle \mathbf{A} \rangle$
14.	$x \sim y_0 \quad \mathbf{V} y_0 y \alpha \quad \mathbf{G} x y$	$\mathbf{G} x y_0 \wedge x \sim y_0$	$\langle \mathbf{K}_0, 12, 13 \rangle$
15.		$\mathbf{G} x y_0 \wedge x \sim y_0 \rightarrow x = y_0$	$\langle$S3.3$\rangle$

16.	$x \sim y_0 \quad \mathbf{V}y_0 y\alpha \quad \mathbf{G}xy$	$x = y_0$	⟨MP, 14, 15⟩
17.	$x = y_0 \quad \neg\mathbf{A}y_0\alpha$	$\neg\mathbf{A}x\alpha$	⟨$\mathbf{ER}_1$⟩
18.	$x = y_0 \quad \mathbf{V}y_0 y\alpha$	$\neg\mathbf{A}x\alpha$	⟨KS, 4, 17⟩
19.	$x \sim y_0 \quad \mathbf{V}y_0 y\alpha \quad \mathbf{G}xy$	$\neg\mathbf{A}x\alpha$	⟨KS, 16, 18⟩
20.	$\mathbf{A}x\alpha$	$\mathbf{A}x\alpha$	⟨A⟩
21.	$\mathbf{A}x\alpha \quad x \sim y_0 \quad \mathbf{V}y_0 y\alpha \quad \mathbf{G}xy$	$\neg\mathbf{G}xy$	⟨W, 19, 20⟩
22.	$\mathbf{A}x\alpha \quad x \sim y_0 \quad \mathbf{V}y_0 y\alpha$	$\neg\mathbf{G}xy$	⟨SW, 21⟩
23.	$\mathbf{A}x\alpha \quad x \sim y_0 \wedge \mathbf{V}y_0 y\alpha$	$\neg\mathbf{G}xy$	⟨VA, 22⟩
24.	$\mathbf{A}x\alpha \quad \exists y_0(x \sim y_0 \wedge \mathbf{V}y_0 y\alpha)$	$\neg\mathbf{G}xy$	⟨$\mathbf{P}''_{y_0}$, 23⟩
25.		$\mathbf{M}xy\alpha \to \exists y_0(x \sim y_0 \wedge \mathbf{V}y_0 y\alpha)$	⟨S6.4⟩
26.	$\mathbf{M}xy\alpha$	$\exists y_0(x \sim y_0 \wedge \mathbf{V}y_0 y\alpha)$	⟨BI, 25⟩
27.	$\mathbf{M}xy\alpha \quad \mathbf{A}x\alpha$	$\neg\mathbf{G}xy$	⟨KS, 26, 24⟩
28.	$\mathbf{M}xy\alpha \wedge \mathbf{A}x\alpha$	$\neg\mathbf{G}xy$	⟨VA, 27⟩
29.		$\mathbf{M}xy\alpha \wedge \mathbf{A}x\alpha \to \neg\mathbf{G}xy$	⟨$\mathbf{I}_1$, 28⟩

S6.7 $\quad \mathbf{M}xy_2\alpha \wedge x < x_1 \sim y_1 < y_2 \to \mathbf{V}y_1y_2\alpha$

Proof

1.	$\mathbf{M}xy_2\alpha \wedge x < x_1 \sim y_1 < y_2 \to$	$\mathbf{M}x_1y_2\alpha$	⟨A6.6⟩
2.	$\mathbf{M}xy_2\alpha \wedge x < x_1 \sim y_1 < y_2$	$\mathbf{M}x_1y_2\alpha$	⟨BI, 1⟩
3.	$\mathbf{M}xy_2\alpha \quad x < x_1 \quad x_1 \sim y_1 \quad y_1 < y_2$	$\mathbf{M}x_1y_2\alpha$	⟨ZA, 2⟩
4.	$x_1 \sim y_1$	$x_1 \sim y_1$	⟨A⟩
5.	$y_1 < y_2$	$y_1 < y_2$	⟨A⟩
6.	$x_1 \sim y_1 \quad y_1 < y_2$	$x_1 \sim y_1 < y_2$	⟨$\mathbf{K}_0$, 4, 5⟩
7.	$\mathbf{M}xy_2\alpha \quad x < x_1 \quad x_1 \sim y_1 \quad y_1 < y_2$	$\mathbf{M}x_1y_2\alpha \wedge x_1 \sim y_1 < y_2$	⟨$\mathbf{K}_0$, 3, 6⟩
8.		$\mathbf{M}x_1y_2\alpha \wedge x_1 \sim y_1 < y_2 \to \mathbf{V}y_1y_2\alpha$	⟨S6.5⟩
9.	$\mathbf{M}xy_2\alpha \quad x < x_1 \quad x_1 \sim y_1 \quad y_1 < y_2$	$\mathbf{V}y_1y_2\alpha$	⟨MP, 7, 8⟩
10.	$\mathbf{M}xy_2\alpha \wedge x < x_1 \sim y_1 < y_2$	$\mathbf{V}y_1y_2\alpha$	⟨VA, 9⟩
11.		$\mathbf{M}xy_2\alpha \wedge x < x_1 \sim y_1 < y_2 \to \mathbf{V}y_1y_2\alpha$	⟨$\mathbf{I}_1$, 10⟩

