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On General Boundedness and Dominating Cardinals

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Abstract For cardinals κ, λ, μ we let $\mathfrak{b}_{\kappa,\lambda,\mu}$ be the smallest size of a subset B of ${}^{\lambda}\mu$ unbounded in the sense of $\leq_\kappa$; that is, such that there is no function $f \in {}^{\lambda}\mu$ such that $\{\alpha < \lambda : g(\alpha) > f(\alpha)\}$ has size less than κ for all $g \in B$. Similarly for $\mathfrak{d}_{\kappa,\lambda,\mu}$, the general dominating number, which is the smallest size of a subset B of ${}^{\lambda}\mu$ such that for every $g \in {}^{\lambda}\mu$ there is an $f \in B$ such that the above set has size less than κ. These cardinals are generalizations of the usual ones for $\kappa = \lambda = \mu = \omega$. When all three are the same regular cardinal, the relationships between them have been completely described by Cummings and Shelah. We also consider some variants of the functions, following van Douwen, in particular the version $\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$ of $\mathfrak{b}_{\kappa,\lambda,\mu}$ in which B is required to consist of strictly increasing functions. Some of the main results of this paper are: (1) $\mathfrak{b}_{\mu,\mu,\mathrm{cf}\mu} \leq \mathfrak{b}_{\mathrm{cf}\mu,\mathrm{cf}\mu,\mathrm{cf}\mu}$; (2) for $\lambda \leq \mu$, $\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$ always exists; (3) if $\mathrm{cf}\lambda = \mathrm{cf}\mu < \lambda \leq \mu$, then $\mathfrak{b}_{\mathrm{cf}\mu,\mathrm{cf}\mu,\mathrm{cf}\mu} = \mathfrak{b}^{\uparrow}_{\lambda,\lambda,\mu}$; (4) $\mathfrak{d}_{\omega,\mu,\mu} = \mathfrak{d}_{1,\mu,\mu}$. For background see Section 1 of the paper. Several open problems are stated.

1 Definitions

We make the standing assumptions that we have cardinals κ, λ, μ with (1) $\kappa = 1$ or κ is infinite, (2) $\kappa \leq \lambda$, and (3) λ and μ are infinite. Note in particular that we allow for the possibility that $\lambda > \mu$. The definitions of our functions depend on some quasi orders defined as follows. For $f, g \in {}^{\lambda}\mu$ we write

$$\begin{aligned}
f \leq_\kappa g &\quad \text{iff} \quad |\{\xi < \lambda : f(\xi) > g(\xi)\}| < \kappa,\\
f <_\kappa g &\quad \text{iff} \quad |\{\xi < \lambda : f(\xi) \geq g(\xi)\}| < \kappa,\\
f \leq g &\quad \text{iff} \quad \forall \xi < \lambda[f(\xi) \leq g(\xi)],\\
f < g &\quad \text{iff} \quad \forall \xi < \lambda[f(\xi) < g(\xi)].
\end{aligned}$$

The following obvious proposition can be used to fill in some details below.

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Proposition 1.1

(i) *If* $f \leq_\kappa g \leq_\kappa h$ *then* $f \leq_\kappa h$.
(ii) *If* $f <_\kappa g <_\kappa h$*, then* $f <_\kappa h$.
(iii) *If* $f \leq g$*, then* $f \leq_\kappa g$.
(iv) *If* $f < g$*, then* $f <_\kappa g$.
(v) *If* $f \leq_\kappa g < h$*, then* $f <_\kappa h$.
(vi) *If* $f < g \leq_\kappa h$*, then* $f <_\kappa h$.
(vii) *If* $f <_\kappa g$*, then* $f \leq_\kappa g$.
(viii) $f < g$ *iff* $f <_1 g$.
(ix) $f \leq g$ *iff* $f \leq_1 g$. $\dashv$[1]

Given $B \subseteq {}^\lambda\mu$, a $\leq_\kappa$*-bound* for B is an element $g \in {}^\lambda\mu$ such that $f \leq_\kappa g$ for all $f \in B$. By Proposition 1.1(v), (vii), this is equivalent to saying that there is an element $g \in {}^\lambda\mu$ such that $f <_\kappa g$ for all $f \in B$.

A subset B of ${}^\lambda\mu$ is $\leq_\kappa$*-dominating* if for every $f \in {}^\lambda\mu$ there is a $g \in B$ such that $f \leq_\kappa g$. Again we can say $f <_\kappa g$ here.

Some other notions enter into the definitions of some of our functions. We say that $B \subseteq {}^\lambda\mu$ is $\leq_\kappa$*-unbounded on* $A \in [\lambda]^\lambda$ provided that for every $f \in {}^\lambda\mu$ there is a $g \in B$ such that $|\{\alpha \in A : f(\alpha) < g(\alpha)\}| \geq \kappa$. A (κ, μ, λ)*-scale* is a subset B of ${}^\lambda\mu$ which is $\leq_\kappa$-dominating and well-ordered by $<_\kappa$.

Now we are in a position to define the various boundedness and dominating numbers. We do not discuss now when the indicated minimums actually exist.

$$\begin{aligned}
\mathfrak{b}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } \leq_\kappa\text{-unbounded subset of } {}^\lambda\mu\};\\
\mathfrak{d}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } \leq_\kappa\text{-dominating subset of } {}^\lambda\mu\};\\
\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } \leq_\kappa\text{-unbounded subset of } {}^\lambda\mu\\
&\qquad \text{consisting of strictly increasing functions}\};\\
\mathfrak{d}^{\uparrow}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } \leq_\kappa\text{-dominating subset of } {}^\lambda\mu\\
&\qquad \text{consisting of strictly increasing functions}\};\\
\mathfrak{b}^{\mathrm{wo}}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } \leq_\kappa\text{-unbounded subset of } {}^\lambda\mu\\
&\qquad \text{and } B \text{ is well-ordered by } <_\kappa\};\\
\mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } (\kappa, \lambda, \mu)\text{-scale}\};\\
\mathfrak{b}^{\uparrow\mathrm{wo}}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } \leq_\kappa\text{-unbounded set of strictly}\\
&\qquad \text{increasing members of } {}^\lambda\mu \text{ well-ordered by } <_\kappa\};\\
\mathfrak{d}^{\uparrow\mathrm{wo}}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is a } \leq_\kappa\text{-dominating set of strictly}\\
&\qquad \text{increasing members of } {}^\lambda\mu \text{ well-ordered by } <_\kappa\};\\
\mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu} &= \min\{|B| : B \text{ is } \leq_\kappa\text{-unbounded on every } A \in [\lambda]^\lambda\}.
\end{aligned}$$

For μ and ν regular, the relationships between $\mathfrak{b}_{\mu,\mu,\mu}$, $\mathfrak{d}_{\mu,\mu,\mu}$, $\mathfrak{b}_{\nu,\nu,\nu}$, and $\mathfrak{d}_{\nu,\nu,\nu}$ have been completely described by Cummings and Shelah [1]. The function $\mathfrak{d}_{\lambda,\lambda,\mu}$ has been investigated extensively by Szymański [3]. He also gives some consistency results when $\mu^\lambda \leq 2^\omega$, extending results of Jech and Prikry [2]. Several of Szymański's results generalize in an obvious way to $\mathfrak{d}_{\kappa,\lambda,\mu}$. To make this paper self-contained, we give proofs for these results as well as our new theorems. We merely state some of the main consistency results given in those papers, however.

2 The Main Functions $\mathfrak{b}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}_{\kappa,\lambda,\mu}$

Clearly ${}^{\lambda}\mu$ itself is $\leq_\kappa$-unbounded and $\leq_\kappa$-dominating, so $\mathfrak{b}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}_{\kappa,\lambda,\mu}$ always exist.

Proposition 2.1 *If a set $B \subseteq {}^{\lambda}\mu$ is $\leq_\kappa$-dominating, then it is also $\leq_\kappa$-unbounded.*

Proof Suppose not: say $f \leq_\kappa g$ for all $f \in B$. Define $h(\xi) = g(\xi) + 1$ for all $\xi < \lambda$. Since B is $\leq_\kappa$-dominating, choose $f \in B$ such that $h \leq_\kappa f$. Thus $h \leq_\kappa f \leq_\kappa g$, so by 1.1(i), $h \leq_\kappa g$. But $\{\xi < \lambda : h(\xi) > g(\xi)\} = \lambda \geq \kappa$, contradiction. □

Corollary 2.2 $\mathfrak{b}_{\kappa,\lambda,\mu} \leq \mathfrak{d}_{\kappa,\lambda,\mu} \leq \mu^{\lambda}$. ⊣

Proposition 2.3

(i) $\mathfrak{b}_{\kappa,\lambda,\mu}$ *is regular.*
(ii) $\mathfrak{b}_{\kappa,\lambda,\mu} \leq \mathrm{cf}\mathfrak{d}_{\kappa,\lambda,\mu}$.

Proof (i) Suppose that $\mathfrak{b}_{\kappa,\lambda,\mu}$ is singular, $B \subseteq {}^{\lambda}\mu$ is $\leq_\kappa$-unbounded, and $|B| = \mathfrak{b}_{\kappa,\lambda,\mu}$. Write $B = \bigcup_{\alpha<\mathrm{cf}\mathfrak{b}_{\kappa,\lambda,\mu}} B_\alpha$ with each $|B_\alpha| < \mathfrak{b}_{\kappa,\lambda,\mu}$. Then there is a $\leq_\kappa$ bound g_α for B_α. A $\leq_\kappa$-bound for $\{g_\alpha : \alpha < \mathrm{cf}\mathfrak{b}_{\kappa,\lambda,\mu}\}$ is a $\leq_\kappa$-bound for B, contradiction.

(ii) By 2.2 we may assume that $\mathfrak{d}_{\kappa,\lambda,\mu}$ is singular. Suppose that $\mathrm{cf}\mathfrak{d}_{\kappa,\lambda,\mu} < \mathfrak{b}_{\kappa,\lambda,\mu}$. Let $B \subseteq {}^{\lambda}\mu$ be $\leq_\kappa$ dominating with $|B| = \mathfrak{d}_{\kappa,\lambda,\mu}$. Write $B = \bigcup_{\alpha<\mathrm{cf}\mathfrak{d}_{\kappa,\lambda,\mu}} C_\alpha$ with $|C_\alpha| < \mathfrak{d}_{\kappa,\lambda,\mu}$ for each $\alpha < \mathrm{cf}\mathfrak{d}_{\kappa,\lambda,\mu}$. For each $\alpha < \mathrm{cf}\mathfrak{d}_{\kappa,\lambda,\mu}$ choose $f^\alpha \in {}^{\lambda}\mu$ such that $f^\alpha \not\leq_\kappa g$ for all $g \in C_\alpha$. Then choose $h \in B$ such that $f^\alpha \leq_\kappa h$ for all $\alpha < \mathrm{cf}\mathfrak{d}_{\kappa,\lambda,\mu}$. Say $h \in C_\alpha$. Hence $f^\alpha \not\leq_\kappa h$, contradiction. □

Proposition 2.4 *Assume that $\kappa \leq \kappa' \leq \lambda$, with each of κ, κ' satisfying the standing condition for κ, namely, equal to 1 or infinite. Then*

(i) *if $B \subseteq {}^{\lambda}\mu$ is $\leq_{\kappa'}$-unbounded, then it is $\leq_\kappa$-unbounded;*
(ii) *if $D \subseteq {}^{\lambda}\mu$ is $\leq_\kappa$-dominating, then it is $\leq_{\kappa'}$-dominating;*
(iii) $\mathfrak{b}_{\kappa,\lambda,\mu} \leq \mathfrak{b}_{\kappa',\lambda,\mu}$;
(iv) $\mathfrak{d}_{\kappa',\lambda,\mu} \leq \mathfrak{d}_{\kappa,\lambda,\mu}$.

