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A TABLEAU PROOF METHOD ADMITTING THE EMPTY DOMAIN

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1 *Introduction.* There have been several papers concerning systems of first order logic whose theorems are valid in all domains including the empty one. Some, for example, [1, 2] do not admit vacuous quantification. If vacuous quantification is allowed, two definitions of validity in the empty domain are possible, depending on how vacuous quantification is interpreted. Mostowski [5] interprets $(\forall x)A$, where x does not occur free in A, as equivalent to A; Hailperin [3] and Quine [6] interpret $(\forall x)A$ as true over the empty domain. All the preceeding proof systems are axiomatic, however see [4] for a natural deduction system.

In this paper we present simple and intuitive modifications of the tableau proof system of [8] (allowing vacuous quantification): one which produces a logic equivalent to that of Hailperin and Quine, and one which produces a logic equivalent to Mostowski's. We first sketch the classical system, then we present our modifications and sketch proofs of correctness and completeness.

2 *The Classical Tableau System.* We use $x, y, z, \ldots$ for individual variables (free and bound); $a, b, c, \ldots$ for individual parameters; and $A, B, C, \ldots$ to represent formulas. The notion of formula is defined as usual, allowing vacuous quantification. By $A(x/a)$ we mean the result of substituting the parameter a for every free occurrence of the variable x in A. A formula with no free variables is called a closed formula, or a sentence. A formula with no parameters is called pure.

We use the unified notation of Smullyan [7, 8] in which α stands for any essentially conjunctive formula, β for any disjunctive, γ for any universal formula, and δ for any existential. In the charts below we list the four α forms, and give the respective components, denoted α_1 and α_2, and the three β forms and their respective components, denoted β_1 and β_2.

α	α_1	α_2
$A \wedge B$	A	B
$\sim(A \vee B)$	$\sim A$	$\sim B$
$\sim(A \supset B)$	A	$\sim B$
$\sim\sim A$	A	A

β	β_1	β_2
$A \vee B$	A	B
$\sim(A \wedge B)$	$\sim A$	$\sim B$
$A \supset B$	$\sim A$	B

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Similarly, the two γ forms and their respective instances, and the two δ forms and their instances, are given in the following charts.

γ	$\gamma(a)$
$(\forall x)A$	$A(x/a)$
$\sim(\exists x)A$	$\sim A(x/a)$

δ	$\delta(a)$
$(\exists x)A$	$A(x/a)$
$\sim(\forall x)A$	$\sim A(x/a)$

Proofs are in tree form, with the origin at the top. If θ is a branch of a tree, by $\langle\theta, A\rangle$ we mean the result of lengthening θ by adding A at the bottom.

If S is a finite set of closed formulas, a tableau for S is any tree constructed by the following recursive rules.

If θ is a sequence of all elements of S, the tree whose only branch is θ is a tableau for S.

Suppose $\mathcal{T}$ is a tableau for S, θ is a branch of $\mathcal{T}$, and α occurs in θ. The result of replacing θ in $\mathcal{T}$ by $\langle\theta, \alpha_1, \alpha_2\rangle$ is a tableau for S.

If $\mathcal{T}$ is a tableau for S, θ is a branch of $\mathcal{T}$, and β occurs in θ, the result of replacing θ in $\mathcal{T}$ by the two branches $\langle\theta, \beta_1\rangle$ and $\langle\theta, \beta_2\rangle$ is a tableau for S.

If $\mathcal{T}$ is a tableau for S, θ is a branch of $\mathcal{T}$, and γ occurs in θ, the result of replacing θ in $\mathcal{T}$ by the branch $\langle\theta, \gamma(a)\rangle$, where a is any parameter, is a tableau for S.

If $\mathcal{T}$ is a tableau for S, θ is a branch of $\mathcal{T}$, δ occurs in θ, and a is a parameter *not occurring* in any formula in θ, the result of replacing θ in $\mathcal{T}$ by $\langle\theta, \delta(a)\rangle$ is a tableau for S.

The above rules for extending branches may be codified as follows.

$$\frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 \mid \beta_2} \qquad \frac{\gamma}{\gamma(a)} \qquad \frac{\delta}{\delta(a)}\ \text{provided } a \text{ is `new'}$$

A branch of a tableau is called *closed* if, for some formula A, both A and $\sim A$ are on the branch. A tableau is called closed if each branch is closed.

If A is a pure sentence, A is a theorem in the above system if there is a closed tableau for $\{\sim A\}$. Correctness and completeness are established in [8].