S6.8 $\quad \mathbf{V}y_0y_2\alpha \leftrightarrow \exists x_1 \exists y_1(x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M}x_1y_2\alpha \wedge \forall y(y_0 \leq y < y_2 \to \neg\mathbf{A}y\alpha))$

Proof

I

1.	$\mathbf{V}y_0y_2\alpha$	$\mathbf{V}y_0y_2\alpha$	⟨A⟩
2.	$\mathbf{V}y_0y_2\alpha$	$y_0 < y_2 \wedge \mathbf{A}y_2\alpha \wedge \forall y(y_0 \leq y < y_2 \to \neg\mathbf{A}y\alpha)$	⟨D5.2, 1⟩
3.	$\mathbf{V}y_0y_2\alpha$	$\forall y(y_0 \leq y < y_2 \to \neg\mathbf{A}y\alpha)$	⟨$\mathbf{K}_2$, 2⟩

4. $x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha$

$x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha$ ⟨**A**⟩

5. $\mathbf{V} y_0 y_2 \alpha \quad x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha$

$x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge$
$\forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha)$ ⟨$\mathbf{K}_0$, 3, 4⟩

6. $\mathbf{V} y_0 y_2 \alpha \quad x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha$

$\exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge$
$\forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha))$ ⟨$\mathbf{P}'_{y_1}$, 5⟩

7. $\mathbf{V} y_0 y_2 \alpha \quad x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha$

$\exists x_1 \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge$
$\forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha))$ ⟨$\mathbf{P}'_{x_1}$, 6⟩

8. $\mathbf{V} y_0 y_2 \alpha \quad \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha)$

$\exists x_1 \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge$
$\forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha))$ ⟨$\mathbf{P}''_{y_1}$, 7⟩

9. $\mathbf{V} y_0 y_2 \alpha \quad \exists x_1 \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha)$

$\exists x_1 \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge$
$\forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha))$ ⟨$\mathbf{P}''_{x_1}$, 8⟩

10. $\mathbf{V} y_0 y_2 \alpha \rightarrow \exists x_1 \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha)$ ⟨A6.7⟩

11. $\mathbf{V} y_0 y_2 \alpha \quad \exists x_1 \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha)$ ⟨**BI**, 10⟩

12. $\mathbf{V} y_0 y_2 \alpha \quad \exists x_1 \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge$
$\forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha))$ ⟨**KS**, 11, 9⟩

13. $\mathbf{V} y_0 y_2 \alpha \rightarrow \exists x_1 \exists y_1 (x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge$
$\forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha))$ ⟨$\mathbf{I}_1$, 12⟩

II

1. $y_0 < y_1 \wedge y_1 < y_2 \rightarrow y_0 < y_2$ ⟨A3.2.3⟩
2. $y_0 < y_1 \wedge y_1 < y_2 \quad y_0 < y_2$ ⟨**BI**, 1⟩
3. $y_0 < y_1 \quad y_1 < y_2 \quad y_0 < y_2$ ⟨**ZA**, 2⟩
4. $y_0 = y_1 \quad y_1 < y_2 \quad y_0 < y_2$ ⟨$\mathbf{ER}_1$⟩
5. $y_0 < y_1 \vee y_0 = y_1 \quad y_1 < y_2$
   $y_0 < y_2$ ⟨$\mathbf{A}_0$, 3, 4⟩
6. $y_0 \leq y_1 \quad y_0 \leq y_1$ ⟨**A**⟩
7. $y_0 \leq y_1 \quad y_0 < y_1 \vee y_0 = y_1$ ⟨D3.1, 6⟩
8. $y_0 \leq y_1 \quad y_1 < y_2 \quad y_0 < y_2$ ⟨**KS**, 7, 5⟩
9. $y_0 \leq y_1 < y_2 \quad y_0 < y_2$ ⟨**VA**, 8⟩
10. $\forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha) \quad \forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha)$ ⟨**A**⟩
11. $\mathbf{M} x_1 y_2 \alpha \rightarrow \mathbf{A} y_2 \alpha$ ⟨A6.3⟩
12. $\mathbf{M} x_1 y_2 \alpha \quad \mathbf{A} y_2 \alpha$ ⟨**BI**, 11⟩
13. $\mathbf{M} x_1 y_2 \alpha \quad \forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha)$
    $\mathbf{A} y_2 \alpha \wedge \forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha)$ ⟨$\mathbf{K}_0$, 12, 10⟩
14. $y_0 \leq y_1 < y_2 \quad \mathbf{M} x_1 y_2 \alpha \quad \forall y (y_0 \leq y < y_2 \rightarrow \neg \mathbf{A} y \alpha)$
    $y_0 < y_2 \wedge \mathbf{A} y_2 \alpha \wedge \forall y (y_0 \leq y < y_2 \rightarrow$
    $\neg \mathbf{A} y \alpha)$ ⟨$\mathbf{K}_0$, 9, 13⟩