Proof (i) Suppose that $f \leq_\kappa g$ for all $f \in B$. Then for any $f \in {}^{\lambda}\mu$ we have

$$|\{\alpha < \lambda : f(\alpha) > g(\alpha)\}| < \kappa \leq \kappa',$$

and this contradicts B being $\leq_{\kappa'}$-unbounded.

(ii) Given $f \in {}^{\lambda}\mu$, choose $g \in D$ such that $f \leq_\kappa g$. Clearly then $f \leq_{\kappa'} g$, as desired.

(iii) and (iv) are immediate from (i) and (ii). □

For the next proposition, cf. [3], Lemma 2.3.

Proposition 2.5 *If $\omega \leq \lambda \leq \lambda'$, then $\mathfrak{b}_{\kappa,\lambda',\mu} \leq \mathfrak{b}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}_{\kappa,\lambda,\mu} \leq \mathfrak{d}_{\kappa,\lambda',\mu}$.*

Proof For $\mathfrak{b}$, suppose that $B \subseteq {}^{\lambda}\mu$ is $\leq_\kappa$-unbounded, with $|B| = \mathfrak{b}_{\kappa,\lambda,\mu}$. For each $f \in {}^{\lambda}\mu$ define $f^+ \in {}^{\lambda'}\mu$ by setting, for any $\xi < \lambda'$,

$$f^+(\xi) = \begin{cases} f(\xi) & \text{if } \xi < \lambda, \\ 0 & \text{otherwise} \end{cases}$$

Let $B' = \{f^+ : f \in B\}$. We claim that B' is $\leq_\kappa$-unbounded (as desired). For, suppose to the contrary that $f^+ \leq_\kappa g \in {}^{\lambda'}\mu$ for all $f \in B$. Thus for any $f \in B$,

$$\kappa > |\{\xi < \lambda' : f^+(\xi) > g(\xi)\}| = |\{\xi < \lambda : f(\xi) > g(\xi)\}|,$$

contradiction.

For $\mathfrak{d}$, suppose that $B \subseteq {}^{\lambda'}\mu$ is $\leq_\kappa$-dominating and $|B| = \mathfrak{d}_{\mu,\lambda',\kappa}$. Let $B' = \{f \restriction \lambda : f \in B\}$. We claim that B' is $\leq_\kappa$-dominating (as desired). For, let $g \in {}^\lambda\mu$ be given. Let g^+ be defined as above. Choose $f \in B$ such that $g^+ \leq_\kappa f$. Then

$$\kappa > |\{\xi < \lambda' : g^+(\xi) > f(\xi)\}| = |\{\xi < \lambda : g(\xi) > f(\xi)\}|,$$

so $g \leq_\kappa (f \restriction \lambda)$. □

For Proposition 2.6(ii), cf. Szymański [3], Lemma 2.1.

Proposition 2.6

(i) $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{b}_{\kappa,\lambda,\mathrm{cf}\mu}$.
(ii) $\mathfrak{d}_{\kappa,\lambda,\mu} = \mathfrak{d}_{\kappa,\lambda,\mathrm{cf}\mu}$.

Proof We may assume that μ is singular. Let $\langle \nu_\xi : \xi < \mathrm{cf}\mu \rangle$ be a strictly increasing sequence of cardinals with supremum μ. For each $f \in {}^\lambda\mu$ and each $\alpha < \lambda$ let $f^-(\alpha)$ be the least $\xi < \mathrm{cf}\mu$ such that $f(\alpha) \leq \nu_\xi$; and for each $g \in {}^\lambda\mathrm{cf}\mu$ and each $\alpha < \lambda$ let $g^+(\alpha) = \nu_{g(\alpha)}$. Thus if $f \in {}^\lambda\mu$, $g \in {}^\lambda\mathrm{cf}\mu$, and $\alpha < \lambda$, then

$$(*)\qquad f(\alpha) > g^+(\alpha) \text{ iff } f(\alpha) > \nu_{g(\alpha)} \text{ iff } \nu_{f^-(\alpha)} > \nu_{g(\alpha)} \text{ iff } f^-(\alpha) > g(\alpha);$$

it follows that $f \leq_\kappa g^+$ if and only if $f^- \leq_\kappa g$. Hence if $B \subseteq {}^\lambda\mu$ is $\leq_\kappa$ unbounded, then $\{f^- : f \in B\}$ is $\leq_\kappa$ unbounded. This gives $\geq$ in (i). Suppose that $C \subseteq {}^\lambda\mathrm{cf}\mu$ is $\leq_\kappa$-unbounded; we claim that $\{g^+ : g \in C\}$ is $\leq_\kappa$-unbounded, thus giving $\leq$ in (i). For, suppose that $g^+ \leq_\kappa f \in {}^\lambda\mu$ for all $g \in C$. Define $h(\alpha) = f(\alpha) + 1$ for all $\alpha \in \lambda$. Then $g^+ <_\kappa h$ for all $g \in C$, and then $(*)$ gives $g <_\kappa h^-$ for all $g \in C$, contradiction.

(ii) follows from $(*)$ similarly. □

By this proposition, we can restrict to the case μ regular for most purposes.

As applications of these simple results we have the following inequalities, which we will discuss later:

$$\begin{aligned}\mathfrak{b}_{1,\aleph_\omega,\omega} &\leq \mathfrak{b}_{\omega,\aleph_\omega,\omega} \leq \cdots \leq \mathfrak{b}_{\omega,\omega_2,\omega} \leq \mathfrak{b}_{\omega,\omega_1,\omega} \leq \mathfrak{b}_{\omega,\omega,\omega} \leq \mathrm{cf}\mathfrak{d}_{\omega,\omega,\omega}\\ &\leq \mathfrak{d}_{\omega,\omega,\omega} \leq \mathfrak{d}_{\omega,\omega_1,\omega} \leq \mathfrak{d}_{\omega,\omega_2,\omega} \leq \cdots \leq \mathfrak{d}_{\omega,\aleph_\omega,\omega} \leq \mathfrak{d}_{1,\aleph_\omega,\omega}.\end{aligned}$$

Proposition 2.7 *Assume that $B \subseteq {}^\lambda\mu$ with $|B| < \mathrm{cf}\mu$. Then B is $\leq$-bounded, and hence is not $\leq$-dominating.*

Proof Define $g(\xi) = \sup\{f(\xi) + 1 : f \in B\}$ for all $\xi < \lambda$. Since $|B| < \mathrm{cf}\mu$, we have $g(\xi) < \mu$ for all $\xi < \lambda$. For any $f \in B$, $\{\xi : f(\xi) > g(\xi)\} = 0$, so $f \leq g$. Thus g is a $\leq$-upper bound for B. So B is $\leq$-bounded. By 2.1 it is not $\leq$-dominating. □

Corollary 2.8 *If μ is regular then $\mathfrak{b}_{\kappa,\lambda,\mu} \geq \mu$.* ⊣

The following result is well known.

Proposition 2.9 *For μ regular we have $\mathfrak{b}_{\mu,\mu,\mu} > \mu$.*

Proof Suppose that $B \subseteq {}^{\mu}\mu$ with $|B| \leq \mu$; we want to find a $\leq_\mu$-bound for B. Let $B = \{f_\xi : \xi < \mu\}$. For any $\eta < \mu$, let $g(\eta) = \sup_{\xi \leq \eta} f_\xi(\eta)$. Then for any $\xi < \mu$ we have $\{\eta : f_\xi(\eta) > g(\eta)\} \subseteq \xi$, and so $|\{\eta : f_\xi(\eta) > g(\eta)\}| < \mu$, as desired. □

Proposition 2.10 *Suppose that $\lambda < \mathrm{cf}\mu$. Then there is a set $B \subseteq {}^{\lambda}\mu$ of strictly increasing functions such that B is $<$-dominating, B is well-ordered by $<$, and $|B| = \mathrm{cf}\mu$.*

Proof Let $\alpha_\xi \uparrow \mu$ for $\xi < \mathrm{cf}\mu$, the α_ξ's ordinals. We define a new sequence $\langle \beta_\xi : \xi < \mathrm{cf}\mu\rangle$ by recursion. $\beta_0 = 0$; β is continuous; and $\beta_{\xi+1} = \max\{\alpha_\xi, \beta_\xi + \lambda\}$. Now for all $\xi < \mathrm{cf}\mu$ and $\eta < \lambda$ let $f_\xi(\eta) = \beta_\xi + \eta$. Let $B = \{f_\xi : \xi < \mathrm{cf}\mu\}$. To see that B is $<$-dominating, let $g \in {}^{\lambda}\mu$ be given. For every $\eta < \lambda$ choose $\xi_\eta < \mathrm{cf}\mu$ such that $g(\eta) < \beta_{\xi_\eta}$. Let $\rho = \sup_{\eta<\lambda} \xi_\eta$. So $\rho < \mathrm{cf}\mu$ by our assumption, and hence for any $\eta < \lambda$, $g(\eta) < \beta_{\xi_\eta} \leq \beta_\rho \leq f_\rho(\eta)$.

Clearly $f_\xi < f_\eta$ if $\xi < \eta < \mathrm{cf}\mu$. □

Corollary 2.11 *If μ is regular and $\lambda < \mu$, then $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{d}_{\kappa,\lambda,\mu} = \mathfrak{d}_{1,\lambda,\mu} = \mu$.*

Proof

$$\begin{aligned} \mu &\leq \mathfrak{b}_{\kappa,\lambda,\mu} && \text{by 2.8} \\ &\leq \mathfrak{d}_{\kappa,\lambda,\mu} && \text{by 2.2} \\ &\leq \mathfrak{d}_{1,\lambda,\mu} && \text{by 2.4} \\ &\leq \mu && \text{by 2.10} \end{aligned}$$

□

For the following result, cf. [3], Lemma 2.2.