3 *The Modified Tableau Systems.* As we remarked in the introduction, two notions of validity in the empty domain are possible. If x has no free occurrences in A, Mostowski interprets $(\forall x)A$ and $(\exists x)A$ to be equivalent to A, even over the empty domain, emphasizing the word *vacuous* in vacuous quantification. Hailperin and Quine take $(\forall x)A$ to be true and $(\exists x)A$ to be false in the empty domain, whether or not x occurs free in A, emphasizing the word *quantifiers*.

We propose a restriction on the γ-rule above. Some of the words in the restriction are subject to two interpretations. Read one way, the resulting system is equivalent to Mostowski's, read the other, to that of Hailperin and Quine. The restriction is simply this:

No parameter may be used in extending a branch by an application of the γ-rule unless it already occurs in a formula on that branch.

The ambiguity referred to above comes in interpreting the words 'used' and 'occurs' in the case of vacuous quantification. Suppose, at some stage in the construction of a branch θ of a tableau, we encountered the formula $(\forall x)A$, where x had no free occurrences in A, and, using the γ-rule, we added $A(x/a)$ to θ. Did we use the parameter a or not? Similarly, if x did not occur free in A, $(\exists x)A$ occurred on θ, and we added $A(x/a)$ to θ using the δ-rule, does the parameter a now occur on the branch or not? Let us call the usage of 'used' and 'occurs' in which we say a *was* used or *does* occur (albeit vacuously) the *liberal* usage, and the usage in which we say a *was not* used or *does not* occur, the *strict* usage.

In the above modified system, if the words of the restriction are interpreted strictly, the resulting system is equivalent to that of Mostowski. If the words are interpreted liberally, the resulting system is equivalent to the system of Hailperin and Quine.

4 *Correctness*. In this section we sketch a proof that anything provable in one of the above systems is valid in an appropriate sense.

We call I a Mostowski interpretation in a domain D if (1) D is not empty and I is an interpretation in the usual sense, or (2) D is empty and I is again an interpretation in the usual sense, but following the convention that if x has no free occurrences in A, $(\forall x)A$ and $(\exists x)A$ are identical with A under the interpretation I. Similarly we call I a Hailperin interpretation in the domain D if either (1) D is not empty and I is an interpretation in the usual sense, or (2) D is empty and I is an interpretation in the usual sense, but with the convention that $(\forall x)A$ is true and $(\exists x)A$ is false under I.

We call a formula Mostowski valid (Hailperin valid) if it is true in *all* domains under all Mostowski interpretations (Hailperin interpretations).

A formula A is Mostowski satisfiable (Hailperin satisfiable) if, under some Mostowski (Hailperin) interpretation I in some domain D, A is true. A branch of a tableau is called Mostowski (Hailperin) satisfiable if the set of signed formulas on it is simultaneously Mostowski (Hailperin) satisfiable. A tableau is Mostowski (Hailperin) satisfiable if some branch is.

If $\mathcal{T}$ is a tableau and $\mathcal{T}'$ results from $\mathcal{T}$ by the application of one of the four tableau rules to a branch of $\mathcal{T}$, using the strict (liberal) restriction on the γ-rule, we call $\mathcal{T}'$ a strict (liberal) extension of $\mathcal{T}$.

Lemma. *Let $\mathcal{T}$ be a tableau for some pure set of sentences S. If $\mathcal{T}$ is Mostowski (Hailperin) satisfiable, and $\mathcal{T}'$ is any strict (liberal) extension of $\mathcal{T}$, $\mathcal{T}'$ is Mostowski (Hailperin) satisfiable.*

The proof of the above lemma is a straightforward variant of the corresponding lemma for the classical system of section 2; for details of that proof, see [8].

Theorem. *For a pure sentence A, if A is provable in the strict (liberal) system, then A is Mostowski (Hailperin) valid.*

Proof: If A is not Mostowski (Hailperin) valid, $\sim A$ is Mostowski (Hailperin) satisfiable. Then if A were strictly (liberally) provable, by the above lemma, some closed tableau must be Mostowski (Hailperin) satisfiable, which is impossible.

