

Continuous Semiring-Semimodule Pairs and Mixed Algebraic Systems*

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Abstract

We associate with every commutative continuous semiring S and alphabet Σ a category whose objects are all sets and a morphism $X \rightarrow Y$ is determined by a function from X into the semiring of formal series $S\langle\langle(Y \uplus \Sigma)^*\rangle\rangle$ of finite words over $Y \uplus \Sigma$, an $X \times Y$ -matrix over $S\langle\langle(Y \uplus \Sigma)^*\rangle\rangle$, and a function from X into the continuous $S\langle\langle(Y \uplus \Sigma)^*\rangle\rangle$ -semimodule $S\langle\langle(Y \uplus \Sigma)^\omega\rangle\rangle$ of series of ω -words over $Y \uplus \Sigma$. When S is also an ω -semiring (equipped with an infinite product operation), then we define a fixed point operation over our category and show that it satisfies all identities of iteration categories. We then use this fixed point operation to give semantics to recursion schemes defining series of finite and infinite words. In the particular case when the semiring is the Boolean semiring, we obtain the context-free languages of finite and ω -words.

1 Introduction

Suppose that S is a continuous semiring and Σ and X are sets. Let X^* and X^ω respectively denote the sets of all finite and ω -words over X . We can form the continuous semiring $S\langle\langle X^* \rangle\rangle$ of series over X^* with coefficients in S and the continuous $S\langle\langle X^* \rangle\rangle$ -semimodule $S\langle\langle X^\omega \rangle\rangle$ of series over X^ω with coefficients in S . If S is equipped with an infinite product operation $S^\omega \rightarrow S, s_1 s_2 \cdots \mapsto \prod_{n \geq 1} s_n$, satisfying certain axioms including a sort of continuity described in the sequel, then we can also define an infinite product operation $(S\langle\langle X^* \rangle\rangle)^\omega \rightarrow S\langle\langle X^\omega \rangle\rangle$. In our first result, we show that the construction of the ‘continuous ω -semiring-semimodule pair’ $(S\langle\langle X^* \rangle\rangle, S\langle\langle X^\omega \rangle\rangle)$ enjoys a universal property, cf. Theorem 5.1.

In the second part of the paper we use the above universality result to give an algebraic treatment of recursion schemes defining series of finite and infinite words over Σ . To this end, we will restrict ourselves to commutative continuous

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ω -semirings S . Suppose that

$$x_i = p_i \quad i \in I \quad (R)$$

is a finite or infinite recursion scheme (or system of fixed point equations), where each p_i is a series in $S\langle\langle X \uplus \Sigma \rangle\rangle^*$. Then we associate with R in a natural way a function

$$F_R : S\langle\langle \Sigma^* \rangle\rangle \times S\langle\langle S^\omega \rangle\rangle \rightarrow S\langle\langle \Sigma^* \rangle\rangle \times S\langle\langle S^\omega \rangle\rangle$$

and define the semantics of R as a ‘canonical’ fixed point of F_R .

In order to facilitate the construction of the canonical fixed point, we introduce a category whose objects are sets (of recursion variables) and a morphism $f : X \rightarrow Y$ has three components:

- a function $f_0 : X \rightarrow S\langle\langle Y \uplus \Sigma \rangle\rangle^*$,
- a matrix $f_M \in (S\langle\langle Y \uplus \Sigma \rangle\rangle^*)^{X \times Y}$,
- a function $f_\omega : X \rightarrow S\langle\langle Y \uplus \Sigma \rangle\rangle^\omega$.

We define composition and the identity morphisms in a natural way to obtain a category $\mathbf{Ser}_{S,\Sigma}^\omega$. By taking the first components of morphisms, this category can be projected onto the category $\mathbf{Ser}_{S,\Sigma}$ having sets as objects and functions $X \rightarrow S\langle\langle Y \uplus \Sigma \rangle\rangle^*$ as morphisms $X \rightarrow Y$. It will be clear from the definition that the coproduct $X_1 \oplus \dots \oplus X_n$ of any finite sequence X_1, \dots, X_n of objects exists in $\mathbf{Ser}_{S,\Sigma}^\omega$, in fact it will be given by disjoint union. (Actually all coproducts will exist, but this fact is not important for the paper.)

Next we define a dagger operation mapping a morphism $f : X \rightarrow X \oplus Y$ to a morphism $f^\dagger : X \rightarrow Y$. We prove that for any $f : X \rightarrow X \oplus Y$, the morphism $f^\dagger : X \rightarrow Y$ is a solution of the fixed point equation

$$\xi = f \circ \langle \xi, \mathbf{id}_Y \rangle.$$

(Here, $\langle -, - \rangle$ denotes the source pairing operation determined by the coproduct structure.) Intuitively, f represents a system of fixed point equations (recursion scheme) in the variables X and parameters Y , and f^\dagger is its canonical solution. In particular, the function F_R associated with a system of fixed point equations (R) can be seen as a morphism $f : X \rightarrow X$ in $\mathbf{Ser}_{S,\Sigma}^\omega$, and the canonical fixed point of F_R can be derived from $f^\dagger : X \rightarrow \emptyset$. Indeed the three components of f^\dagger are a function $X \rightarrow S\langle\langle \Sigma^* \rangle\rangle$, the empty $X \times \emptyset$ -matrix over $S\langle\langle \Sigma^* \rangle\rangle$, and a function $X \rightarrow S\langle\langle \Sigma^\omega \rangle\rangle$. The first and third components form the canonical fixed point of F_R . Our approach generalizes the construction of context-free languages of finite and ω -words [10].

Categories with finite coproducts (or dually, products) equipped with a parametric fixed point operation have been studied since the late 1960’s. A class of structures, called iteration theories, or iteration categories, have been identified. It has been shown that most (if not all) of the major fixed point structures used in computer science give rise to iteration categories. As a main technical contribution,

we prove that for each commutative continuous semiring S and alphabet Σ , $\mathbf{Ser}_{S,\Sigma}^\omega$ is also an iteration category, cf. Theorem 7.1. A few consequences of this fact are discussed in the conclusion.