15.	$y_0 \leq y_1 < y_2 \quad \mathbf{M} x_1 y_2 \alpha \quad \forall y(y_0 \leq y < y_2 \to \neg \mathbf{A} y \alpha)$	$\mathbf{V} y_0 y_2 \alpha$	⟨D5.2, 14⟩
16.	$x_1 \sim y_1 \quad y_0 \leq y_1 < y_2 \quad \mathbf{M} x_1 y_2 \alpha \quad \forall y(y_0 \leq y < y_2 \to \neg \mathbf{A} y \alpha)$	$\mathbf{V} y_0 y_2 \alpha$	⟨**EA**, 15⟩
17.	$x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge \forall y(y_0 \leq y < y_2 \to \neg \mathbf{A} y \alpha)$	$\mathbf{V} y_0 y_2 \alpha$	⟨**VA**, 16⟩
18.	$\exists y_1(x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge \forall y(y_0 \leq y < y_2 \to \neg \mathbf{A} y \alpha))$	$\mathbf{V} y_0 y_2 \alpha$	⟨$\mathbf{P}''_{y_1}$, 17⟩
19.	$\exists x_1 \exists y_1(x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge \forall y(y_0 \leq y < y_2 \to \neg \mathbf{A} y \alpha))$	$\mathbf{V} y_0 y_2 \alpha$	⟨$\mathbf{P}''_{x_1}$, 18⟩
20.	$\exists x_1 \exists y_1(x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge \forall y(y_0 \leq y < y_2 \to \neg \mathbf{A} y \alpha)) \to \mathbf{V} y_0 y_2$		⟨$\mathbf{I}_1$, 19⟩

III

1.	$\mathbf{V} y_0 y_2 \alpha \longleftrightarrow \exists x_1 \exists y_1(x_1 \sim y_1 \wedge y_0 \leq y_1 < y_2 \wedge \mathbf{M} x_1 y_2 \alpha \wedge \forall y(y_0 \leq y < y_2 \to \neg \mathbf{A} y_2))$	⟨I, II⟩

S7.1 $\neg \mathbf{W} x x \alpha$

Proof

1.	$\mathbf{W} xy \alpha$	$\mathbf{W} xy \alpha$	⟨**A**⟩
2.	$\mathbf{W} xy \alpha$	$\mathbf{B} xy \alpha \wedge \neg \mathbf{B} yx \alpha$	⟨D7.1, 1⟩
3.	$\mathbf{W} xy \alpha$	$\mathbf{B} xy \alpha$	⟨$\mathbf{K}_1$, 2⟩
4.	$\mathbf{W} xx \alpha$	$\mathbf{B} xx \alpha$	⟨$\mathbf{S}^x_y$, 3⟩
5.	$\mathbf{W} xy \alpha$	$\neg \mathbf{B} yx \alpha$	⟨$\mathbf{K}_2$, 2⟩
6.	$\mathbf{W} xx \alpha$	$\neg \mathbf{B} xx \alpha$	⟨$\mathbf{S}^x_y$, 5⟩
7.	$\mathbf{W} xx \alpha$	$\neg \mathbf{W} xx \alpha$	⟨**W**, 4, 6⟩
8.	$\neg \mathbf{W} xx \alpha$		⟨**SW**, 7⟩

S7.2 $\mathbf{W} xy \alpha \to \neg \mathbf{W} yx \alpha$

Proof

1.	$\mathbf{W} xy \alpha$	$\mathbf{W} xy \alpha$	⟨**A**⟩
2.	$\mathbf{W} xy \alpha$	$\mathbf{B} xy \alpha \wedge \neg \mathbf{B} yx \alpha$	⟨D7.1, 1⟩
3.	$\mathbf{W} xy \alpha$	$\neg \mathbf{B} yx \alpha$	⟨$\mathbf{K}_2$, 2⟩
4.	$\mathbf{W} yx \alpha$	$\mathbf{W} yx \alpha$	⟨**A**⟩
5.	$\mathbf{W} yx \alpha$	$\mathbf{B} yx \alpha \wedge \neg \mathbf{B} xy \alpha$	⟨D7.1, 4⟩
6.	$\mathbf{W} yx \alpha$	$\mathbf{B} yx \alpha$	⟨$\mathbf{K}_1$, 5⟩
7.	$\mathbf{W} xy \alpha \quad \mathbf{W} yx \alpha$	$\neg \mathbf{W} yx \alpha$	⟨**W**, 3, 6⟩
8.	$\mathbf{W} xy \alpha$	$\neg \mathbf{W} yx \alpha$	⟨**SW**, 7⟩
9.	$\mathbf{W} xy \alpha \to \neg \mathbf{W} yx \alpha$		⟨$\mathbf{I}_1$, 8⟩