Proposition 2.12 *Every $\leq_\kappa$-dominating subset B of ${}^{\lambda}\mu$ has size $> \lambda$.*

Proof Suppose, to the contrary, that $B \subseteq {}^{\lambda}\mu$ is $\leq_\kappa$-dominating and $B = \{g_\xi : \xi < \lambda\}$. Let k be a bijection from λ onto $\lambda \times \lambda$. For a member x of $\lambda \times \lambda$ we write $x = (x_0, x_1)$. Now we define $f : \lambda \to \mu$ by setting, for any $\eta < \lambda$, $f(\eta) = g_{k(\eta)_0}(\eta) + 1$. Choose $\xi < \lambda$ such that $f \leq_\kappa g_\xi$. Now if $k(\eta)_0 = \xi$, then $f(\eta) = g_\xi(\eta) + 1$. Hence $|\{\eta < \lambda : f(\eta) > g_\xi(\eta)\}| = \lambda$, contradiction. □

Corollary 2.13 $\mathfrak{d}_{\kappa,\lambda,\mu} > \lambda$. ⊣

We shall see that the situation is different for $\mathfrak{b}_{\kappa,\lambda,\mu}$. For the next result we need a simple set-theoretic lemma.

Lemma 2.14 *Suppose that μ is regular. Assume that $\langle \Gamma_\xi : \xi < \mu\rangle$ is a system of subsets of λ such that $\bigcup_{\xi<\mu} \Gamma_\xi = \lambda$ and for all $\xi < \eta$ with both in μ we have $\Gamma_\xi \subseteq \Gamma_\eta$. Also suppose that one of the following conditions holds:*

(i) $\kappa, \mu < \lambda$.
(ii) $\kappa < \mu$.
(iii) $\mu < \kappa = \lambda$ *and* $\mathrm{cf}\lambda \neq \mu$.

Then there is a $\xi < \mu$ such that $|\Gamma_\xi| \geq \kappa$.

Proof Suppose to the contrary that $|\Gamma_\xi| < \kappa$ for all $\xi < \mu$. If (i) holds, then

$$\lambda = \bigcup_{\xi<\mu} \Gamma_\xi = \left|\bigcup_{\xi<\mu} \Gamma_\xi\right| \leq \sum_{\xi<\mu} |\Gamma_\xi| \leq \kappa \cdot \mu < \lambda,$$

contradiction. For (ii) and (iii), first note:

(1) $\forall \xi < \mu \exists \eta \in (\xi, \mu)[\Gamma_\xi \subset \Gamma_\eta]$. In fact, otherwise we get $\xi < \mu$ such that $\Gamma_\xi = \bigcup_{\eta<\mu} \Gamma_\eta = \lambda$, while $|\Gamma_\xi| < \kappa \leq \lambda$, contradiction. So (1) holds.

By (1) we may assume that if $\xi < \eta < \mu$ then $\Gamma_\xi \subset \Gamma_\eta$. If (ii) holds, then $|\Gamma_\kappa| \geq \kappa$, contradiction.

Now suppose that (iii) holds. If $\mu < \mathrm{cf}\lambda$ a contradiction is immediate. So, suppose that $\mathrm{cf}\lambda < \mu$. Thus λ is singular. Let $\beta_\xi \uparrow \lambda$ for $\xi < \mathrm{cf}\lambda$, the β_ξ's cardinals. If there is a cardinal $\rho < \lambda$ such that $|\Gamma_\xi| < \rho$ for all $\xi < \mu$, then $\lambda \leq \rho \cdot \mu$, contradiction. Thus

(2) For every cardinal $\rho < \lambda$ there is a $\xi < \mu$ such that $|\Gamma_\xi| \geq \rho$. For each $\eta < \mathrm{cf}\lambda$ choose $\alpha_\eta < \mu$ such that $|\Gamma_{\alpha_\eta}| \geq \beta_\eta$. Let $\gamma = \sup_{\eta<\mathrm{cf}\lambda} \alpha_\eta$; so $\gamma < \mu$. Then $|\Gamma_\gamma| \geq \lambda$, contradiction. $\square$

Proposition 2.15 *For μ regular the following conditions are equivalent:*

(a) *conditions (i), (ii), and (iii) of 2.14 all fail to hold;*
(b) $\kappa = \lambda$ *and* $\mu = \mathrm{cf}\lambda$.

Proof (a)⇒(b) Assume (a). Suppose that $\kappa < \lambda$. Since (i) fails, it follows that $\lambda \leq \mu$. But this contradicts (ii) failing. Thus $\kappa = \lambda$.

Suppose that $\mu \neq \mathrm{cf}\lambda$. If $\lambda < \mu$, this contradicts (ii) failing. So $\mu \leq \lambda$. Since (iii) fails, $\mu = \lambda$. But then $\mu = \mathrm{cf}\lambda$, contradiction.

(b)⇒(a). Assume (b). Clearly then (i) fails. We have $\mu = \mathrm{cf}\lambda \leq \lambda = \kappa$, so (ii) fails. Obviously (iii) fails. $\square$

Proposition 2.16 *Suppose that $\lambda \leq \mu$ and that one of the following conditions holds:*

(i) $\kappa, \mathrm{cf}\mu < \lambda$*;*
(ii) $\kappa < \mathrm{cf}\mu$*;*
(iii) $\mathrm{cf}\mu < \kappa = \lambda$ *and* $\mathrm{cf}\lambda \neq \mathrm{cf}\mu$.

Then there is a $B = \{f_\xi : \xi < \mathrm{cf}\mu\} \subseteq {}^\lambda\mu$ consisting of strictly increasing functions, with $f_\xi \neq f_\rho$ and $f_\xi \leq f_\rho$ for $\xi < \rho < \mathrm{cf}\mu$, such that B is $\leq_\kappa$-unbounded.

Proof Let $\alpha_\xi \uparrow \mu$ for $\xi < \mathrm{cf}\mu$, the α_ξ's ordinals. For all $\xi < \mathrm{cf}\mu$ and $\eta < \lambda$ let $f_\xi(\eta) = \alpha_\xi + \eta$. Let $B = \{f_\xi : \xi < \mathrm{cf}\mu\}$. We just need to show that B is $\leq_\kappa$ unbounded. Suppose that $g \in {}^\lambda\mu$ and $f_\xi \leq_\kappa g$ for all $\xi < \mathrm{cf}\mu$. For each $\xi < \mathrm{cf}\mu$ let $\Gamma_\xi = \{\eta < \lambda : f_\xi(\eta) > g(\eta)\}$. The conditions of Lemma 2.14 then hold, contradiction. $\square$

Proposition 2.17 *Suppose that μ is regular and that one of the following conditions holds:*

(i) $\kappa, \mu < \lambda$*;*
(ii) $\kappa < \mu$*;*
(iii) $\mu < \kappa = \lambda$ *and* $\mathrm{cf}\lambda \neq \mu$.

Then $\mathfrak{b}_{\kappa,\lambda,\mu} = \mu$.

Proof By 2.7 the inequality $\geq$ holds. Now choose a singular cardinal ρ such that $\rho \geq \lambda$ and $\mathrm{cf}\rho = \mu$. Then by 2.6 and 2.16 we have $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{b}_{\kappa,\lambda,\rho} = \mu$. $\square$

Proposition 2.18 *Suppose that μ is regular, $\mu < \lambda$, and $\mathrm{cf}\lambda = \mu$. Then $\mathfrak{b}_{\lambda,\lambda,\mu} > \mu$.*

Proof Suppose that $\{f_\xi : \xi < \mu\}$ is a system of members of ${}^\lambda\mu$. Let $\nu_\xi \uparrow \lambda$ for $\xi < \mu$, continuous, with $\nu_0 = 0$. Given $\eta < \lambda$, choose $\xi < \mu$ such that $\nu_\xi \leq \eta < \nu_{\xi+1}$, and define $g(\eta) = \sup_{\tau\leq\xi} f_\tau(\eta)$. We claim that $f_\rho \leq_\lambda g$ for all $\rho < \mu$. For, if $\eta < \lambda$ and $f_\rho(\eta) > g(\eta)$, choose ξ so that $\nu_\xi \leq \eta < \nu_{\xi+1}$. Then by the definition of g it follows that $\xi < \rho$. Hence $\eta < \nu_{\xi+1} \leq \nu_\rho$. So $\{\eta : f_\rho(\eta) > g(\eta)\} \subseteq \nu_\rho$, which has size less than λ, as desired. $\square$

The following proposition summarizes our results so far for $\mathfrak{b}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}_{\kappa,\lambda,\mu}$; recall also 2.6.

Proposition 2.19 *Let μ be regular.*

(i) *If $\lambda < \mu$, then $\mathfrak{b}_{\kappa,\lambda,\mu} = \mu$.*
(ii) *If $\mu \leq \lambda$ and $\kappa < \lambda$, then $\mathfrak{b}_{\kappa,\lambda,\mu} = \mu$.*
(iii) *If $\mu \leq \lambda = \kappa$ and $\mu \neq \mathrm{cf}\lambda$, then $\mathfrak{b}_{\kappa,\lambda,\mu} = \mu$.*
(iv) *If $\mu \leq \lambda = \kappa$ and $\mu = \mathrm{cf}\kappa$, then $\mu < \mathfrak{b}_{\kappa,\lambda,\mu}$.*
(v) *If $\lambda < \mu$, then $\mathfrak{d}_{\kappa,\lambda,\mu} = \mu$.*
(vi) *If $\mu \leq \lambda$, then $\lambda < \mathfrak{d}_{\kappa,\lambda,\mu}$.*

Proof

(i) 2.17(ii).
(ii) 2.17(i) if $\mu < \lambda$, and 2.17(ii) if $\mu = \lambda$.
(iii) 2.17(iii).
(iv) 2.9 if $\mu = \lambda$, 2.18 if $\mu < \lambda$.
(v) 2.11.
(vi) 2.12. $\square$

Thus it remains to indicate possibilities for $\mathfrak{b}_{\kappa,\lambda,\mu}$ in case (iv) and possibilities for $\mathfrak{d}_{\kappa,\lambda,\mu}$ in case (vi). Note that case (iv) can be more simply expressed as concerning $\mathfrak{b}_{\lambda,\lambda,\mu}$ with $\mu = \mathrm{cf}\lambda$. Some special cases are $\mathfrak{b}_{\mu,\mu,\mu}$ with μ regular and $\mathfrak{b}_{\aleph_\omega,\aleph_\omega,\omega}$. Also, the relationships between the numbers $\mathfrak{b}_{\kappa,\lambda,\mu}$ as the subscripts vary, and similarly for $\mathfrak{d}_{\kappa,\lambda,\mu}$, have not yet been fully described.

Proposition 2.20 *If $\mathrm{cf}\lambda = \mu$, then $\mathfrak{b}_{\lambda,\lambda,\mu} \leq \mathfrak{b}_{\mu,\mu,\mu}$.*

Proof We may assume that λ is singular. Let $\nu_\xi \uparrow \lambda$, $\xi < \mu$, be a sequence of regular cardinals such that $\mu < \nu_0$. For each $f \in {}^\mu\mu$ we define $f^+ \in {}^\lambda\mu$ as follows. For any $\alpha < \lambda$, let ξ be the supremum of all ordinals such that $\nu_\xi \leq \alpha$, and let $f^+(\alpha) = f(\xi)$.