2 Continuous semirings

Recall that a commutative monoid $(V, +, 0)$ is *continuous* (cf. Section 2.2 of [6]) if it is equipped with a partial order \leq such that the supremum of any chain (or equivalently, directed set [12]) exists and 0 is the least element. Moreover, the sum operation $+$ is continuous:

$$x + \sup Y = \sup(x + Y)$$

for all nonempty chains or nonempty directed sets $Y \subseteq V$, where $x + Y = \{x + y : y \in Y\}$. (Actually this also holds when the set is empty.) It follows that the sum operation is monotonic: if $x \leq y$ in V , then $x + z \leq y + z$ for all $z \in V$.

Suppose that V is a continuous commutative monoid and $x_i \in V$ for all $i \in I$. We define $\sum_{i \in I} x_i$ as the supremum of all finite sums $x_{i_1} + \dots + x_{i_n}$ where i_1, \dots, i_n are pairwise different elements of I . It is well-known that this summation operation is completely associative and commutative:

$$\sum_{i \in I} x_i = \sum_{j \in J} \sum_{i \in I_j} x_i,$$

whenever I is the disjoint union of the sets $I_j, j \in J$ and $x_i \in V$ for all $i \in I$. Moreover,

$$\sum_{i \in I} x_i = \sum_{i \in I} x_{\pi(i)},$$

whenever π is a permutation $I \rightarrow I$ and $x_i \in V$ for all $i \in I$.

Continuous commutative monoids are closed under several constructions including direct product. Suppose that V_i is a continuous commutative monoid for all $i \in I$. Then $V = \prod_{i \in I} V_i$, equipped with the pointwise sum operation and pointwise ordering, is also a continuous commutative monoid. It follows that the summation operation in the product V is the pointwise summation. In particular, if V is a continuous commutative monoid, then so is V^I for any set I .

Suppose now that $S = (S, +, \cdot, 0, 1)$ is a semiring [8, 11]. We say that S is a *continuous semiring* (cf. Section 2.2 of [6]) if $(S, +, 0)$ is a continuous commutative monoid equipped with a partial order \leq^1 and the product operation is continuous (hence, also monotonic), i.e., it preserves the supremum of nonempty chains (or nonempty directed subsets) in either argument:

$$\begin{aligned} (\sup X)y &= \sup(Xy) \\ y(\sup X) &= \sup(yX), \end{aligned}$$

¹Unlike at some other places, we do not require that $x \leq y$ iff there is some z with $x + z = y$.

for all nonempty chains (or directed sets) $X \subseteq S$, where $Xy = \{xy : x \in X\}$ and yX is defined in the same way. It follows that product distributes over all sums:

$$\begin{aligned} \left(\sum_{i \in I} x_i\right)y &= \sum_{i \in I} x_i y \\ y\left(\sum_{i \in I} x_i\right) &= \sum_{i \in I} y x_i, \end{aligned}$$

whenever $x_i \in S$ for all $i \in I$.

Suppose that S is a continuous semiring and I is a set. Then the *matrix semiring* $S^{I \times I}$, equipped with the pointwise sum operation, the usual matrix product operation and the pointwise ordering is also a continuous semiring. The matrix product is meaningful since the sum of any I -indexed family of elements of S exists.

Continuous semirings are also closed under the formation of *power series semirings*. Suppose that S is a continuous semiring and X is a set. As usual, let $S\langle\langle X^* \rangle\rangle$ denote the semiring of all power series $s = \sum_{u \in X^*} \langle s, u \rangle u$ over X with coefficients in S . Each series s may be viewed as a function $X^* \rightarrow S$ mapping a word $u \in X^*$ to $\langle s, u \rangle$. Equipped with the pointwise order relation $s \leq s'$ iff $\langle s, u \rangle \leq \langle s', u \rangle$ for all $u \in X^*$, $S\langle\langle X^* \rangle\rangle$ is a continuous semiring. The sum of any family of series is the pointwise sum.

Theorem 2.1. *Suppose that S is a continuous semiring. Then for each set X , the continuous semiring $S\langle\langle X^* \rangle\rangle$ has the following universal property. Given a continuous semiring S' , a continuous semiring morphism $h_S : S \rightarrow S'$ and any function $h_X : X \rightarrow S'$ such that the elements of $h_S(S)$ commute with the elements of S' , there is a unique continuous semiring morphism $h^\sharp : S\langle\langle X^* \rangle\rangle \rightarrow S'$ extending h_S and h_X .*

Proof. We provide an outline of the proof. For details, see [7].

First we extend h_X to a monoid morphism $X^* \rightarrow S'$, denoted just h . Then, for a series $s \in S\langle\langle X^* \rangle\rangle$, we define $h^\sharp(s) = \sum_{u \in X^*} h_S(\langle s, u \rangle) h(u)$. It is clear that h^\sharp extends h_S and h_X and preserves the constants 0 and 1. Then we prove that h^\sharp is continuous and preserves the binary sum operation. It then follows that h^\sharp preserves all finite and infinite sums. Last, we prove that h^\sharp preserves the product operation. Since the definition of h^\sharp was forced, it is unique. \square

3 Continuous semiring-semimodule pairs

Suppose now that S is a semiring and V is a commutative monoid as above. We call V a (left) S -*semimodule* [9] if there is an action $S \times V \rightarrow S$ subject to the usual

associativity, distributivity and unitary conditions:

$$\begin{aligned}
 (s_1 s_2)v &= s_1(s_2 v) \\
 (s_1 + s_2)v &= s_1 v + s_2 v \\
 s(v_1 + v_2) &= s v_1 + s v_2 \\
 1v &= v \\
 0v &= 0 \\
 s0 &= 0
 \end{aligned}$$

for all $s, s_1, s_2 \in S$ and $v, v_1, v_2 \in V$. When V is an S -semimodule, we also say that (S, V) is a *semiring-semimodule* pair.

We call a semiring-semimodule pair (S, V) continuous if S is a continuous semiring and V is a continuous commutative monoid such that the action is continuous in either argument:

$$\begin{aligned}
 (\sup X)v &= \sup(Xv) \\
 x(\sup Y) &= \sup(xY)
 \end{aligned}$$

for all $x \in S, y \in V$ and nonempty chains (or nonempty directed sets) $X \subseteq S$ and $Y \subseteq V$. Of course, $Xv = \{sv : s \in X\}$ and $xY = \{xw : w \in Y\}$. It follows that action distributes over summation on either side:

$$\begin{aligned}
 \left(\sum_{i \in I} x_i\right)v &= \sum_{i \in I} x_i v \\
 x\left(\sum_{i \in I} v_i\right) &= \sum_{i \in I} x v_i
 \end{aligned}$$

for all $x, x_i \in S, v, v_i \in V, i \in I$, where I is any index set.