S7.3 $\mathbf{W} xy \alpha \wedge \mathbf{W} yz \alpha \to \mathbf{W} xz \alpha$

Proof

1.	$\mathbf{W} xy \alpha$	$\mathbf{W} xy \alpha$	⟨**A**⟩
2.	$\mathbf{W} xy \alpha$	$\mathbf{B} xy \alpha \wedge \neg \mathbf{B} yx \alpha$	⟨D7.1, 1⟩
3.	$\mathbf{W} yz \alpha$	$\mathbf{W} yz \alpha$	⟨**A**⟩

4.	$\mathbf{W}yz\alpha$			$\mathbf{B}yz\alpha \wedge \neg\mathbf{B}zy\alpha$	⟨D7.1, 3⟩
5.	$\mathbf{W}xy\alpha$			$\mathbf{B}xy\alpha$	⟨$\mathbf{K}_1$, 2⟩
6.	$\mathbf{W}yz\alpha$			$\mathbf{B}yz\alpha$	⟨$\mathbf{K}_1$, 4⟩
7.	$\mathbf{W}xy\alpha$	$\mathbf{W}yz\alpha$		$\mathbf{B}xy\alpha \wedge \mathbf{B}yz\alpha$	⟨$\mathbf{K}_0$, 5, 6⟩
8.				$\mathbf{B}xy\alpha \wedge \mathbf{B}yz\alpha \to \mathbf{B}xz\alpha$	⟨A7.2⟩
9.	$\mathbf{W}xy\alpha$	$\mathbf{W}yz\alpha$		$\mathbf{B}xz\alpha$	⟨**MP**, 7, 8⟩
10.	$\mathbf{B}zx\alpha$			$\mathbf{B}zx\alpha$	⟨**A**⟩
11.	$\mathbf{W}xy\alpha$	$\mathbf{B}zx\alpha$		$\mathbf{B}zx\alpha \wedge \mathbf{B}xy\alpha$	⟨$\mathbf{K}_0$, 10, 5⟩
12.				$\mathbf{B}zx\alpha \wedge \mathbf{B}xy\alpha \to \mathbf{B}zy\alpha$	⟨A7.2⟩
13.	$\mathbf{W}xy\alpha$	$\mathbf{B}zx\alpha$		$\mathbf{B}zy\alpha$	⟨**MP**, 11, 12⟩
14.	$\mathbf{W}yz\alpha$			$\neg\mathbf{B}zy\alpha$	⟨$\mathbf{K}_2$, 4⟩
15.	$\mathbf{W}xy\alpha$	$\mathbf{W}yz\alpha$	$\mathbf{B}zx\alpha$	$\neg\mathbf{B}zx\alpha$	⟨**W**, 13, 14⟩
16.	$\mathbf{W}xy\alpha$	$\mathbf{W}yz\alpha$		$\neg\mathbf{B}zx\alpha$	⟨**SW**, 15⟩
17.	$\mathbf{W}xy\alpha$	$\mathbf{W}yz\alpha$		$\mathbf{B}xz\alpha \wedge \neg\mathbf{B}zx\alpha$	⟨$\mathbf{K}_0$, 9, 16⟩
18.	$\mathbf{W}xy\alpha$	$\mathbf{W}yz\alpha$		$\mathbf{W}xz\alpha$	⟨D7.1, 17⟩
19.	$\mathbf{W}xy\alpha \wedge \mathbf{W}yz\alpha$			$\mathbf{W}xz\alpha$	⟨**VA**, 18⟩
20.				$\mathbf{W}xy\alpha \wedge \mathbf{W}yz\alpha \to \mathbf{W}xz\alpha$	⟨$\mathbf{I}_1$, 19⟩