Now we prove $\leq$. Suppose that $B \subseteq {}^\mu\mu$ is $\leq_\mu$-unbounded. For each $g \in {}^\lambda\mu$ we define $\langle M_{g\xi} : \xi < \mu\rangle$ and $g^- \in {}^\mu\mu$ as follows. Write

$$\nu_{\xi+1}\backslash\nu_\xi = \bigcup_{\eta<\mu}\{\alpha \in \nu_{\xi+1}\backslash\nu_\xi : g(\alpha) = \eta\}.$$

Since $\nu_{\xi+1}$ is a regular cardinal greater than μ, choose the least $g^-(\xi) < \mu$ such that the set

$$M_{g\xi} \stackrel{\text{def}}{=} \{\alpha \in \nu_{\xi+1}\backslash\nu_\xi : g(\alpha) = g^-(\xi)\}$$

has size $\nu_{\xi+1}$.

(1) If $f \in {}^{\mu}\mu$, $g \in {}^{\lambda}\mu$, and $f^+ \leq_\lambda g$, then $f \leq_\mu g^-$.

For, assume the hypothesis, and suppose that $f \not\leq_\mu g^-$. Thus $N \stackrel{\text{def}}{=} \{\xi \in \mu : f(\xi) > g^-(\xi)\}$ has size μ. If $\xi \in N$ and $\alpha \in M_{g\xi}$, then

$$f^+(\alpha) = f(\xi) > g^-(\xi) = g(\alpha),$$

so $|\{\gamma < \lambda : f^+(\gamma) > g(\gamma)\}| = \lambda$, contradiction. So (1) holds. It follows that $\{f^+ : f \in B\}$ is $\leq_\lambda$-unbounded, proving $\leq$. □

Proposition 2.21 $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{b}^{\text{wo}}_{\kappa,\lambda,\mu}$.

Proof Let $B \subseteq {}^{\lambda}\mu$ be $\leq_\kappa$-unbounded with $|B| = \mathfrak{b}_{\kappa,\lambda,\mu}$. Write $B = \{f_\xi : \xi < \mathfrak{b}_{\kappa,\lambda,\mu}\}$. We now define $\langle g_\xi : \xi < \mathfrak{b}_{\kappa,\lambda,\mu}\rangle$ by recursion. Suppose that g_ξ has been defined for all $\xi < \eta$. Then $\{g_\xi : \xi < \eta\}$ is $\leq_\kappa$-bounded; say that $g_\xi \leq_\kappa h$ for all $\xi < \eta$. Define $g_\eta(\alpha) = \max\{h(\alpha)+1, f_\eta(\alpha)+1\}$ for all $\alpha < \lambda$. Then $g_\xi <_\kappa g_\eta$ for all $\xi < \eta$. Since $f_\eta < g_\eta$ for all $\eta < \lambda$, clearly $\{g_\eta : \eta < \mathfrak{b}_{\kappa,\lambda,\mu}\}$ is $\leq_\kappa$-unbounded. □

The following lemma is due to Szymański [3], Lemma 4.

Lemma 2.22 *Suppose that μ is regular, $\kappa \leq \lambda' \leq \lambda$, and $\nu = \mathfrak{b}_{\kappa,\lambda',\mu}$. Then $\mathfrak{d}_{1,\lambda,\nu} \leq \mathfrak{d}_{\kappa,\lambda,\mu}$.*

Proof By 2.21 let $\langle g_\alpha : \alpha < \nu\rangle$ be a $<_\kappa$-increasing, unbounded sequence of elements of ${}^{\lambda'}\mu$. Let $\langle R_\alpha : \alpha < \lambda\rangle$ be a partition of λ into sets of size λ', and write $R_\alpha = \{\gamma_{\alpha\beta} : \beta < \lambda'\}$. Now with each $f \in {}^{\lambda}\mu$ we will associate a function $h_f \in {}^{\lambda}\nu$. Let $\alpha < \lambda$. For each $\beta < \lambda'$ let $\varphi_\alpha(\beta) = f(\gamma_{\alpha\beta})$. Thus $\varphi_\alpha \in {}^{\lambda'}\mu$. We let $h_f(\alpha)$ be the least $\gamma < \nu$ such that $g_\gamma \not\leq_\kappa \varphi_\alpha$.

Let $B \subseteq {}^{\lambda}\mu$ be $\leq_\kappa$-dominating, of size $\mathfrak{d}_{\kappa,\lambda,\mu}$. We claim that $\{h_f : f \in B\}$ is $\leq_1$-dominating; this will prove the lemma. Suppose that $s \in {}^{\lambda}\nu$. Define $t \in {}^{\lambda}\mu$ by setting $t(\gamma_{\alpha\beta}) = g_{s(\alpha)}(\beta)$ for all $\alpha < \lambda, \beta < \lambda'$. Choose $f \in B$ such that $t \leq_\kappa f$. We claim that $s \leq h_f$. Suppose that $\alpha < \lambda$ and $s(\alpha) > h_f(\alpha)$. Now $g_{h_f(\alpha)} \not\leq_\kappa \varphi_\alpha$, so also $g_{s(\alpha)} \not\leq_\kappa \varphi_\alpha$. So $|\{\beta < \lambda' : g_{s(\alpha)}(\beta) > \varphi_\alpha(\beta)\}| \geq \kappa$, that is, $|\{\beta < \lambda' : t(\gamma_{\alpha\beta}) > f(\gamma_{\alpha\beta})\}| \geq \kappa$, contradiction. □

Corollary 2.23 *Let μ be regular. If one of the conditions 2.19(i)–(iii) holds, then $\mathfrak{d}_{\kappa,\lambda,\mu} = \mathfrak{d}_{1,\lambda,\mu}$.* ⊣

The remaining case concerns $\mathfrak{d}_{\lambda,\lambda,\mu}$ with $\text{cf}\lambda = \mu$. Here we have the following result.

Proposition 2.24 *Suppose that μ is regular, $\text{cf}\lambda = \mu$, and $\lambda^{<\lambda} \leq \mathfrak{d}_{\lambda,\lambda,\mu}$. Then $\mathfrak{d}_{\lambda,\lambda,\mu} = \mathfrak{d}_{1,\lambda,\mu}$.*

Proof By 2.4, $\mathfrak{d}_{\lambda,\lambda,\mu} \leq \mathfrak{d}_{1,\lambda,\mu}$. Now suppose that $X \subseteq {}^{\lambda}\mu$ is $\leq_\lambda$-dominating, with $|X| = \mathfrak{d}_{\lambda,\lambda,\mu}$. Let M be the set of all functions from a subset of λ of size less than λ

into μ. Then

$$\begin{aligned}|M| &= \left|\bigcup_{\Gamma\in[\lambda]^{<\lambda}} {}^{\Gamma}\mu\right| \\ &\leq \sum_{\Gamma\in[\lambda]^{<\lambda}} \mu^{|\Gamma|} \\ &\leq \sum_{\Gamma\in[\lambda]^{<\lambda}} \mu^{<\lambda} \\ &= \lambda^{<\lambda}\cdot\mu^{<\lambda} = \lambda^{<\lambda} \leq \mathfrak{d}_{\lambda,\lambda,\mu}\,.\end{aligned}$$

Now for any $g \in X$ and $h \in M$ we define $l(g,h) \in {}^{\lambda}\mu$ by setting, for any $\alpha < \lambda$,

$$l(g,h)(\alpha) = \begin{cases} h(\alpha)+1 & \text{if } \alpha \in \mathrm{dmn}(h), \\ g(\alpha) & \text{otherwise.}\end{cases}$$

Let $Y = \{l(g,h) : g \in X \text{ and } h \in M\}$. Clearly $|X| = |Y|$. We claim that Y is $\leq_1$-dominating (as desired). For, suppose that $f \in {}^{\lambda}\mu$. Choose $g \in X$ such that $f \leq_\lambda g$. Thus $F \stackrel{\text{def}}{=} \{\alpha < \mu : f(\alpha) > g(\alpha)\}$ has size less than λ. Clearly $f \leq_1 l(g, f \restriction F)$. □

Corollary 2.25 $\mathfrak{d}_{\omega,\mu,\mu} = \mathfrak{d}_{1,\mu,\mu}$. ⊣

Corollary 2.26 (GCH) *If* $\mathrm{cf}\lambda = \mu$*, then* $\mathfrak{d}_{\lambda,\lambda,\mu} = \mathfrak{d}_{1,\lambda,\mu}$.

Proof By 2.13 we have $\lambda^{<\lambda} = \lambda^+ \leq \mathfrak{d}_{\lambda,\lambda,\mu}$, so the result follows by 2.24. □

Proposition 2.27 (GCH) *If* $\kappa < \mu$ *are infinite regular cardinals, then* $\mathfrak{b}_{\kappa,\kappa,\kappa} = \kappa^+ < \mu^+ = \mathfrak{b}_{\mu,\mu,\mu}$ *and* $\mathfrak{d}_{\kappa,\kappa,\kappa} = \kappa^+ < \mu^+ = \mathfrak{d}_{\mu,\mu,\mu}$. ⊣

We have now described our main results concerning these cardinals. Concerning the inequalities mentioned in $(*)$ following Proposition 2.6, we see now that all of the cardinals up to and including $\mathfrak{b}_{\omega,\omega_1,\omega}$ are equal to ω; then $\omega < \mathfrak{b}_{\omega,\omega,\omega}$ and $\mathfrak{d}_{\omega,\aleph_\omega,\omega} = \mathfrak{d}_{1,\aleph_\omega,\omega}$. Note that under CH, $\mathfrak{d}_{\omega,\omega,\omega} < \mathfrak{d}_{\omega,\omega_1,\omega}$, by 2.13.

The results of Cummings and Shelah completely take care of the important case when all the indices are regular and equal. Their main result, in our terminology, is as follows.

Theorem 2.28 (Cummings, Shelah) *Suppose that* **F** *is a class function assigning to each regular cardinal* κ *a triple* $(\beta(\kappa), \delta(\kappa), \lambda(\kappa))$ *of cardinals so that the following conditions hold:*

(i) $\kappa < \mathrm{cf}(\lambda(\kappa))$*;*
(ii) *if* $\kappa < \kappa'$*, then* $\lambda(\kappa) \leq \lambda(\kappa')$*;*
(iii) $\kappa^+ \leq \beta(\kappa) = \mathrm{cf}(\beta(\kappa)) \leq \mathrm{cf}(\delta(\kappa)) \leq \delta(\kappa) \leq \lambda(\kappa)$.

Then there is a class forcing poset $\mathbb{P}$ preserving all cardinals and cofinalities such that in the generic extension, $\mathfrak{b}_{\kappa\kappa\kappa} = \beta(\kappa)$, $\mathfrak{d}_{\kappa\kappa\kappa} = \delta(\kappa)$, and $2^\kappa = \lambda(\kappa)$, for all regular κ.