5 *Hintikka Sets.* By a vacuous γ we mean a formula of the form $(\forall x)A$ or $\sim(\exists x)A$ where x does not occur free in A; similarly for vacuous δ. For vacuous γ and δ formulas we define null instances as follows:

γ	$\gamma(\phi)$
$(\forall x)A$	A
$\sim(\exists x)A$	$\sim A$

δ	$\delta(\phi)$
$(\exists x)A$	A
$\sim(\forall x)A$	$\sim A$

Adapting a definition from [8], if S is a set of sentences and P is a set of parameters, we call S a Mostowski-Hintikka set with respect to P if:

(0) for no atomic formula A do both A and $\sim A$ belong to S.
(1) (a) $\alpha \in S \Longrightarrow \alpha_1 \in S$ and $\alpha_2 \in S$
 (b) $\beta \in S \Longrightarrow \beta_1 \in S$ or $\beta_2 \in S$
(2) if $P \neq \phi$,
 (a) $\gamma \in S \Longrightarrow \gamma(a) \in S$ for all $a \in P$
 (b) $\delta \in S \Longrightarrow \delta(a) \in S$ for some $a \in P$
(3) if $P = \phi$ and δ is not vacuous, $\delta \notin S$
(4) if $P = \phi$ and γ and δ are vacuous,
 (a) $\gamma \in S \Longrightarrow \gamma(\phi) \in S$
 (b) $\delta \in S \Longrightarrow \delta(\phi) \in S$

Similarly we call S a Hailperin-Hintikka set with respect to P if (0) - (2) as above, and

(3′) if $P = \phi$, $\delta \notin S$.

Let us call a formula $A \in S$ Mostowski-minimal if (1) A is atomic or the negation of an atomic formula, or (2) $P = \phi$ and A is a non-vacuous γ. Similarly, we call $A \in S$ Hailperin-minimal if (1) as above, or (2) $P = \phi$ and A is some γ.

Theorem. *If S is a Mostowski-Hintikka set with respect to P, S is Mostowski satisfiable.*

Proof: Let the domain D of our model be the set P. We may begin to define a Mostowski interpretation I so that if A is a Mostowski-minimal element of S, A is true under the Mostowski interpretation we are defining. It can be shown that such an assignment of truth values to Mostowski-minimal formulas of S can always be extended to a Mostowski interpretation (in general, many) for all formulas, and that, under any such Mostowski interpretation I, for any formula A, if $A \in S$, A is true under I.

Remark: If P is not empty, this becomes Hintikka's lemma for first order logic, and a proof in detail may be found in [8, chapter 5, section 3].

Theorem. *If S is a Hailperin-Hintikka set with respect to P, S is Hailperin satisfiable.*

The proof of this theorem is like the above except that we now use Hailperin-minimal elements of S instead of Mostowski-minimal ones.

6 *Completeness.* We describe a systematic procedure for constructing a tableau for $\{\sim A\}$ which will either close, and so prove A, or provide us with an appropriate Hintikka set containing $\sim A$. The procedure is taken from [8] with only minor changes. It involves designating some formulas on a tableau 'finished'.

Let A be a pure closed formula. Begin a tableau for $\{\sim A\}$ by placing $\sim A$ at the origin. Then we apply the four branch extension rules systematically as follows. Suppose, at the n^{th} stage, the tableau we have is closed, then stop. Also, if no branch extension rule applies to any formula which is not finished, stop. If neither of these is the case, pick an occurrence of a formula B, as high up on the tree as possible, which is not yet finished, and extend the tableau as follows: take every unclosed branch θ passing through B (that is, on which B lies) and

(1) If B is an α, replace θ by the branch $\langle \theta, \alpha_1, \alpha_2 \rangle$
(2) If B is a β, replace θ by the two branches $\langle \theta, \beta_1 \rangle$ and $\langle \theta, \beta_2 \rangle$
(3) If B is a δ, take the first parameter a which does not occur in θ and replace θ by $\langle \theta, \delta(a) \rangle$
(4) If B is a γ, let $\{a, b, c, \ldots, m\}$ be the (finite) set of parameters which occur in formulas of θ, and replace θ by the branch $\langle \theta, \gamma(a), \gamma(b), \gamma(c), \ldots, \gamma(m) \rangle$

Having done one of (1) - (4) for each branch passing through B, declare that occurrence of B finished. This concludes the $n+1^{\text{st}}$ stage of the systematic procedure.

Suppose A is a pure closed formula and we construct a systematic tableau for $\{\sim A\}$ using the strict interpretation of the γ-rule restriction (which affects the word 'occur' in clauses (3) and (4) above). If the tableau is closed, A is a theorem of the strict system. If the tableau is not closed, any open branch is a Mostowski-Hintikka set with respect to the set of parameters strictly occurring on that branch. Then, by the theorem of section 5, there is a Mostowski interpretation in which A is false. So we have the completeness of the strict system relative to Mostowski validity.

Similarly, if we use the liberal interpretation of the γ-rule restriction, we may establish completeness of the liberal system with respect to Hailperin interpretations.

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