S7.4 $\quad \mathbf{M}xy\alpha \wedge \mathbf{P}z\alpha \to \mathbf{W}zx\alpha$

Proof

1.					$\mathbf{M}xy\alpha \wedge \mathbf{A}z\alpha \to \mathbf{B}zx\alpha$	⟨A7.5⟩
2.	$\mathbf{M}xy\alpha \wedge \mathbf{A}z\alpha$				$\mathbf{B}zx\alpha$	⟨**BI**, 1⟩
3.	$\mathbf{M}xy\alpha$	$\mathbf{A}z\alpha$			$\mathbf{B}zx\alpha$	⟨**ZA**, 2⟩
4.	$\mathbf{M}xy\alpha$	$\mathbf{A}y\alpha$			$\mathbf{B}yx\alpha$	⟨$\mathbf{S}_z^y$, 3⟩
5.					$\mathbf{P}x\alpha \wedge \mathbf{A}y\alpha \to \mathbf{W}xy\alpha$	⟨A7.3⟩
6.	$\mathbf{P}x\alpha \wedge \mathbf{A}y\alpha$				$\mathbf{W}xy\alpha$	⟨**BI**, 5⟩
7.	$\mathbf{P}x\alpha$	$\mathbf{A}y\alpha$			$\mathbf{W}xy\alpha$	⟨**ZA**, 6⟩
8.	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$			$\mathbf{W}zy\alpha$	⟨$\mathbf{S}_x^z$, 7⟩
9.	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$			$\mathbf{B}zy\alpha \wedge \neg\mathbf{B}yz\alpha$	⟨D7.1, 8⟩
10.	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$			$\neg\mathbf{B}yz\alpha$	⟨$\mathbf{K}_2$, 9⟩
11.	$\mathbf{B}xz\alpha$				$\mathbf{B}xz\alpha$	⟨**A**⟩
12.	$\mathbf{M}xy\alpha$	$\mathbf{A}y\alpha$	$\mathbf{B}xz\alpha$		$\mathbf{B}yx\alpha \wedge \mathbf{B}xz\alpha$	⟨$\mathbf{K}_0$, 4, 11⟩
13.					$\mathbf{B}yx\alpha \wedge \mathbf{B}xz\alpha \to \mathbf{B}yz\alpha$	⟨A7.2⟩
14.	$\mathbf{M}xy\alpha$	$\mathbf{A}y\alpha$	$\mathbf{B}xz\alpha$		$\mathbf{B}yz\alpha$	⟨**MP**, 12, 13⟩
15.	$\mathbf{M}xy\alpha$	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$	$\mathbf{B}xz\alpha$	$\neg\mathbf{B}xz\alpha$	⟨**W**, 10, 14⟩
16.	$\mathbf{M}xy\alpha$	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$		$\neg\mathbf{B}xz\alpha$	⟨**SW**, 15⟩
17.	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$			$\mathbf{B}zy\alpha$	⟨$\mathbf{K}_1$, 9⟩
18.	$\mathbf{M}xy\alpha$	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$		$\mathbf{B}zy\alpha \wedge \mathbf{B}yx\alpha$	⟨$\mathbf{K}_0$, 4, 17⟩
19.					$\mathbf{B}zy\alpha \wedge \mathbf{B}yx\alpha \to \mathbf{B}zx\alpha$	⟨A7.2⟩
20.	$\mathbf{M}xy\alpha$	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$		$\mathbf{B}zx\alpha$	⟨**MP**, 18, 19⟩
21.	$\mathbf{M}xy\alpha$	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$		$\mathbf{B}zx\alpha \wedge \neg\mathbf{B}xz\alpha$	⟨$\mathbf{K}_0$, 20, 16⟩
22.	$\mathbf{M}xy\alpha$	$\mathbf{P}z\alpha$	$\mathbf{A}y\alpha$		$\mathbf{W}zx\alpha$	⟨D7.1, 21⟩
23.					$\mathbf{M}xy\alpha \to \mathbf{A}y\alpha$	⟨A6.3⟩
24.	$\mathbf{M}xy\alpha$				$\mathbf{A}y\alpha$	⟨**BI**, 23⟩
25.	$\mathbf{M}xy\alpha$	$\mathbf{P}z\alpha$			$\mathbf{W}zx\alpha$	⟨**KS**, 24, 22⟩

26.	$\mathsf{M}xy\alpha \wedge \mathsf{P}z\alpha$	$\mathsf{W}zx\alpha$	$\langle \mathsf{VA}, 25\rangle$
27.		$\mathsf{M}xy\alpha \wedge \mathsf{P}z\alpha \rightarrow \mathsf{W}zx\alpha$	$\langle \mathsf{I}_1, 26\rangle$

S7.5 $\mathsf{M}xy\alpha \rightarrow \neg\mathsf{P}x\alpha$

Proof

1.		$\mathsf{M}xy\alpha \wedge \mathsf{P}z\alpha \rightarrow \mathsf{W}zx\alpha$	$\langle$S7.4$\rangle$
2.	$\mathsf{M}xy\alpha \wedge \mathsf{P}z\alpha$	$\mathsf{W}zx\alpha$	$\langle \mathsf{BI}, 1\rangle$
3.	$\mathsf{M}xy\alpha \quad \mathsf{P}z\alpha$	$\mathsf{W}zx\alpha$	$\langle \mathsf{ZA}, 2\rangle$
4.	$\mathsf{M}xy\alpha \quad \mathsf{P}x\alpha$	$\mathsf{W}xx\alpha$	$\langle \mathsf{S}^x_z, 3\rangle$
5.		$\neg\mathsf{W}xx\alpha$	$\langle$S7.1$\rangle$
6.	$\mathsf{M}xy\alpha \quad \mathsf{P}x\alpha$	$\neg\mathsf{P}x\alpha$	$\langle \mathsf{W}, 4, 5\rangle$
7.	$\mathsf{M}xy\alpha$	$\neg\mathsf{P}x\alpha$	$\langle \mathsf{SW}, 6\rangle$
8.		$\mathsf{M}xy\alpha \rightarrow \neg\mathsf{P}x\alpha$	$\langle \mathsf{I}_1, 7\rangle$