For example the following is relatively consistent:

$$\mathfrak{b}_{\omega,\omega,\omega} = \omega_3, \quad \mathfrak{d}_{\omega,\omega,\omega} = \omega_5, \quad 2^\omega = \omega_7, \quad \mathfrak{b}_{\omega_1,\omega_1,\omega_1} = \omega_2,$$
$$\mathfrak{d}_{\omega_1,\omega_1,\omega_1} = \omega_4, \quad 2^{\omega_1} = \omega_7.$$

Two consistency results of Szymański are also relevant; they generalize work of Jech and Prikry. Szymański [3], Theorem 2.1 shows that if 2^ω is real-valued measurable, $\omega \leq \lambda \leq 2^\omega$, and λ is regular, then $\mathfrak{d}_{\lambda,\lambda,\omega} = 2^\omega$. In Theorem 2.3 he shows that if 2^ω is real-valued measurable, $\omega \leq \mu \leq \lambda < 2^\omega$, and λ, μ are regular, then $\mathfrak{d}_{\lambda,\lambda,\mu} < 2^\omega$.

By the above, the main open problem concerning $\mathfrak{b}_{\kappa,\lambda,\mu}$ is as follows.

Problem 1 For $\mathrm{cf}\lambda = \mu$, λ singular, is $\mathfrak{b}_{\lambda,\lambda,\mu} = \mathfrak{b}_{\mu,\mu,\mu}$?

In particular, we do not know whether $\mathfrak{b}_{\aleph_\omega,\aleph_\omega,\omega} = \mathfrak{b}_{\omega,\omega,\omega}$. Under CH, this equality holds.

Our results concerning $\mathfrak{d}_{\kappa,\lambda,\mu}$ are fragmentary. We mention just one definite problem.

Problem 2 Is it consistent to have an uncountable regular cardinal μ such that $\mathfrak{d}_{\mu,\mu,\mu} < \mathfrak{d}_{1,\mu,\mu}$?

3 $\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}^{\uparrow}_{\kappa,\lambda,\mu}$

Naturally we have to assume that $\lambda \leq \mu$ for most of the considerations in this section.

As a corollary of Proposition 2.1 we have

Corollary 3.1 *If $\mathfrak{d}^{\uparrow}_{\kappa,\lambda,\mu}$ exists, then so does $\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$, and $\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu} \leq \mathfrak{d}^{\uparrow}_{\kappa,\lambda,\mu}$.* $\dashv$

Lemma 3.2 *If $\lambda \leq \mathrm{cf}\mu$, and $f \in {}^\lambda\mu$, then there is a strictly increasing $g \in {}^\lambda\mu$ such that $f < g$.*

Proof For all $\alpha < \lambda$ let $g(\alpha) = \sup(\{f(\alpha)+1\} \cup \{g(\beta)+1 : \beta < \alpha\}$. $\square$

This lemma does not extend to the case $\mathrm{cf}\mu < \lambda \leq \mu$. In fact, if $\mathrm{cf}\mu < \lambda \leq \mu$, then there is an $f \in {}^\lambda\mu$ such that there is no strictly increasing $g \in {}^\lambda\mu$ with $f <_\lambda g$. To see this, let $h : \mathrm{cf}\mu \to \mu$ be strictly increasing and continuous with range cofinal in μ. We define $f : \lambda \to \mu$ as follows. For each $\alpha < \lambda$ and each $\xi < \mathrm{cf}\mu$, let $f(\mathrm{cf}\mu \cdot \alpha + \xi) = h(\xi)$. Suppose that $g \in {}^\lambda\mu$ is strictly increasing and $f <_\lambda g$. For each $\alpha < \lambda$, choose $\beta_\alpha < \mathrm{cf}\mu$ such that $h(\beta_\alpha) \geq g(\mathrm{cf}\mu \cdot (\alpha+1))$. Then for any $\alpha < \lambda$ and $\xi \in [\beta_\alpha, \mathrm{cf}\mu)$ we have

$$f(\mathrm{cf}\mu \cdot \alpha + \xi) = h(\xi) \geq h(\beta_\alpha) \geq g(\mathrm{cf}\mu \cdot (\alpha+1) > g(\mathrm{cf}\mu \cdot \alpha + \xi).$$

It follows that $|\{\eta < \lambda : f(\eta) \geq g(\eta)\}| \geq \lambda$, contradiction. Thus if $\mathrm{cf}\mu < \lambda \leq \mu$, then $\mathfrak{d}^{\uparrow}_{\kappa,\lambda,\mu}$ does not exist.

Corollary 3.3 *If $\lambda \leq \mathrm{cf}\mu$, then $\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}^{\uparrow}_{\kappa,\lambda,\mu}$ exist, and equal $\mathfrak{b}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}_{\kappa,\lambda,\mu}$, respectively.* $\dashv$

By Corollary 3.3 and the above remarks, the study of $\mathfrak{d}^{\uparrow}_{\kappa,\lambda,\mu}$ completely reduces to that of $\mathfrak{d}_{\kappa,\lambda,\mu}$. Moreover, from 2.19(vi) we see that only in the case $\mathrm{cf}\mu = \lambda$ is there a possibility that the notion $\mathfrak{d}^{\uparrow}_{\kappa,\lambda,\mu}$ is useful in studying $\mathfrak{d}_{\kappa,\lambda,\mu}$.

Propositions 2.7 and 2.16 give the following corollary.

Corollary 3.4 *Suppose that $\lambda \leq \mu$ and that one of the following conditions holds:*

(i) $\kappa, \mathrm{cf}\mu < \lambda$;
(ii) $\kappa < \mathrm{cf}\mu$;
(iii) $\mathrm{cf}\mu < \kappa = \lambda$ *and* $\mathrm{cf}\lambda \neq \mathrm{cf}\mu$.

Then $\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu} = \mathrm{cf}\mu$. $\dashv$

Proposition 3.5 *Suppose that $\lambda \leq \mu$ and that one of the following conditions holds:*

(i) $\kappa, \mathrm{cf}\mu < \lambda$;
(ii) $\kappa < \mathrm{cf}\mu$;
(iii) $\mathrm{cf}\mu < \kappa = \lambda$ *and* $\mathrm{cf}\lambda \neq \mathrm{cf}\mu$;
(iv) $\kappa = \lambda = \mathrm{cf}\mu$.

Then $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$.

Proof If one of (i), (ii), (iii) holds, the conclusion follows by 2.6, 2.17, and 3.4. Assume that (iv) holds. Obviously $\mathfrak{b}_{\mathrm{cf}\mu,\mathrm{cf}\mu,\mu} \leq \mathfrak{b}^{\uparrow}_{\mathrm{cf}\mu,\mathrm{cf}\mu,\mu}$. Now assume that $B \subseteq {}^{\mathrm{cf}\mu}\mu$ is $\leq_{\mathrm{cf}\mu}$-unbounded and $|B| = \mathfrak{b}_{\mathrm{cf}\mu,\mathrm{cf}\mu,\mu}$. By 3.2 we may assume that each member of B is strictly increasing. So obviously $\mathfrak{b}^{\uparrow}_{\mathrm{cf}\mu,\mathrm{cf}\mu,\mu} \leq |B|$. $\square$

Suppose that (i) – (iv) of 3.5 fail. Then $\mathrm{cf}\mu \leq \kappa$ because (ii) fails. Hence $\kappa = \lambda$ since (i) fails. Because (iv) fails, we have $\mathrm{cf}\mu < \kappa$. Hence $\mathrm{cf}\lambda = \mathrm{cf}\mu$ since (iii) fails. Thus the case not covered by 3.5 is $\mathrm{cf}\mu = \mathrm{cf}\lambda < \kappa = \lambda \leq \mu$. Here the following result is relevant.

Lemma 3.6 *Suppose that μ is singular, $\mathrm{cf}\mu = \mathrm{cf}\lambda < \lambda \leq \mu$, and $f \in {}^{\lambda}\mu$. Then there is a strictly increasing $g \in {}^{\lambda}\mu$ such that $g \not\leq_{\lambda} f$.*

Proof Let $\langle \varphi(\xi) : \xi < \mathrm{cf}\mu \rangle$ be a strictly increasing sequence of cardinals which is continuous, with supremum λ, and with $\varphi(0) = 0$ and $\varphi(1)$ infinite. Then let $\langle \pi(\xi) : \xi < \mathrm{cf}\mu \rangle$ be a strictly increasing sequence of cardinals which is continuous, with supremum μ, with $\pi(0) = 0$ and $\pi(1)$ infinite, and such that $\varphi(\xi) \leq \pi(\xi)$ for all $\xi < \mathrm{cf}\mu$.

For all $\xi < \mathrm{cf}\mu$ let

$$\Gamma(\xi) = \{\alpha < \lambda : \pi(\xi) \leq f(\alpha) < \pi(\xi + 1)\}.$$

If $|\Gamma(\xi)| = \lambda$ for some $\xi < \mathrm{cf}\mu$, let $g = \langle \pi(\xi + 1) + \eta : \eta < \lambda \rangle$; clearly g is as desired. So, assume that $|\Gamma(\xi)| < \lambda$ for all $\xi < \mathrm{cf}\mu$. Note that $\lambda = \bigcup_{\xi<\mathrm{cf}\mu} \Gamma(\xi)$.

Now we claim:

(1) $\forall \xi < \mathrm{cf}\mu \exists \eta < \mathrm{cf}\mu (\xi < \eta$ and $\varphi(\xi) < |\Gamma(\eta)|)$.

For, otherwise there is a $\xi < \mathrm{cf}\mu$ such that for every $\eta < \mathrm{cf}\mu$, if $\xi < \eta$ then $|\Gamma(\eta)| \leq \varphi(\xi)$. So

$$\lambda = \left| \bigcup_{\rho<\mathrm{cf}\mu} \Gamma(\rho) \right| \leq \sum_{\rho\leq\xi} |\Gamma(\rho)| + \sum_{\xi<\rho<\mathrm{cf}\mu} |\Gamma(\rho)| < \lambda,$$

contradiction. So (1) holds.

Now we define $\nu \in {}^{\mathrm{cf}\mu}\mathrm{cf}\mu$ by recursion. First choose $\nu(0)$ so that $\Gamma(\nu(0))$ is infinite. Now suppose that $0 < \sigma$ and $\nu(\rho)$ has been defined for all $\rho < \sigma$. First let ξ be minimum such that for all $\rho < \sigma$ we have $\nu(\rho) < \xi$ and $|\Gamma(\nu(\rho))| < \varphi(\xi)$. Apply (1) to get $\nu(\sigma)$ such that $\xi < \nu(\sigma)$ and $\varphi(\xi + 1) < |\Gamma(\nu(\sigma))|$. Thus we have

(2) $\nu : \mathrm{cf}\mu \to \mathrm{cf}\mu$ is strictly increasing.
(3) If $\rho < \sigma < \mathrm{cf}\mu$, then $|\Gamma(\nu(\rho))|^{++} < |\Gamma(\nu(\sigma))|$. and $\sup_{\rho<\mathrm{cf}\mu} |\Gamma(\nu(\rho))| = \lambda$.
For each $\xi < \mathrm{cf}\mu$ let $\beta(\xi)$ be the least element of $\Gamma(\nu(\xi + 1))$ such that
(4) $|\{\gamma < \beta(\xi) : \gamma \in \Gamma(\nu(\xi + 1))\}| > |\Gamma(\nu(\xi))|$;
it follows that

(5) $|\{\gamma \geq \beta(\xi) : \gamma \in \Gamma(\nu(\xi+1))\}| > |\Gamma(\nu(\xi))|$.
This is possible since $|\Gamma(\nu(\xi))|^{++} < |\Gamma(\nu(\xi+1))|$. Thus
(6) $\sup_{\xi<\mathrm{cf}\mu} \beta(\xi) = \lambda$.