Moreover, we call a continuous semiring-semimodule pair (S, V) a continuous ω -semiring-semimodule pair if it is equipped with an *infinite product operation* $\prod_{n>1} x_n$ mapping an ω -sequence (or ω -word) $x_1 x_2 \dots \in S^\omega$ to $\prod_{n \geq 1} x_n \in V$. The infinite product is subject to the following axioms:

Ax1

$$x \prod_{n \geq 1} x_n = \prod_{n \geq 1} y_n,$$

where $y_1 = x$ and $y_{n+1} = x_n$ for all $n \geq 1$.

Ax2

$$\prod_{n \geq 1} x_n = \prod_{n \geq 1} x_{i_n} \dots x_{i_{n+1}-1}$$

where the sequence $i_1 = 1 \leq i_2 \leq \dots$ increases without a bound and the product of an empty family is 1.

Ax3

$$\prod_{n \geq 1} (x_n + y_n) = \sum_{z_n = x_n \text{ or } z_n = y_n} \prod_{n \geq 1} z_n$$

Ax4

$$\prod_{n \geq 1} \sup X_n = \sup_{x_n \in X_n} \prod_{n \geq 1} x_n$$

where for each n , $X_n \subseteq S$ is a nonempty chain (or a nonempty directed set).

It follows that

$$\prod_{n \geq 1} \sum_{i_n \in I_n} x_{i_n} = \sum_{i_1 \in I_1, i_2 \in I_2, \dots} \prod_{n \geq 1} x_{i_n} \quad (1)$$

where $\{x_{i_n} : i_n \in I_n\}$ is a family of elements of S for all $n \geq 1$. Indeed,

$$\begin{aligned} \prod_{n \geq 1} \sum_{i_n \in I_n} x_{i_n} &= \prod_{n \geq 1} \sup_{F_n \subseteq I_n} \left\{ \sum_{i_n \in F_n} x_{i_n} \right\} \\ &= \sup_{F_n \subseteq I_n} \prod_{n \geq 1} \left\{ \sum_{i_n \in F_n} x_{i_n} \right\} \\ &= \sup_{F_n \subseteq I_n} \sum_{i_n \in F_n} \prod_{n \geq 1} x_{i_n} \\ &= \sup_{F \subseteq I_1 \times I_2 \times \dots} \sum_{(i_1, i_2, \dots) \in F} \prod_{n \geq 1} x_{i_n} \\ &= \sum_{(i_1, i_2, \dots) \in I_1 \times I_2 \times \dots} \prod_{n \geq 1} x_{i_n}. \end{aligned}$$

In particular, note that $\prod_{n \geq 1} x_n = 0$ whenever there is some m such that x_m is 0. (This also follows from Ax1.) Moreover, infinite product is monotonic: if $x_n \leq y_n$ in S for all $n \geq 1$, then $\prod_{n \geq 1} x_n \leq \prod_{n \geq 1} y_n$.

Suppose that S is a continuous semiring. Then we define a star operation $S \rightarrow S$ as usual: $s^* = \sum_{n \geq 0} s^n$ for all $s \in S$. It is known that s^* is the least solution of the fixed point equation $x = sx + 1$ (and also of $x = xs + 1$) over S . And if (S, V) is a continuous ω -semiring-semimodule pair, we define an omega operation $S^\omega \rightarrow V$ by $s^\omega = \prod_{n \geq 1} s$ for all $s \in S$. It is known that for each $s \in S$, s^ω is a solution of the equation $v = sv$ over V .

Complete equational and quasi-equational axiomatization of the equational properties of the star operation in continuous semirings has been given in [2]. Among the identities satisfied by continuous semirings are the sum star and product star identities [3, 1]:

$$\begin{aligned} (x + y)^* &= (x^* y)^* x^* \\ (xy)^* &= 1 + x(yx)^* y \end{aligned}$$

Also $0^* = 1$ and $1^* = 1^{**}$ hold.

When (S, V) is a continuous ω -semiring-semimodule pair, the omega operation satisfies the sum omega and product omega identities [1]:

$$\begin{aligned} (x + y)^\omega &= (x^* y)^\omega + (x^* y)^* x^\omega \\ (xy)^\omega &= x(yx)^\omega \end{aligned}$$

for all $x, y \in S$. Also $0^\omega = 0$.

4 Matrices and series

4.1 Matrices

Suppose that (S, V) is a continuous ω -semiring-semimodule pair and I is a set. Then, as mentioned above, $S^{I \times I}$ is a continuous semiring, and V^I is a continuous commutative monoid. There is a natural left action of $S^{I \times I}$ on V^I defined similarly to matrix multiplication:

$$(MN)_i = \sum_{j \in I} M_{i,j} N_j, \quad i \in I,$$

for all $M \in S^{I \times I}$ and $N \in V^I$. Moreover, we may define an infinite product operation by

$$\left(\prod_{n \geq 1} M_n \right)_i = \sum_{i=i_1, i_2, \dots} \prod_{n \geq 1} (M_n)_{i_n, i_{n+1}}, \quad i \in I.$$

Proposition 4.1. *Suppose that (S, V) is a continuous ω -semiring-semimodule pair and I is a set. Then, equipped with the pointwise orderings, $(S^{I \times I}, V^I)$ is also a continuous ω -semiring-semimodule pair.*

Proof. The fact that $S^{I \times I}$ is a continuous semiring is proved in [7]. It is clear that V^I is a continuous commutative monoid and that equipped with the action, $(S^{I \times I}, V^I)$ is a semiring-semimodule pair. The action is continuous in either argument, so that $(S^{I \times I}, V^I)$ is also a continuous semiring-semimodule pair. This can be proved by an argument similar to that used in [7] to establish that product in $S^{I \times I}$ is continuous in either argument.