S7.6 $\mathsf{M}xy\alpha \rightarrow \neg\mathsf{G}xy$

Proof

1.		$\mathsf{M}xy\alpha \wedge \mathsf{P}z\alpha \rightarrow \mathsf{W}zx\alpha$	$\langle$S7.4$\rangle$
2.	$\mathsf{M}xy\alpha \wedge \mathsf{P}z\alpha$	$\mathsf{W}zx\alpha$	$\langle \mathsf{BI}, 1\rangle$
3.	$\mathsf{M}xy\alpha \quad \mathsf{P}z\alpha$	$\mathsf{W}zx\alpha$	$\langle \mathsf{ZA}, 2\rangle$
4.	$\mathsf{M}xy\alpha \quad \mathsf{P}y_0\alpha$	$\mathsf{W}y_0x\alpha$	$\langle \mathsf{S}^{y_0}_z, 3\rangle$
5.		$\mathsf{V}y_0y\alpha \rightarrow \mathsf{P}y_0\alpha$	$\langle$S5.8$\rangle$
6.	$\mathsf{V}y_0y\alpha$	$\mathsf{P}y_0\alpha$	$\langle \mathsf{BI}, 5\rangle$
7.	$\mathsf{M}xy\alpha \quad \mathsf{V}y_0y\alpha$	$\mathsf{W}y_0x\alpha$	$\langle \mathsf{KS}, 6, 4\rangle$
8.	$\mathsf{M}xy\alpha \quad \mathsf{V}y_0y\alpha \quad x = y_0$	$\mathsf{W}xx\alpha$	$\langle \mathsf{I}^x_{y_0}, 7\rangle$
9.		$\mathsf{G}xy_0 \wedge x \sim y_0 \rightarrow x = y_0$	$\langle$S3.3$\rangle$
10.	$\mathsf{G}xy_0 \wedge x \sim y_0$	$x = y_0$	$\langle \mathsf{BI}, 9\rangle$
11.	$\mathsf{G}xy_0 \quad x \sim y_0$	$x = y_0$	$\langle \mathsf{ZA}, 10\rangle$
12.	$\mathsf{M}xy\alpha \quad x \sim y_0 \quad \mathsf{V}y_0y\alpha \quad \mathsf{G}xy_0$	$\mathsf{W}xx\alpha$	$\langle \mathsf{KS}, 8, 11\rangle$
13.		$\neg\mathsf{W}xx\alpha$	$\langle$S7.1$\rangle$
14.	$\mathsf{M}xy\alpha \quad x \sim y_0 \quad \mathsf{V}y_0y\alpha \quad \mathsf{G}xy_0$	$\neg\mathsf{G}xy_0$	$\langle \mathsf{W}, 12, 13\rangle$
15.	$\mathsf{M}xy\alpha \quad x \sim y_0 \quad \mathsf{V}y_0y\alpha$	$\neg\mathsf{G}xy_0$	$\langle \mathsf{SW}, 14\rangle$
16.		$\mathsf{V}y_0y\alpha \rightarrow \mathsf{G}y_0y$	$\langle$S5.5$\rangle$
17.	$\mathsf{V}y_0y\alpha$	$\mathsf{G}y_0y$	$\langle \mathsf{BI}, 16\rangle$
18.	$\mathsf{M}xy\alpha \quad x \sim y_0 \quad \mathsf{V}y_0y\alpha$	$\neg\mathsf{G}xy_0 \wedge \mathsf{G}y_0y$	$\langle \mathsf{K}_0, 15, 17\rangle$
19.		$\neg\mathsf{G}xy_0 \wedge \mathsf{G}y_0y \rightarrow \neg\mathsf{G}xy$	$\langle$S3.5$\rangle$
20.	$\mathsf{M}xy\alpha \quad x \sim y_0 \quad \mathsf{V}y_0y\alpha$	$\neg\mathsf{G}xy$	$\langle \mathsf{MP}, 18, 19\rangle$
21.	$\mathsf{M}xy\alpha \quad x \sim y_0 \wedge \mathsf{V}y_0y\alpha$	$\neg\mathsf{G}xy$	$\langle \mathsf{VA}, 20\rangle$
22.	$\mathsf{M}xy\alpha \quad \exists y_0(x \sim y_0 \wedge \mathsf{V}y_0y\alpha)$	$\neg\mathsf{G}xy$	$\langle \mathsf{P}''_{y_0}, 21\rangle$
23.		$\mathsf{M}xy\alpha \rightarrow \exists y_0(x \sim y_0 \wedge \mathsf{V}y_0y\alpha)$	$\langle$S6.4$\rangle$

24. $\mathsf{M}xy\alpha \quad \exists y_0(x \sim y_0 \wedge \mathsf{V}y_0 y\alpha)$ ⟨BI, 23⟩
25. $\mathsf{M}xy\alpha \quad \neg\mathsf{G}xy$ ⟨KS, 24, 22⟩
26. $\mathsf{M}xy\alpha \rightarrow \neg\mathsf{G}xy$ ⟨I_1, 25⟩

S7.7 $\mathsf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1 \rightarrow \neg\mathsf{G}xy$