For, if $\delta < \lambda$ and $\beta(\xi) \leq \delta$ for all $\xi < \mathrm{cf}\mu$, then

$$\sup_{\xi<\mathrm{cf}\mu} |\{\gamma < \delta : \gamma \in \Gamma(\nu(\xi+1))\}| \leq |\delta| < \lambda,$$

while

$$\begin{aligned} & \sup_{\xi<\mathrm{cf}\mu} |\{\gamma < \delta : \gamma \in \Gamma(\nu(\xi+1))\}| \\ \geq \quad & \sup_{\xi<\mathrm{cf}\mu} |\{\gamma < \beta(\xi) : \gamma \in \Gamma(\nu(\xi+1))\}| \\ \geq \quad & \sup_{\xi<\mathrm{cf}\mu} |\Gamma(\nu(\xi))| = \lambda, \end{aligned}$$

contradiction.

So it is easy to find $\rho \in {}^{\mathrm{cf}\mu}\mathrm{cf}\mu$ such that if $\sigma < \tau < \mathrm{cf}\mu$, then $\varphi(\tau) \leq \beta(\rho(\tau))$, $\rho(\sigma) < \rho(\tau)$, and $\beta(\rho(\sigma)) < \beta(\rho(\tau))$. Then $\sup_{\sigma<\mathrm{cf}\mu} \beta(\rho(\sigma)) = \lambda$. Thus

(7) $\rho : \mathrm{cf}\mu \to \mathrm{cf}\mu$ is strictly increasing.
(8) $\beta \circ \rho : \mathrm{cf}\mu \to \lambda$ is strictly increasing.

For each $\xi < \lambda$ let $h(\xi) < \mathrm{cf}\mu$ be such that $\varphi(h(\xi)) \leq \xi < \varphi(h(\xi)+1)$.

Now we define $\sigma : \mathrm{cf}\mu \to \mathrm{cf}\mu$. If $\sigma(\eta)$ has been defined for all $\eta < \xi$, let $\sigma(\xi)$ be an ordinal greater than all $\sigma(\eta)$ and $h(\beta(\rho(\sigma(\eta))))$ for $\eta < \xi$. Thus

(9) $\sigma : \mathrm{cf}\mu \to \mathrm{cf}\mu$ is strictly increasing.

Also, clearly $\sup_{\xi<\mathrm{cf}\mu} \beta(\rho(\sigma(\xi))) = \lambda$. For each $\xi < \lambda$ let $\tau(\xi) < \mathrm{cf}\mu$ be minimum such that $\xi < \beta(\rho(\sigma(\tau(\xi))))$.

(10) If $\xi < \eta < \lambda$, then $\tau(\xi) \leq \tau(\eta)$.

For, $\xi < \eta < \beta(\rho(\sigma(\tau(\eta))))$, so $\tau(\xi) \leq \tau(\eta)$.

Now we define $g : \lambda \to \mu$ by setting, for any $\xi < \lambda$,

$$g(\xi) = \pi(\nu(\rho(\sigma(\tau(\xi))))) + \xi.$$

We claim that g is as desired in the proposition. To show that g is strictly increasing, suppose that $\xi < \eta < \lambda$. Hence by (10), $\tau(\xi) \leq \tau(\eta)$. If $\tau(\xi) = \tau(\eta)$, clearly $g(\xi) < g(\eta)$. So assume that $\tau(\xi) < \tau(\eta)$. Note that $\pi \circ \nu \circ \rho \circ \sigma$ is strictly increasing. Hence

$$\begin{aligned} g(\xi) &= \pi(\nu(\rho(\sigma(\tau(\xi))))) + \xi \\ &\leq \pi(\nu(\rho(\sigma(\tau(\eta))))) + \xi \\ &< g(\eta). \end{aligned}$$

We also claim

(11) If $\theta < \mathrm{cf}\mu, \xi \in \Gamma(\nu(\rho(\sigma(\theta))+1))$, and $\beta(\rho(\sigma(\theta))) \leq \xi$, then $f(\xi) < g(\xi)$.

For, assume the hypotheses of (11). Now $\xi < \beta(\rho(\sigma(\tau(\xi))))$, and hence it follows that $\beta(\rho(\sigma(\theta))) < \beta(\rho(\sigma(\tau(\xi))))$, so $\theta < \tau(\xi)$. Therefore

$$\begin{aligned} g(\xi) &\geq \pi(\nu(\rho(\sigma(\tau(\xi))))) \\ &\geq \pi(\nu(\rho(\sigma(\theta))) + 1) \\ &> f(\xi), \end{aligned}$$

as desired in (11). Now the desired conclusion follows from (11) and (5). □

Corollary 3.7 *Assume that $\lambda \leq \mu$. Then the set of all strictly increasing functions in ${}^{\lambda}\mu$ is $\leq_\kappa$-unbounded.*

Proof This is true by 2.16 if its hypotheses hold. Otherwise, by 2.15 we have $\kappa = \lambda$ and $\mathrm{cf}\mu = \mathrm{cf}\lambda$. So, if in addition $\lambda = \mathrm{cf}\mu$, then the conclusion holds by 3.2. In the remaining case, with $\kappa = \lambda$ and $\mathrm{cf}\mu = \mathrm{cf}\lambda < \lambda \leq \mu$, 3.6 applies. □

Corollary 3.8 *Assume that $\lambda \leq \mu$. Then $\mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$ always exists, and $\mathfrak{b}_{\kappa,\lambda,\mu} \leq \mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$.* ⊣

Lemma 3.9 *Suppose that $\kappa \leq \mathrm{cf}\lambda$, $B \subseteq {}^{\lambda}\mu$ is $\leq_\kappa$-unbounded, and $\forall f \in B \forall \alpha, \beta \in \lambda[\alpha < \beta \Rightarrow f(\alpha) \leq f(\beta)]$. Then B is $\leq_\kappa$-unbounded on every member A of $[\lambda]^{\lambda}$.*

Proof Take any $f \in {}^{\lambda}\mu$; we define $\hat{f} \in {}^{\lambda}\mu$ by setting, for each $\alpha < \lambda$,

$$\hat{f}(\alpha) = f(\min\{\gamma \in A : \alpha \leq \gamma\}).$$

Since B is $\leq_\kappa$-unbounded, choose $g \in B$ so that $g \not\leq_\kappa \hat{f}$. Thus $J \overset{\mathrm{def}}{=} \{\alpha < \lambda : g(\alpha) > \hat{f}(\alpha)\}$ has size $\geq \kappa$. We claim

$$(*) \qquad \forall \alpha \in J \exists \beta \in A[\alpha < \beta \text{ and } f(\beta) < g(\beta)].$$

For, let $\beta = \min\{\gamma \in A : \alpha \leq \gamma\}$. Then $g(\beta) \geq g(\alpha) > \hat{f}(\alpha) = f(\beta)$, as desired.

By $(*)$, $|\{\beta \in A : f(\beta) < g(\beta)\}| \geq \mathrm{cf}\lambda \geq \kappa$, as desired. □

Concerning the following result, recall 2.20.

Proposition 3.10 *Suppose that* $\mathrm{cf}\lambda = \mathrm{cf}\mu < \lambda \leq \mu$. *Then*

$$\mathfrak{b}_{\mathrm{cf}\mu,\mathrm{cf}\mu,\mathrm{cf}\mu} = \mathfrak{b}^{\uparrow}_{\lambda,\lambda,\mu}.$$

Proof Let $\rho_\xi \uparrow \lambda$ for $\xi < \mathrm{cf}\mu$ be continuous, each $\rho_{\xi+1}$ a regular cardinal, $\rho_0 = 0$, and $\mathrm{cf}\mu < \rho_1$. Similarly get $\nu_\xi \uparrow \mu$ for $\xi < \mathrm{cf}\mu$, continuous each $\nu_{\xi+1}$ a regular cardinal, $\nu_0 = 0$, each $\rho_\xi \leq \nu_\xi$.

First we take $\leq$. Suppose that $B \subseteq {}^{\lambda}\mu$ is a $\leq_\lambda$-unbounded set of strictly increasing functions with $|B| = \mathfrak{b}^{\uparrow}_{\lambda,\lambda,\mu}$. For each $f \in B$ we define $f^{-} \in {}^{\mathrm{cf}\mu}\mathrm{cf}\mu$ by setting, for each $\xi < \mathrm{cf}\mu$,

$$f^{-}(\xi) = \text{least } \eta < \mathrm{cf}\mu \text{ such that } f(\rho_\xi) \leq \nu_\eta.$$

We claim that $\{f^{-} : f \in B\}$ is $\leq_{\mathrm{cf}\mu}$-unbounded; this will prove $\leq$. For, suppose that $f^{-} \leq_{\mathrm{cf}\mu} g$ for all $f \in B$. Define $h \in {}^{\lambda}\mu$ as follows. Suppose that $\alpha < \lambda$. Let ξ be minimum such that $\alpha \leq \rho_\xi$, and set $h(\alpha) = \nu_{g(\xi)+1}$. We claim that $f \leq_\lambda h$ for all $f \in B$ (contradiction). For, suppose that $f \in B$. Then $f^{-} \leq_{\mathrm{cf}\mu} g$. Choose

$\sigma < \mathrm{cf}\mu$ such that $f^-(\xi) \le g(\xi)$ for all $\xi \in [\sigma, \mathrm{cf}\mu)$. Suppose that $\rho_\sigma \le \alpha$. Choose ξ minimum such that $\alpha \le \rho_\xi$. Thus $\sigma \le \xi$, so

$$f(\alpha) \le f(\rho_\xi) \le \nu_{f^-(\xi)} \le \nu_{g(\xi)} < \nu_{g(\xi)+1} = h(\alpha);$$

this shows that $f \le_\lambda h$, as desired.

Now we turn to $\ge$. Let $B \subseteq {}^{\mathrm{cf}\mu}\mathrm{cf}\mu$ be an $\le_{\mathrm{cf}\mu}$-unbounded set of strictly increasing functions, with $|B| = \mathfrak{b}_{\mathrm{cf}\mu,\mathrm{cf}\mu,\mathrm{cf}\mu}$, using 3.3. By 3.9, B is $\le_{\mathrm{cf}\mu}$-unbounded on each $A \in [\mathrm{cf}\mu]^{\mathrm{cf}\mu}$. For each $f \in B$ and $\alpha < \lambda$ we define

$$h_f(\alpha) = \nu_{f(\xi+1)} + \alpha \quad \text{if} \quad \alpha \in \rho_{\xi+1} \backslash \rho_\xi.$$

Note that for any $\alpha \in \rho_{\xi+1} \backslash \rho_\xi$ we have $\alpha < \rho_{\xi+1} \le \nu_{\xi+1} \le \nu_{f(\xi+1)}$. Clearly each h_f is strictly increasing. It suffices to show that $\{h_f : f \in B\}$ is $\le_\lambda$-unbounded. Suppose that $h_f \le_\lambda g \in {}^\lambda\mu$ for all $f \in B$. For each $\xi < \mathrm{cf}\mu$ let

$$M_\xi = \{\alpha < \lambda : g(\alpha) \in [\nu_\xi, \nu_{\xi+1})\}.$$

Thus $\lambda = \bigcup_{\xi < \mathrm{cf}\mu} M_\xi$.