In order to conclude, we still need to prove that the infinite product over matrices satisfies the axioms Ax1-Ax4. Let $M_n \in S^{I \times I}$ for all $n \geq 1$, and define $M'_1 = M_1$ and $M'_n = M_{n+1}$ for all $n \geq 1$. Then for every $i \in I$, the i th component of $\prod_{n \geq 1} M'_n$ is

$$\sum_{j \in I} (M_1)_{i,j} \sum_{j=j_1, j_2, \dots \in I} \left(\prod_{n \geq 1} (M_{n+1})_{j_n, j_{n+1}} \right) = \sum_{i=i_1, i_2, \dots \in I} \prod_{n \geq 1} (M_n)_{i_n, i_{n+1}}$$

which is the i th component of $\prod_{n \geq 1} M_n$. Hence, Ax1 holds.

Suppose now that the sequence $1 = k_1 \leq k_2 \leq \dots$ increases without a bound and define $M'_n = M_{i_n} \cdots M_{i_{n+1}-1}$ for all $n \geq 1$. Then for each $i \in I$, the j th component of $\prod_{n \geq 1} M'_n$ is

$$\begin{aligned} & \sum_{j=j_1, j_2, \dots \in I} \prod_{n \geq 1} (M'_n)_{j_n, j_{n+1}} \\ &= \sum_{j=j_1, j_2, \dots \in I} \prod_{n \geq 1} \sum_{j_n=\ell_1, \ell_2, \dots, \ell_{i_{n+1}-i_n}=j_{n+1}} (M_{i_n})_{\ell_1, \ell_2} \cdots (M_{i_{n+1}-1})_{\ell_{i_{n+1}-i_n-1}, \ell_{i_{n+1}-i_n}} \\ &= \sum_{j=j_1, j_2, \dots} \prod_{n \geq 1} (M_j)_{j_n, j_{n+1}}, \end{aligned}$$

proving Ax2.

In order to prove that Ax3 holds, let $M_n, M'_n \in S^{I \times I}$ for all $n \geq 1$. Then for every $i \in I$, the i th component of $\prod_{n \geq 1} (M_n + M'_n)$ is

$$\begin{aligned} \sum_{i=i_1, i_2, \dots} \prod_{n \geq 1} (M_n + M'_n)_{i_n, i_{n+1}} &= \sum_{P_n=M_n \text{ or } P_n=M'_n} \sum_{i=i_1, i_2, \dots} \prod_{n \geq 1} P_{i_n, i_{n+1}} \\ &= \left(\sum_{P_n=M_n \text{ or } P_n=M'_n} \prod_{n \geq 1} P_n \right)_i, \end{aligned}$$

i.e., the i th component $\sum_{P_n=M_n \text{ or } P_n=M'_n} \prod_{n \geq 1} P_n$. This proves Ax3.

Suppose now that for each $n \geq 1$, \mathcal{M}_n is a nonempty chain w.r.t. the pointwise ordering of matrices in $S^{I \times I}$. We want to prove that

$$\prod_{n \geq 1} \sup \mathcal{M}_n = \sup_{M_n \in \mathcal{M}_n} \prod_{n \geq 1} M_n. \quad (2)$$

Let $i \in I$ be fixed. Then the i th component of $\prod_{n \geq 1} \sup \mathcal{M}_n$ is

$$\begin{aligned} \sum_{i=i_1, i_2, \dots} \prod_{n \geq 1} (\sup \mathcal{M})_{i_n, i_{n+1}} &= \sum_{i=i_1, i_2, \dots} \sup_{M_n \in \mathcal{M}_n} \prod_{n \geq 1} (M_n)_{i_n, i_{n+1}} \\ &= \sup_{M_n \in \mathcal{M}_n} \sum_{i=i_1, i_2, \dots} \prod_{n \geq 1} (M_n)_{i_n, i_{n+1}} \\ &= \sup_{M_n \in \mathcal{M}_n} \left(\prod_{n \geq 1} M_n \right)_i. \end{aligned}$$

Here, we used the fact, proved in [7], that summation is continuous. It follows that (2) holds. \square

Hence, if (S, V) is a continuous ω -semiring-semimodule pair, then $(S^{I \times I}, V^I)$ comes with a star operation and an omega operation.

4.2 Series

We may also construct continuous ω -semiring-semimodule pairs of power series. To this end, we assume that S is a continuous ω -semiring equipped with an infinite product $S^\omega \rightarrow S$ subject to axioms similar to those defining continuous ω -semiring-semimodule pairs. This amounts to requiring that, equipped with left multiplication as the action and the infinite product, (S, S) is a continuous ω -semiring-semimodule pair.

Let X be any set. We already know that $S\langle\langle X^* \rangle\rangle$ is a continuous semiring. In a similar way, $S\langle\langle X^\omega \rangle\rangle$, equipped with the pointwise sum operation and pointwise ordering, is a continuous commutative monoid, and the action of $S\langle\langle X^* \rangle\rangle$ on $S\langle\langle X^\omega \rangle\rangle$ defined by

$$sr = \sum_{w \in A^\omega} \sum_{w=uv} \langle s, u \rangle \langle r, v \rangle uv$$

turns $S\langle\langle X^\omega \rangle\rangle$ into an $S\langle\langle X^* \rangle\rangle$ -semimodule. Moreover, the action is continuous in either argument and it is easy to check that Ax1-Ax4 hold.

Let $s_n \in S\langle\langle X^* \rangle\rangle$ for all $n \geq 1$. We define $r = \prod_{n \geq 1} s_n \in S\langle\langle X^\omega \rangle\rangle$ as follows. Given $v \in X^\omega$, we define

$$\langle r, v \rangle = \sum_{v=v_1 v_2 \dots} \prod_{n \geq 1} \langle s_n, v_n \rangle$$

Proposition 4.2. *Let S be a continuous ω -semiring and X be a set. Then $(S\langle\langle X^* \rangle\rangle, S\langle\langle X^\omega \rangle\rangle)$ is a continuous ω -semiring-semimodule pair.*

Proof. First we establish Ax1. Let $s \in S\langle\langle X^* \rangle\rangle$ and $s_n \in S\langle\langle X^* \rangle\rangle$ for all $n \geq 1$. Then, for all $w \in X^\omega$,

$$\begin{aligned} \langle s \prod_{n \geq 1} s_n, w \rangle &= \sum_{w=uv} \langle s, u \rangle \langle \prod_{n \geq 1} s_n, v \rangle \\ &= \sum_{w=uv} \langle s, u \rangle \sum_{v=v_1 v_2 \dots} \prod_{n \geq 1} \langle s_n, v_n \rangle \\ &= \sum_{w=uv_1 v_2 \dots} \langle s, u \rangle \prod_{n \geq 1} \langle s_n, v_n \rangle \\ &= \sum_{w=v_1 v_2 \dots} \langle s'_n, v_n \rangle \\ &= \langle \prod_{n \geq 1} s'_n, w \rangle, \end{aligned}$$

where $s'_1 = s$ and $s'_{n+1} = s_n$ for all $n \geq 1$.