Proof

1. $\mathsf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1 \rightarrow$
   $\mathsf{W}yx\alpha$ ⟨S7.8⟩
2. $\mathsf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1$
   $\mathsf{W}yx\alpha$ ⟨BI, 1⟩
3. $\mathsf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1$
   $\mathsf{W}yx\alpha$ ⟨ZA, 2⟩
4. $\mathsf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1 \quad x = y$
   $\mathsf{W}xx\alpha$ ⟨I^x_y, 3⟩
5. $\mathsf{G}xy \wedge x \sim y \rightarrow x = y$ ⟨S3.3⟩
6. $\mathsf{G}xy \wedge x \sim y \quad x = y$ ⟨BI, 5⟩
7. $\mathsf{G}xy \quad x \sim y \quad x = y$ ⟨ZA, 6⟩
8. $\mathsf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1 \quad \mathsf{G}xy$
   $\mathsf{W}xx\alpha$ ⟨KS, 4, 7⟩
9. $\neg\mathsf{W}xx\alpha$ ⟨S7.1⟩
10. $\mathsf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1 \quad \mathsf{G}xy$
    $\neg\mathsf{G}xy$ ⟨W, 8, 9⟩
11. $\mathsf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1$
    $\neg\mathsf{G}xy$ ⟨SW, 10⟩
12. $\mathsf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1$
    $\neg\mathsf{G}xy$ ⟨VA, 11⟩
13. $\mathsf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1 \rightarrow$
    $\neg\mathsf{G}xy$ ⟨I_1, 12⟩

S7.8 $\mathsf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1 \rightarrow \mathsf{W}yx\alpha$

Proof

1. $\mathsf{M}x_1y_1\alpha \wedge x_1 < x \sim y < y_1 \rightarrow$
   $\mathsf{M}xy_1\alpha$ ⟨A6.6⟩
2. $\mathsf{M}x_1y_1\alpha \wedge x_1 < x \sim y < y_1$
   $\mathsf{M}xy_1\alpha$ ⟨BI, 1⟩
3. $\mathsf{M}x_1y_1\alpha \quad x_1 < x \quad x \sim y \quad y < y_1$
   $\mathsf{M}xy_1\alpha$ ⟨ZA, 2⟩
4. $x_1 = x \quad \mathsf{M}x_1y_1\alpha \quad \mathsf{M}xy_1\alpha$ ⟨ER_1⟩
5. $\mathsf{M}x_1y_1\alpha \quad x_1 < x \vee x_1 = x \quad x \sim y \quad y < y_1$
   $\mathsf{M}xy_1\alpha$ ⟨A_0, 3, 4⟩
6. $x_1 \leq x \quad x_1 \leq x$ ⟨A⟩
7. $x_1 \leq x \quad x_1 < x \vee x_1 = x$ ⟨D3.1, 6⟩
8. $\mathsf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1$
   $\mathsf{M}xy_1\alpha$ ⟨KS, 7, 5⟩
9. $\mathsf{M}xy_1\alpha \wedge \mathsf{P}z\alpha \rightarrow \mathsf{W}zx\alpha$ ⟨S7.4⟩
10. $\mathsf{M}xy_1\alpha \wedge \mathsf{P}z\alpha \quad \mathsf{W}zx\alpha$ ⟨BI, 9⟩

11. $\mathbf{M}xy_1\alpha \quad \mathbf{P}z\alpha \qquad \mathbf{W}zx\alpha$ $\langle\mathbf{ZA}, 10\rangle$
12. $\mathbf{M}xy_1\alpha \quad \mathbf{P}y\alpha \qquad \mathbf{W}yx\alpha$ $\langle\mathbf{S}^y_z, 11\rangle$
13. $\mathbf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1 \quad \mathbf{P}y\alpha$
$\qquad \mathbf{W}yx\alpha$ $\langle\mathbf{KS}, 8, 12\rangle$
14. $\mathbf{M}x_1y_1\alpha \wedge x_1 < x \sim y < y_1 \to$
$\qquad \mathbf{V}yy_1\alpha$ $\langle S6.7\rangle$
15. $\mathbf{M}x_1y_1\alpha \wedge x_1 < x \sim y < y_1$
$\qquad \mathbf{V}yy_1\alpha$ $\langle\mathbf{BI}, 14\rangle$
16. $\mathbf{M}x_1y_1\alpha \quad x_1 < x \quad x \sim y \quad y < y_1$
$\qquad \mathbf{V}yy_1\alpha$ $\langle\mathbf{ZA}, 15\rangle$
17. $\mathbf{M}x_1y_1\alpha \wedge x_1 \sim y < y_1 \to \mathbf{V}yy_1\alpha$ $\langle S6.5\rangle$
18. $\mathbf{M}x_1y_1\alpha \wedge x_1 \sim y < y_1 \qquad \mathbf{V}yy_1\alpha$ $\langle\mathbf{BI}, 17\rangle$
19. $\mathbf{M}x_1y_1\alpha \quad x_1 \sim y \quad y < y_1$
$\qquad \mathbf{V}yy_1\alpha$ $\langle\mathbf{ZA}, 18\rangle$
20. $x_1 = x \quad x \sim y \qquad x_1 \sim y$ $\langle\mathbf{ER}_1\rangle$
21. $\mathbf{M}x_1y_1\alpha \quad x_1 = x \quad x \sim y \quad y < y_1$
$\qquad \mathbf{V}yy_1\alpha$ $\langle\mathbf{KS}, 19, 20\rangle$
22. $\mathbf{M}x_1y_1\alpha \quad x_1 < x \vee x_1 = x \quad x \sim y \quad y < y_1$
$\qquad \mathbf{V}yy_1\alpha$ $\langle\mathbf{A}_0, 16, 21\rangle$
23. $\mathbf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1$
$\qquad \mathbf{V}yy_1\alpha$ $\langle\mathbf{KS}, 7, 22\rangle$
24. $\mathbf{V}yy_1\alpha \to \mathbf{P}y\alpha$ $\langle S5.8\rangle$
25. $\mathbf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1$
$\qquad \mathbf{P}y\alpha$ $\langle\mathbf{MP}, 23, 24\rangle$
26. $\mathbf{M}x_1y_1\alpha \quad x_1 \leq x \quad x \sim y \quad y < y_1$
$\qquad \mathbf{W}yx\alpha$ $\langle\mathbf{KS}, 25, 13\rangle$
27. $\mathbf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1$
$\qquad \mathbf{W}yx\alpha$ $\langle\mathbf{VA}, 26\rangle$
28. $\mathbf{M}x_1y_1\alpha \wedge x_1 \leq x \sim y < y_1 \to$
$\qquad \mathbf{W}yx\alpha$ $\langle\mathbf{I}_1, 27\rangle$