(1) $\forall \xi < \mathrm{cf}\mu(|M_\xi| < \lambda)$.

For, suppose that $|M_\xi| = \lambda$. Then for every $\alpha \in [\rho_{\xi+1}, \lambda) \cap M_\xi$ and all $f \in B$, there is an $\eta \ge \xi$ such that $h_f(\alpha) = \nu_{f(\eta+1)} + \alpha \ge \nu_{f(\xi+1)} \ge \nu_{\xi+1} > g(\alpha)$, so $\{\alpha : h_f(\alpha) > g(\alpha)\}| = \lambda$, contradiction. So (1) holds.

By (1), an obvious construction gives an increasing sequence $\langle \eta_\xi : \xi < \mathrm{cf}\mu \rangle$ of ordinals less than $\mathrm{cf}\mu$ such that for all $\xi < \mathrm{cf}\mu$ we have $\rho_{\xi+1} \le |M_{\eta_\xi}| < \lambda$. Now if $\xi < \mathrm{cf}\mu$, then

$$M_{\eta_\xi} = \bigcup_{\tau < \mathrm{cf}\mu} (M_{\eta_\xi} \cap [\rho_\tau, \rho_{\tau+1})),$$

and $\mathrm{cf}\mu < \rho_{\xi+1}$, so there is a $\tau_\xi < \mathrm{cf}\mu$ such that $|M_{\eta_\xi} \cap [\rho_{\tau_\xi}, \rho_{\tau_\xi+1})| \ge \rho_{\xi+1}$.

(2) For each $\sigma < \mathrm{cf}\mu$ we have $|\{\xi < \mathrm{cf}\mu : \tau_\xi = \sigma\}| < \mathrm{cf}\mu$.

For, suppose otherwise. Choose $\xi < \mathrm{cf}\mu$ such that $\tau_\xi = \sigma$ and $\sigma < \xi$. Then

$$|M_{\eta_\xi} \cap [\rho_{\tau_\xi}, \rho_{\tau_\xi+1})| \ge \rho_{\xi+1} > \rho_{\sigma+1} = \rho_{\tau_\xi+1},$$

contradiction. So (2) holds.

By (2) we can define $l \in {}^{\mathrm{cf}\mu}\mathrm{cf}\mu$ such that for every $\sigma \in \mathrm{rng}(\tau)$ we have $l(\sigma) >$ each η_ξ such that $\tau_\xi = \sigma$. Now $|\mathrm{rng}(\tau)| = \mathrm{cf}\mu$ by (2). The set B is $\le_{\mathrm{cf}\mu}$ unbounded on $\mathrm{rng}(\tau)$, so choose $f \in B$ such that $|\{\sigma \in \mathrm{rng}(\tau) : f(\sigma) > l(\sigma)\}| = \mathrm{cf}\mu$. Now take any $\sigma \in \mathrm{rng}(\tau)$ such that $f(\sigma) > l(\sigma)$, take any $\xi < \mathrm{cf}\mu$ such that $\tau_\xi = \sigma$, and take any $\alpha \in M_{\eta_\xi} \cap [\rho_{\tau_\xi}, \rho_{\tau_\xi+1})$. Then

$$h_f(\alpha) = \nu_{f(\tau_\xi+1)} + \alpha > \nu_{f(\tau_\xi)} = \nu_{f(\sigma)} > \nu_{l(\sigma)} \ge \nu_{\eta_\xi+1} > g(\alpha).$$

This shows that $|\{\alpha : h_f(\alpha) > g(\alpha)\}| = \lambda$, contradiction. □

Proposition 3.11 *Suppose that* $\mathrm{cf}\mu < \lambda \le \mu$. *Then the set* B *of all strictly increasing functions in* ${}^\lambda\mu$ *is not* $\le_\kappa$*-dominating, but it* $\le_\kappa$*-dominates every strictly increasing member of* ${}^\lambda\mu$.

Proof This is immediate from the remark following 3.2. □

4 $\mathfrak{b}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$

Proposition 4.1 *The following conditions are equivalent:*

(i) *There is a* (κ, λ, μ)*-scale.*
(ii) $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{d}_{\kappa,\lambda,\mu}$.
(iii) $\mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$ *exists.*
(iv) $\mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$ *exists and equals* $\mathfrak{b}_{\kappa,\lambda,\mu}$.

Proof (i)⇔(iii) Obvious.

(iii)⇒(ii) and (iii)⇒(iv) Assume (iii). Clearly then $\mathfrak{d}_{\kappa,\lambda,\mu} \leq \mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$, so by 2.2 it suffices to show that $\mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu} \leq \mathfrak{b}_{\kappa,\lambda,\mu}$. Suppose that $B \subseteq {}^{\lambda}\mu$ is $\leq_\kappa$-unbounded and $|B| < \mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$. Let C be a (κ, λ, μ)-scale of size $\mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$. Clearly $\mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$ is regular, so we can take a cofinal subset in order to assume that C has order type $\mathfrak{d}^{\mathrm{wo}}_{\kappa,\lambda,\mu}$ under $<_\kappa$. For each $f \in B$ choose $g_f \in C$ such that $f <_\kappa g_f$. Hence there is an $h \in C$ such that $g_f <_\kappa h$ for all $f \in B$. But then h is a $\leq_\kappa$-bound of B, contradiction.

(iv)⇒(ii) Obvious.

(ii)⇒(i) Let $B \subseteq {}^{\lambda}\mu$ be $\leq_\kappa$-dominating and of size $\mathfrak{d}_{\kappa,\lambda,\mu}$. List it as $\langle f_\alpha : \alpha < \mathfrak{d}_{\kappa,\lambda,\mu}\rangle$. Now we define a new sequence $\langle g_\alpha : \alpha < \mathfrak{d}_{\kappa,\lambda,\mu}\rangle$. Suppose it is defined for all $\alpha < \beta$. Then $\{g_\alpha : \alpha < \beta\}$ has size less than $\mathfrak{b}_{\kappa,\lambda,\mu}$ by (ii), so it is bounded; say that $g_\alpha <_\kappa h$ for all $\alpha < \beta$. Let g_β be any function such $h \leq_\kappa g_\beta$ and $f_\beta \leq_\kappa g_\beta$. Then $\{g_\alpha : \alpha < \mathfrak{d}_{\kappa,\lambda,\mu}\}$ is a (κ, λ, μ)-scale, as desired. □

Corollary 4.2

(i) *If* $\lambda < \mathrm{cf}\mu$ *then there is a* (κ, λ, μ)*-scale.*
(ii) *If* $\mathrm{cf}\mu \leq \lambda$ *and* $\kappa < \lambda$*, then there does not exist a* (κ, λ, μ)*-scale.*
(iii) *If* $\mathrm{cf}\mu \leq \lambda = \kappa$ *and* $\mathrm{cf}\mu \neq \mathrm{cf}\lambda$*, then there does not exist a* (κ, λ, μ)*-scale.*

Proof By 2.19 and 4.1. □

The possibility not covered by 4.2 can be expressed, by 2.6 and 4.1, as the case of scales (λ, λ, μ) with μ regular and $\mathrm{cf}\lambda = \mu$. Consistently there are no scales in this case when λ is singular, as by the Cummings, Shelah result we can have $\mathfrak{b}_{\mu,\mu,\mu} < \mathfrak{d}_{\mu,\mu,\mu}$ for μ regular, while for λ singular with $\mathrm{cf}\lambda = \mu$ we can have $\mathfrak{b}_{\lambda,\lambda,\mu} \leq \mathfrak{b}_{\mu,\mu,\mu} = \mu^+ < \lambda < \mathfrak{d}_{\lambda,\lambda,\mu}$ by 2.13 and 2.20. In the other direction, by their theorem it is consistent to have (μ, μ, μ)-scales for any regular μ. But the following problem appears to be open.

Problem 3 Suppose that λ is singular with $\mathrm{cf}\lambda = \mu$. Is it consistent to have a (λ, λ, μ)-scale?

5 $\mathfrak{b}^{\uparrow\mathrm{wo}}_{\kappa,\lambda,\mu}$ and $\mathfrak{d}^{\uparrow\mathrm{wo}}_{\kappa,\lambda,\mu}$

Proposition 5.1 *If* $\lambda \leq \mathrm{cf}\mu$*, then* $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{b}^{\uparrow\mathrm{wo}}_{\kappa,\lambda,\mu}$.

Proof Let $B \subseteq {}^{\lambda}\mu$ be $\leq_\kappa$-unbounded with $|B| = \mathfrak{b}_{\kappa,\lambda,\mu}$. Write $B = \{f_\alpha : \alpha < \mathfrak{b}_{\kappa,\lambda,\mu}\}$. Now we define $\langle g_\alpha : \alpha < \mathfrak{b}_{\kappa,\lambda,\mu}\rangle$ by recursion. Suppose that g_α has been defined for all $\alpha < \beta$, with $\beta < \mathfrak{b}_{\kappa,\lambda,\mu}$. Then $\{g_\alpha : \alpha < \beta\}$ is $\leq_\kappa$-bounded, say by h. Now we define

$$g_\beta(\xi) = \max\{\sup\{g_\beta(\eta) + 1 : \eta < \xi\}, h(\xi) + 1\}.$$

Clearly then $\{g_\alpha : \alpha < \mathfrak{b}_{\kappa,\lambda,\mu}\}$ satisfies the conditions defining $\mathfrak{b}^{\uparrow \mathrm{wo}}_{\kappa,\lambda,\mu}$. □

Proposition 5.2 *Assume the following conditions (i)–(ii):*

(i) $\mathrm{cf}\mu < \lambda < \mu$;
(ii) $\kappa < \lambda$, *or else* $\kappa = \lambda$ *and* $\mathrm{cf}\lambda \neq \mathrm{cf}\mu$.

Then $\mathfrak{b}^{\uparrow \mathrm{wo}}_{\kappa,\lambda,\mu} = \mathrm{cf}\mu$.

Proof Let $\nu_\xi \uparrow \mu$ for $\xi < \mathrm{cf}\mu$, the ν_ξ cardinals, with $\nu_\xi + \lambda < \nu_{\xi+1}$ for each ξ (ordinal addition). Define $f_\xi(\eta) = \nu_\xi + \eta$ for all $\xi < \mathrm{cf}\mu$ and $\eta < \lambda$. Clearly each f_ξ is strictly increasing, and $\langle f_\xi : \xi < \mathrm{cf}\mu\rangle$ is $<_1$-increasing. By the proof of 2.16, $\{f_\xi : \xi < \mathrm{cf}\mu\}$ is $\leq_\kappa$-unbounded. □

The following two situations are not covered by 5.1 and 5.2:

1. $\mathrm{cf}\mu < \lambda = \mu$;
2. $\kappa = \lambda$ and $\mathrm{cf}\lambda = \mathrm{cf}\mu < \lambda < \mu$.