Let again $s_n \in S\langle\langle X^* \rangle\rangle$ for all $n \geq 1$. Suppose that the sequence $i_1 = 1 \leq i_2 \leq \dots$ increases without a bound. For each $n \geq 1$, define $s'_n = s_{i_n} \dots s_{i_{n+1}-1}$. Then for all $w \in X^\omega$,

$$\begin{aligned} \langle \prod_{n \geq 1} s_n, w \rangle &= \sum_{w=v_1 v_2 \dots} \prod_{n \geq 1} \langle s_n, v_n \rangle \\ &= \sum_{w=v_1 v_2 \dots} \prod_{n \geq 1} \langle s_{i_n}, v_{i_n} \rangle \dots \langle s_{i_{n+1}-1}, v_{i_{n+1}-1} \rangle \\ &= \sum_{w=u_1 u_2 \dots} \prod_{n \geq 1} \langle s'_n, u_n \rangle \\ &= \langle \prod_{n \geq 1} s'_n, w \rangle, \end{aligned}$$

proving Ax2.

Next, suppose that $s_n, s'_n \in S\langle\langle X^* \rangle\rangle$ for all $n \geq 1$. Then for all $w \in X^\omega$,

$$\begin{aligned} \langle \prod_{n \geq 1} (s_n + s'_n), w \rangle &= \sum_{w=v_1 v_2 \dots} \prod_{n \geq 1} \langle s_n + s'_n, v_n \rangle \\ &= \sum_{w=v_1 v_2 \dots} \sum_{r_n=s_n \text{ or } r_n=s'_n} \prod_{n \geq 1} \langle r_n, v_n \rangle \\ &= \sum_{r_n=s_n \text{ or } r_n=s'_n} \langle \prod_{n \geq 1} r_n, w \rangle, \end{aligned}$$

proving Ax3.

Finally, suppose that (I_n, \leq) is a nonempty directed partially ordered set for each $n \geq 1$ and $s_i \in S\langle\langle X^* \rangle\rangle$ for all $i \in I_n, n \geq 1$ such that $s_i \leq s_j$ whenever $i \leq j$ in I_n . We want to prove that

$$\prod_{n \geq 1} \sup_{i \in I_n} s_i = \sup_{i_1 \in I_1, i_2 \in I_2, \dots} \prod_{n \geq 1} s_{i_n}.$$

Let $w \in X^\omega$. Then

$$\begin{aligned} \langle \prod_{n \geq 1} \sup_{i \in I_n} s_i, w \rangle &= \sum_{w=v_1 v_2 \dots} \prod_{n \geq 1} \langle \sup_{i \in I_n} s_i, v_n \rangle \\ &= \sum_{w=v_1 v_2 \dots} \prod_{n \geq 1} \sup_{i \in I_n} \langle s_i, v_n \rangle \\ &= \sum_{w=v_1 v_2 \dots} \sup_{i_1 \in I_1, i_2 \in I_2, \dots} \prod_{n \geq 1} \langle s_{i_n}, v_n \rangle \\ &= \sup_{i_1 \in I_1, i_2 \in I_2, \dots} \sum_{w=v_1 v_2 \dots} \prod_{n \geq 1} \langle s_{i_n}, v_n \rangle \\ &= \sup_{i_1 \in I_1, i_2 \in I_2, \dots} \langle \prod_{n \geq 1} s_{i_n}, w \rangle \\ &= \langle \sup_{i_1 \in I_1, i_2 \in I_2, \dots} \prod_{n \geq 1} s_{i_n}, w \rangle, \end{aligned}$$

proving Ax4. □

Hence, if S is a continuous semiring, then for any set X , $(S\langle\langle X^* \rangle\rangle, S\langle\langle X^\omega \rangle\rangle)$ has a star and an omega operation.

5 Freeness

Suppose now that (S, V) and (S', V') are continuous ω -semiring-semimodule pairs. We say that a pair of functions $h = (h_S, h_V)$ with $h_S : S \rightarrow S'$ and $h_V : V \rightarrow V'$ is a continuous ω -semiring-semimodule pair morphism if h_S is a continuous semiring homomorphism, h_V is a continuous monoid homomorphism, h_S and h_V jointly preserve the action, moreover, infinite product is preserved:

$$h_V(\prod_{n \geq 1} x_n) = \prod_{n \geq 1} h_S(x_n)$$

for all $x_n \in S, n \geq 1$. In this section we prove:

Theorem 5.1. *Suppose that S is a continuous ω -semiring. Then for each set X , the continuous ω -semiring-semimodule pair $(S\langle\langle X^* \rangle\rangle, S\langle\langle X^\omega \rangle\rangle)$ has the following universal property. Let (S', V') be a continuous ω -semiring-semimodule pair, $h_S : S \rightarrow S'$ a continuous semiring morphism and $h_X : X \rightarrow S'$ a function. Suppose that the elements of $h_S(S)$ commute with the elements of S' and the following infinite commutativity holds:*

$$\prod_{n \geq 1} s_n s'_n = (\prod_{n \geq 1} s_n) (\prod_{n \geq 1} s'_n)$$

for all $s_n \in h_S(S)$ and $s'_n \in S'$. Then there is a unique continuous semiring-semimodule morphism $h^\sharp = (h_S^\sharp, h_V^\sharp)$ extending h_S and h_X .

Proof. We have already shown that $(S\langle\langle X^* \rangle\rangle, S\langle\langle X^\omega \rangle\rangle)$ is a continuous semiring-semimodule pair. Let us first extend h to a function $X^* \rightarrow S$, denoted just h , so that it becomes a (multiplicative) monoid homomorphism. Then let $h_S^\sharp : S\langle\langle X^* \rangle\rangle \rightarrow S'$ be defined by

$$h_S^\sharp(s) = \sum_{u \in X^*} h_S(\langle s, u \rangle)h(u).$$

It is known that h_S^\sharp is a continuous semiring homomorphism.