S7.9 $\quad \mathbf{V}y_1y_2\alpha \to \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathbf{M}xy_2\alpha \wedge \neg\mathbf{G}xy_2)$

Proof

1. $\mathbf{M}xy_2\alpha \qquad \mathbf{M}xy_2\alpha$ $\langle\mathbf{A}\rangle$
2. $\mathbf{M}xy_2\alpha \to \neg\mathbf{G}xy_2$ $\langle S7.6\rangle$
3. $\mathbf{M}xy_2\alpha \qquad \neg\mathbf{G}xy_2$ $\langle\mathbf{MP}, 1, 2\rangle$
4. $\mathbf{M}xy_2\alpha \qquad \mathbf{M}xy_2\alpha \wedge \neg\mathbf{G}xy_2$ $\langle\mathbf{K}_0, 1, 3\rangle$
5. $x \sim y \wedge y_1 \leq y < y_2 \qquad x \sim y \wedge y_1 \leq y < y_2$ $\langle\mathbf{A}\rangle$
6. $x \sim y \wedge y_1 \leq y < y_2 \quad \mathbf{M}xy_2\alpha$
$\qquad x \sim y \wedge y_1 \leq y < y_2 \wedge \mathbf{M}xy_2\alpha \wedge \neg\mathbf{G}xy_2$
$\langle\mathbf{K}_0, 5, 4\rangle$
7. $x \sim y \wedge y_1 \leq y < y_2 \quad \mathbf{M}xy_2\alpha$
$\qquad \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathbf{M}xy_2\alpha \wedge \neg\mathbf{G}xy_2)$
$\langle\mathbf{P}'_y, 6\rangle$
8. $x \sim y \wedge y_1 \leq y < y_2 \quad \mathbf{M}xy_2\alpha$
$\qquad \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathbf{M}xy_2\alpha \wedge$
$\qquad \neg\mathbf{G}xy_2)$ $\langle\mathbf{P}'_x, 7\rangle$

9. $x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha$ $\quad \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha \wedge \neg \mathsf{G} x y_2)$ $\langle \mathsf{VA}, 8\rangle$

10. $\exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha)$ $\quad \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha \wedge \neg \mathsf{G} x y_2)$ $\langle \mathsf{P}''_y, 9\rangle$

11. $\exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha)$ $\quad \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha \wedge \neg \mathsf{G} x y_2)$ $\langle \mathsf{P}''_x, 10\rangle$

12. $\mathsf{V} y_1 y_2 \alpha \rightarrow \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha)$ $\langle A6.7\rangle$

13. $\mathsf{V} y_1 y_2 \alpha$ $\quad \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha)$ $\langle \mathsf{BI}, 12\rangle$

14. $\mathsf{V} y_1 y_2 \alpha$ $\quad \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha \wedge \neg \mathsf{G} x y_2)$ $\langle \mathsf{KS}, 13, 11\rangle$

15. $\mathsf{V} y_1 y_2 \alpha \rightarrow \exists x \exists y(x \sim y \wedge y_1 \leq y < y_2 \wedge \mathsf{M} x y_2 \alpha \wedge \neg \mathsf{G} x y_2)$ $\langle \mathsf{I}_1, 14\rangle$

REFERENCES

References [1]-[8] and [9]-[12] are given at the ends of the first and second parts of this paper respectively. See *Notre Dame Journal of Formal Logic*, vol. IX (1968), pp. 371-384, and vol. X (1969), pp. 277-284. They are now supplemented by:

[I] Larouche, L., "Examination of the axiomatic foundations of a theory of change. I," in *Notre Dame Journal of Formal Logic*, IX (1968), pp. 371-384.

[II] Larouche, L., "Examination of the axiomatic foundations of a theory of change. II," in *Notre Dame Journal of Formal Logic*, X (1969), pp. 277-284.

[13] Hermes, H., "Einfürung in die mathematische Logik," Stuttgart (1963).

(To be continued).

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