Proposition 5.3 *If* $\kappa < \mu$, *then* $\mathfrak{b}^{\uparrow \mathrm{wo}}_{\kappa,\mu,\mu} = \mathrm{cf}\mu$.

Proof If μ is regular, this holds by 5.1 and 2.17. So, suppose that μ is singular. Let $\nu_\xi \uparrow \mu$ for $\xi < \mathrm{cf}\mu$, with each ν_ξ an infinite cardinal. Let $\preceq$ be the standard ordering of $\mu \times \mu$:

$$\begin{aligned}(\alpha,\beta) \preceq (\gamma,\delta) \quad &\text{iff} \quad \max\{\alpha,\beta\} < \max\{\gamma,\delta\}\\ &\text{iff} \quad \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \beta < \delta\\ &\text{iff} \quad \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \beta = \delta \text{ and } \alpha < \gamma.\end{aligned}$$

Clearly if $\alpha < \beta$ then $(\alpha,\gamma) \prec (\beta,\gamma)$, and if $\beta < \gamma$ then $(\alpha,\beta) \prec (\alpha,\gamma)$.

Let $g : \mu \times \mu \to \mu$ be the order isomorphism given in elementary set theory. Now for each $\xi < \mathrm{cf}\mu$ we define $f_\xi(\beta) = g(\nu_\xi,\beta)$ for any $\beta < \mu$. Clearly each f_ξ is strictly increasing, and $f_\xi <_1 f_\eta$ if $\xi < \eta$. We claim that $\{f_\xi : \xi < \mathrm{cf}\mu\}$ is $\leq_\kappa$-unbounded. For, suppose that $f_\xi <_\kappa h$ for all $\xi < \mathrm{cf}\mu$. Thus the set $M_\xi \overset{\mathrm{def}}{=} \{\alpha < \mu : f_\xi(\alpha) \geq h(\alpha)\}$ has size less than κ. Choose $\alpha \in \mu \setminus \bigcup_{\xi<\mathrm{cf}\mu} M_\xi$. Then $g(\nu_\xi,\alpha) = f_\xi(\alpha) < h(\alpha)$ for all $\xi < \mathrm{cf}\mu$, contradiction. So our claim holds.

Now the proposition follows by 2.7. □

Proposition 5.4 *If* $\lambda < \mathrm{cf}\mu$, *then* $\mathfrak{d}^{\uparrow \mathrm{wo}}_{\kappa,\lambda,\mu} = \mathrm{cf}\mu$.

Proof By 2.19(v), let $\{f_\alpha : \alpha < \mathrm{cf}\mu\}$ be $\leq_\kappa$-dominating. Now we define $\langle g_\alpha : \alpha < \mathrm{cf}\mu\rangle$ by induction:

$$g_\alpha(\xi) = \max\{\sup_{\eta<\xi}(g_\alpha(\eta)+1),\ \sup_{\beta<\alpha}(g_\beta(\xi)+1),\ f_\alpha(\xi)\}.$$

Clearly $\{g_\alpha : \alpha < \mathrm{cf}\mu\}$ shows that $\mathfrak{d}^{\uparrow \mathrm{wo}}_{\kappa,\lambda,\mu} = \mathrm{cf}\mu$. □

Recall from the remark after 3.2 that $\mathfrak{d}^{\uparrow \mathrm{wo}}_{\kappa,\lambda,\mu}$ does not exist if $\mathrm{cf}\mu < \lambda$.

Proposition 5.5 *For* $\lambda \leq \mathrm{cf}\mu$ *the following conditions are equivalent:*

(i) *There is a* (κ,λ,μ)-*scale.*
(ii) $\mathfrak{d}^{\uparrow \mathrm{wo}}_{\kappa,\lambda,\mu}$ *exists.*
(iii) $\mathfrak{d}^{\uparrow \mathrm{wo}}_{\kappa,\lambda,\mu}$ *exists and equals* $\mathfrak{b}_{\kappa,\lambda,\mu}$.

Proof This is clear by 2.11, 4.1, and 5.4 if $\lambda < \mathrm{cf}\mu$. For $\mathrm{cf}\mu = \lambda$, one can modify the proof of (ii)⇒(i) in the proof of 4.1 by applying 3.2 to g_β there. □

In case $\mathrm{cf}\mu < \lambda$, (ii) and (iii) of 5.5 are both false. If $\mathrm{cf}\mu < \lambda$, and either $\kappa < \lambda$ or else $\kappa = \lambda$ and $\mathrm{cf}\lambda \neq \mathrm{cf}\mu$, then also (i) is false. In the remaining case, $\mathrm{cf}\mu < \lambda = \kappa$ and $\mathrm{cf}\mu = \mathrm{cf}\lambda$ it is unknown whether (i) holds.

The following question is open.

Problem 4 Suppose that $\mathrm{cf}\mu < \lambda \leq \mu$ and $\mathrm{cf}\lambda = \mathrm{cf}\mu$. Does $\mathfrak{b}^{\uparrow\mathrm{wo}}_{\lambda,\lambda,\mu}$ exist?

Simple cases of this problem are whether $\mathfrak{b}^{\uparrow\mathrm{wo}}_{\aleph_\omega,\aleph_\omega,\aleph_{\omega+\omega}}$ or $\mathfrak{b}^{\uparrow\mathrm{wo}}_{\aleph_\omega,\aleph_\omega,\aleph_\omega}$ exist.

6 $\mathfrak{b}^{\mathbf{sub}}_{\kappa,\lambda,\mu}$

Clearly ${}^\lambda\mu$ itself is $\leq_\kappa$-unbounded on every $A \in [\lambda]^\lambda$, so $\mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu}$ is always defined. Clearly $\mathfrak{b}_{\kappa,\lambda,\mu} \leq \mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu}$.

Proposition 6.1 $\mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu} = \mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mathrm{cf}\mu}$.

Proof We use the notation of the proof of Proposition 2.6. If $B \subseteq {}^\lambda\mu$ is unbounded on each $A \in [\lambda]^\lambda$, it is clear from $(*)$ that $\{g^- : g \in B\}$ is also unbounded on each $A \in [\lambda]^\lambda$. This proves $\geq$. Now suppose that $B \subseteq {}^\lambda\mathrm{cf}\mu$ is unbounded on each $A \in [\lambda]^\lambda$; we claim that $\{g^+ : g \in B\}$ is also unbounded on each $A \in [\lambda]^\lambda$. So, suppose that $A \in [\lambda]^\lambda$ and $f \in {}^\lambda\mu$. Choose $g \in B$ such that $|\{\alpha \in A : f^-(\alpha) < g(\alpha)\}| \geq \kappa$. It suffices now to show that

$$\{\alpha \in A : f^-(\alpha) < g(\alpha)\} \subseteq \{\alpha \in A : f(\alpha) < g^+(\alpha)\}.$$

Suppose that this inclusion fails; let $\alpha \in A$ be such that $f^-(\alpha) < g(\alpha)$ but $g^+(\alpha) \leq f(\alpha)$. Then

$$f(\alpha) \leq \nu_{f^-(\alpha)} < \nu_{g(\alpha)} = g^+(\alpha) \leq f(\alpha),$$

contradiction. □

Proposition 6.2 *Suppose that $\lambda \leq \mu$ and that one of the following conditions holds:*

(i) $\kappa, \mathrm{cf}\mu < \lambda$.
(ii) $\kappa < \mathrm{cf}\mu$.
(iii) $\mathrm{cf}\mu < \kappa = \lambda$ *and* $\mathrm{cf}\lambda \neq \mathrm{cf}\mu$.

Then $\mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu} = \mathrm{cf}\mu$.

Proof We repeat the proof of 2.16 through the definition of B. We just need to show that B is $\leq_\kappa$-unbounded on every $A \in [\lambda]^\lambda$. Suppose to the contrary that $f \in {}^\lambda\mu$ and for every $g \in B$ we have $|\{\alpha \in A : f(\alpha) < g(\alpha)\}| < \kappa$. Let F be a bijection from λ to A, and for each $\xi < \mathrm{cf}\mu$ let $\Gamma_\xi = \{\eta < \lambda : f_\xi(F(\eta)) > f(F(\eta))\}$. The conditions of 2.14 hold, contradiction. □

Proposition 6.3 *Suppose that $\lambda \leq \mu$ and $\kappa \leq \mathrm{cf}\lambda$. Then* $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu} = \mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu}$.

Proof By 3.8 we have $\mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu} \leq \mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu}$. Obviously $\mathfrak{b}_{\kappa,\lambda,\mu} \leq \mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu}$. By 3.5 and the remark after it, our conclusion follows. □

Proposition 6.4 *Suppose that $\lambda \leq \mu$ and that one of the following conditions holds:*

(i) $\kappa, \mathrm{cf}\mu < \lambda$.
(ii) $\kappa < \mathrm{cf}\mu$.
(iii) $\mathrm{cf}\mu < \kappa = \lambda$ *and* $\mathrm{cf}\lambda \neq \mathrm{cf}\mu$.
(iv) $\kappa = \lambda = \mathrm{cf}\mu$.

Then $\mathfrak{b}_{\kappa,\lambda,\mu} = \mathfrak{b}^{\uparrow}_{\kappa,\lambda,\mu} = \mathfrak{b}^{\mathrm{sub}}_{\kappa,\lambda,\mu}$.

Proof If one of (i)–(iii) holds, this is true by 6.2, 3.4, and 2.17. Suppose that each of (i)–(iii) fails, and (iv) holds. By 2.15, $\kappa = \lambda$ and $\mathrm{cf}\mu = \mathrm{cf}\lambda$. Hence $\kappa = \lambda = \mathrm{cf}\mu = \mathrm{cf}\lambda$, and 6.3 gives the result. □

Note that (i)–(iv) all fail iff $\kappa = \lambda$, $\mathrm{cf}\mu = \mathrm{cf}\lambda$, and $\lambda \neq \mathrm{cf}\mu$; in other words, if and only if $\kappa = \lambda$, $\mathrm{cf}\mu = \mathrm{cf}\lambda$, and λ is singular. Thus the following problem is open.

Problem 5 For λ singular and $\mathrm{cf}\lambda = \mathrm{cf}\mu$, is $\mathfrak{b}_{\lambda,\lambda,\mu} = \mathfrak{b}^{\mathrm{sub}}_{\lambda,\lambda,\mu}$?

For example, we do not know whether $\mathfrak{b}_{\aleph_\omega,\aleph_\omega,\aleph_\omega} = \mathfrak{b}^{\mathrm{sub}}_{\aleph_\omega,\aleph_\omega,\aleph_\omega}$.

Note

1. This symbol will appear following propositions, corollaries, and so on where the author feels that a proof is not necessary.

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