Next we extend h to a function $X^\omega \rightarrow V'$ by defining $h(v) = \prod_{i \geq 1} h_X(x_i)$ for each $v = x_1 x_2 \dots$ in X^ω . Finally, when $s \in S\langle\langle X^\omega \rangle\rangle$, let $h_{V'}^\sharp(s) = \sum_{v \in X^\omega} h_S(\langle s, v \rangle)h(v)$.

Suppose that $s_i \in S\langle\langle X^\omega \rangle\rangle$ for all $i \in I$, where I is nonempty directed partially ordered set, ordered by the relation \leq . Moreover, suppose that $s_i \leq s_j$ whenever $i \leq j$ in I and let $s = \sup_{i \in I} s_i$. Then

$$\begin{aligned} h_{V'}^\sharp(s) &= \sum_{v \in X^\omega} h_S(\langle s, v \rangle)h(v) \\ &= \sum_{v \in X^\omega} \sup_{i \in I} h_S(\langle s_i, v \rangle)h(v) \\ &= \sup_{i \in I} \sum_{v \in X^\omega} h_S(\langle s_i, v \rangle)h(v) \\ &= \sup_{i \in I} h_{V'}^\sharp(s_i), \end{aligned}$$

proving that $h_{V'}^\sharp$ is continuous. To prove that $h_{V'}^\sharp$ preserves the sum operation, let $s_1, s_2 \in S\langle\langle X^\omega \rangle\rangle$. Then

$$\begin{aligned} h_{V'}^\sharp(s_1 + s_2) &= \sum_{v \in X^\omega} h_S(\langle s_1 + s_2, v \rangle)h(v) \\ &= \sum_{v \in X^\omega} h_S(\langle s_1, v \rangle)h(v) + h_S(\langle s_2, v \rangle)h(v) \\ &= \sum_{v \in X^\omega} h_S(\langle s_1, v \rangle)h(v) + \sum_{v \in X^\omega} h_S(\langle s_2, v \rangle)h(v) \\ &= h_{V'}^\sharp(s_1) + h_{V'}^\sharp(s_2). \end{aligned}$$

It is clear that $h_{V'}^\sharp$ preserves 0. In order to prove that h_S^\sharp and $h_{V'}^\sharp$ jointly preserve the action, let $s \in S\langle\langle X^* \rangle\rangle$ and $r \in S\langle\langle X^\omega \rangle\rangle$. Then

$$\begin{aligned} h_{V'}^\sharp(sr) &= \sum_{v \in X^\omega} h_S(\langle sr, v \rangle)h(v) \\ &= \sum_{v \in X^\omega} \sum_{v=uw} h_S(\langle s, u \rangle)h_S(\langle s, w \rangle)h(u)h(w) \\ &= \sum_{v \in X^\omega} \sum_{v=uw} h_S(\langle s, u \rangle)h(u)h_S(\langle r, w \rangle)h(w) \\ &= \sum_{u \in X^*} h_S(\langle s, u \rangle)h(u) \sum_{w \in X^\omega} h_S(\langle r, w \rangle)h(w) \\ &= h_S^\sharp(s)h_{V'}^\sharp(r). \end{aligned}$$

Finally, we prove that h_V^\sharp preserves the infinite product. To this end, let $s_n \in S\langle\langle X^* \rangle\rangle$ for all $n \geq 1$. We want to prove that $h_V^\sharp(\prod_{n \geq 1} s_n) = \prod_{n \geq 1} h_S^\sharp(s_n)$.

$$\begin{aligned}
h_V^\sharp(\prod_{n \geq 1} s_n) &= \sum_{v \in X^\omega} h_S(\langle \prod_{n \geq 1} s_n, v \rangle) h(v) \\
&= \sum_{v \in X^\omega} \sum_{v=v_1 v_2 \dots} \prod_{n \geq 1} h_S(\langle s_n, v_n \rangle) h(v_n) \\
&= \sum_{v \in X^\omega} \sum_{v=v_1 v_2 \dots} \prod_{n \geq 1} h_S(\langle s_n, v_n \rangle) \prod_{n \geq 1} h(v_n) \\
&= \prod_{n \geq 1} (\sum_{v_n \in X^*} h_S(\langle s_n, v_n \rangle) h(v_n)) \\
&= \prod_{n \geq 1} h_S(s_n).
\end{aligned}$$

It is clear that h_S extends h . Since the definitions of h_S and h_V were forced, they are unique. \square

6 The category $\text{Matr}(S, V)$

All categories \mathcal{C} in the paper will have sets as objects. The composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ will be denoted $f \circ g$. We usually let \mathbf{id}_X denote the identity morphism $X \rightarrow X$.

Our categories will have finite coproducts. For a sequence X_1, \dots, X_n of objects, the coproduct $X_1 \oplus \dots \oplus X_n$ will be given by disjoint union $X_1 \uplus \dots \uplus X_n$. In particular, the empty set \emptyset will serve as initial object.

Let X_1, \dots, X_n be objects. For each $i = 1, \dots, n$, the i th coproduct injection $\mathbf{in}_{X_i} : X_i \rightarrow X_1 \oplus \dots \oplus X_n$ will always be determined by the embedding of X_i into $X_1 \uplus \dots \uplus X_n$. We will let $!_X$ denote the unique morphism $\emptyset \rightarrow X$. Moreover, if $f_i : X_i \rightarrow X$ for $i = 1, \dots, n$, then we will let $\langle f_1, \dots, f_n \rangle$ denote the unique morphism $f : X_1 \oplus \dots \oplus X_n \rightarrow X$ with $\mathbf{in}_{X_i} \circ f = f_i$ for all i . And when $f_i : X_i \rightarrow Y_i$, where $i \in \{1, \dots, n\}$, then we let $f_1 \oplus \dots \oplus f_n$ denote the unique morphism $f : X_1 \oplus \dots \oplus X_n \rightarrow Y_1 \oplus \dots \oplus Y_n$ with $\mathbf{in}_{X_i} \circ f = f_i \circ \mathbf{in}_{Y_i}$ for all i .

For any X, Y , the hom-set $\mathcal{C}(X, Y)$ of morphisms $X \rightarrow Y$ will be both a complete partial order (\mathcal{C}, \leq) and a commutative monoid $(\mathcal{C}(X, Y), +, 0_{X, Y})$ such that the zero morphism $0_{X, Y}$ is also least w.r.t. \leq and the operation $+$ is continuous in both of its arguments. Also, the operation of composition will be continuous in both arguments. Moreover, the partial order will be related to the coproduct structure so that for any $f, g : X_1 \oplus \dots \oplus X_n \rightarrow Y$, $f \leq g$ iff $\mathbf{in}_{X_i} \circ f \leq \mathbf{in}_{X_i} \circ g$. It follows that when $f_i : X_i \rightarrow Y_i$, where $i \in \{1, \dots, n\}$, then $f_1 \oplus \dots \oplus f_n \leq g_1 \oplus \dots \oplus g_n$ iff $f_i \leq g_i$ for all i .

The following identities will hold for all $f, g : X \rightarrow Y$ and $h : Y \rightarrow Z$:

$$\begin{aligned}
(f + g) \circ h &= f \circ h + g \circ h \\
0_{X, Y} \circ h &= 0_{X, Z}
\end{aligned}$$

Finally, our categories will be equipped with a dagger operation mapping a morphism $f : X \rightarrow X \oplus Y$ to a morphism $f^\dagger : X \rightarrow Y$. This operation will always be a fixed point operation, so that the following fixed point identity will hold:

$$f^\dagger = f \circ \langle f^\dagger, \mathbf{id}_Y \rangle$$

for all $f : X \rightarrow X \oplus Y$.

Iteration categories are categories with finite coproducts and a dagger operation satisfying certain identities including the above fixed point identity, the parameter identity

$$(f \circ (\mathbf{id}_X \oplus g))^\dagger = f^\dagger \circ g$$

where $f : X \rightarrow X \oplus Y$ and $g : Y \rightarrow Z$, the double dagger identity

$$f^{\dagger\dagger} = (f \circ (\langle \mathbf{id}_X, \mathbf{id}_X \rangle \oplus \mathbf{id}_Y))^\dagger,$$

where $f : X \rightarrow X \oplus X \oplus Y$, to name a few, and some other identities including the group identities that we will described later. All of our categories will be iteration categories.

Suppose now that (S, V) is a continuous ω -semiring-semimodule pair. Then (S, V) determines a category $\mathbf{Matr}_{(S, V)}$ whose objects are all sets and a morphism $I \rightarrow J$ is an ordered pair (A, u) , where $A \in S^{I \times J}$ and $u \in V^I$. Hence a morphism $I \rightarrow I$ is an element of the semiring-semimodule pair $(S^{I \times I}, V^I)$.

Composition is defined as follows. Suppose that $(A, u) : I \rightarrow J$ and $(B, v) : J \rightarrow K$. Then we define $(A, u) \circ (B, v) = (AB, u + Av) : I \rightarrow K$. It is easy to check that composition is associative with the morphisms $(E^{I \times I}, 0^I) : I \rightarrow I$ serving as identities, where $E^{I \times I}$ is the unit matrix in $S^{I \times I}$ and 0^I denotes the zero element of V^I . (For finite sets, this category is defined in [1].)

The partial order \leq on a hom-set of $\mathbf{Matr}_{(S, V)}$ is defined pointwise, so that when $(A, u), (B, v) : I \rightarrow J$, then $(A, u) \leq (B, v)$ iff $A_{i,j} \leq B_{i,j}$ and $u_i \leq v_i$ for all $i \in I$ and $j \in J$. Clearly, each hom-set forms a complete partial order, and it is not difficult to verify that composition is continuous.

We can also impose a commutative monoid structure on the hom-sets by defining $(A, u) + (B, v)$ pointwise, for all $(A, u), (B, v) : I \rightarrow J$. Hence $(A, u) + (B, v) = (C, w)$ with $C_{i,j} = A_{i,j} + B_{i,j}$ and $w_i = u_i + v_i$ for all $i \in I$ and $j \in J$. The zero morphism $I \rightarrow J$ is the morphism $(0^{I \times J}, 0^I)$ consisting of two zero matrices. We denote it by $0_{I, J}$, or just 0. We have

$$\begin{aligned} ((A, u) + (B, v)) \circ (C, w) &= (A, u) \circ (C, w) + (B, v) \circ (C, w) \\ 0_{I, J} \circ (C, w) &= 0_{I, K} \end{aligned}$$

for all $(C, w) : J \rightarrow K$. It is not difficult to prove that composition is continuous.

Coproduct is given by disjoint union on objects. When X_1, \dots, X_n is a sequence of sets and $i \in \{1, \dots, n\}$, then i th coproduct embedding \mathbf{in}_i consists of an $X_i \times (X_1 \uplus \dots \uplus X_n)$ matrix whose $x_i \times X_i$ submatrix is an identity matrix and whose $X_i \times X_j$ matrices are all zero matrices for $j \neq i$, together with the column matrix $0^{X_i} \in V^{X_i}$.

We have already noted that for each set I , $(S^{I \times I}, V^I)$ is a continuous ω -semiring-semimodule pair. Hence it comes with a star and an omega operation: For each $A \in S^{I \times I}$, $A^* = \sum_{n \geq 0} A^n \in S^{I \times I}$ and $A^\omega = \prod_{n \geq 1} A$ in V^I . These operations satisfy the identities mentioned above. And in fact,

$$\begin{aligned} (A + B)^* &= (A^* B)^* A^*, & A, B \in S^{I \times I} \\ (AB)^* &= E_I + A(BA)^* B, & A \in S^{I \times J}, B \in S^{J \times I} \end{aligned}$$

and

$$\begin{aligned} (A + B)^\omega &= (A^* B)^* A^\omega + (A^* B)^\omega, & A, B \in S^{I \times I} \\ (AB)^\omega &= A(BA)^\omega, & A \in S^{I \times J}, B \in S^{J \times I} \end{aligned}$$

Suppose now that $I = J \uplus K$ and $M \in S^{I \times I}$ is partitioned as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$M^* = \begin{pmatrix} (a + bd^*c)^* & (a + bd^*c)^* bd^* \\ (d + ca^*b)^* ca^* & (d + ca^*b)^* \end{pmatrix}$$

and

$$M^\omega = \begin{pmatrix} (a + bd^*c)^\omega + (a + bd^*c)^* bd^\omega \\ (d + ca^*b)^\omega + (d + ca^*b)^* ca^\omega \end{pmatrix}$$

The star and omega operations together give rise to a dagger operation over $\mathbf{Matr}_{(S,V)}$ that map a morphism $X \rightarrow X \oplus Y$ to a morphism $X \rightarrow Y$. To define it, let $(A, u) : X \rightarrow I \oplus J$, and partition A as (a, b) with $a \in S^{I \times I}$ and $b \in S^{I \times J}$. Then we define $(A, u)^\dagger = (a^*b, a^\omega + a^*v) : I \rightarrow J$. Clearly, $(A, u)^\dagger$ is a solution of the equation

$$(X, x) = (A, u) \begin{pmatrix} (X, x) \\ (E_J, 0) \end{pmatrix} = (aX + b, ax + u)$$

where (X, x) ranges over the morphisms $I \rightarrow J$. It is known that equipped with dagger, $\mathbf{Matr}_{(S,V)}$ is an iteration category.

7 Categories of Series

7.1 The category $\mathbf{Ser}_{S,\Sigma}$

Suppose that S is a commutative continuous semiring and Σ is a set. We define the category $\mathbf{Ser}_{S,\Sigma}$ whose objects are all sets and a morphism $X \rightarrow Y$ is a function $f : X \rightarrow S\langle\langle Y \uplus \Sigma \rangle\rangle^*$, or alternatively, a tuple $(f_x)_{x \in X}$ of series $f_x \in S\langle\langle Y \uplus \Sigma \rangle\rangle^*$. Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Then we define $f \circ g$ as the function composition of f and g^\sharp , the extension of the function $Y \uplus \Sigma$ to $S\langle\langle Z \uplus \Sigma \rangle\rangle^*$ which

agrees with g on Y and is the identity function on Σ to a continuous semiring homomorphism $S\langle\langle(Y \uplus \Sigma)^*\rangle\rangle \rightarrow S\langle\langle(Z \uplus \Sigma)^*\rangle\rangle$. For each X , the identity morphism \mathbf{id}_X is the embedding of X into $X \uplus \Sigma$.

Each hom-set of morphisms $X \rightarrow Y$ of the category $\mathbf{Ser}_{S,\Sigma}$ has the structure of a complete partial order and commutative monoid. For any $f, g : X \rightarrow Y$, we define $f \leq g$ iff $f_x \leq g_x$ for all x , and similarly, $(f + g)_x = f_x + g_x$ for all x . The neutral element is the series $0_{X,Y}$ whose components are all 0. This is also the least morphism $X \rightarrow Y$. Composition of morphisms is continuous as is the sum operation.

Also, \mathbf{Ser}_S has finite coproducts are given by disjoint sum on objects. The coproduct $X_1 \oplus \dots \oplus X_n$ of a sequence X_1, \dots, X_n of sets is given by disjoint union, and for each $i \in \{1, \dots, n\}$, $\mathbf{in}_{X_i} : X_i \rightarrow X_1 \oplus \dots \oplus X_n$ is the embedding of X_i into $S\langle\langle(X_1 \uplus \dots \uplus X_n \uplus \Sigma)^*\rangle\rangle$.

We define a dagger operation on $\mathbf{Ser}_{S,\Sigma}$ which maps a morphism $f : X \rightarrow X \oplus Y$ to $f^\dagger : X \rightarrow Y$. The morphism f^\dagger is given as the least solution of the fixed point equation

$$\xi = f \circ \langle \xi, \mathbf{id}_Y \rangle.$$

In more detail, $f^\dagger = \sup_{n \geq 0} f^{(n)}$, where $f^{(0)} = 0$ and $f^{(n+1)} = f \circ \langle f^{(n)}, \mathbf{id}_Y \rangle$. It is known that equipped with this dagger operation, $\mathbf{Ser}_{S,\Sigma}$ is an iteration category.

7.2 The category $\mathbf{Ser}_{S,\Sigma}^\omega$.

Suppose now that S is a commutative continuous ω -semiring satisfying the infinite commutativity identity. Then we define another category $\mathbf{Ser}_{S,\Sigma}^\omega$ with sets as objects as above. However, a morphism $f : X \rightarrow Y$ is now a triplet (f_0, f_M, f_ω) with $f_0 : X \rightarrow Y$ in $\mathbf{Ser}_{S,\Sigma}$ and $(f_M, f_\omega) : X \rightarrow Y$ in $\mathbf{Matr}_{(S\langle\langle(Y \uplus \Sigma)^*\rangle\rangle, S\langle\langle(Y \uplus \Sigma)^\omega\rangle\rangle)}$. Hence, $f_0 : X \rightarrow S\langle\langle(Y \uplus \Sigma)^*\rangle\rangle$, $f_M \in S\langle\langle(Y \uplus \Sigma)^*\rangle\rangle^{X \times Y}$ and $f_\omega : X \rightarrow S\langle\langle(Y \uplus \Sigma)^\omega\rangle\rangle$.

Composition is defined as follows. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. Then the components of $h = f \circ g : X \rightarrow Z$ are given by

- $h_0 = f_0 \circ g_0$, where the composition is taken from $\mathbf{Ser}_{S,\Sigma}$, and
- $(h_M, h_\omega) = g_0^\sharp((f_M, f_\omega)) \circ (g_M, g_\omega) = (g_0^\sharp(f_M), g_0^\sharp(f_\omega)) \circ (g_M, g_\omega)$ where the composition is taken from the category $\mathbf{Matr}_{(S\langle\langle(Z \uplus \Sigma)^*\rangle\rangle, S\langle\langle(Z \uplus \Sigma)^\omega\rangle\rangle)}$.

Note that the definition is legitimate, since $(S\langle\langle(Z \uplus \Sigma)^*\rangle\rangle, S\langle\langle(Z \uplus \Sigma)^\omega\rangle\rangle)$ is a continuous ω -semiring-semimodule pair and $g_0^\sharp((f_M, f_\omega)) = (g_0^\sharp(f_M), g_0^\sharp(f_\omega))$ is a morphism $X \rightarrow Y$ and (g_M, g_ω) is a morphism $Y \rightarrow Z$ in $\mathbf{Matr}_{(S\langle\langle(Z \uplus \Sigma)^*\rangle\rangle, S\langle\langle(Z \uplus \Sigma)^\omega\rangle\rangle)}$. Of course, g_0^\sharp is the extension of g_0 to a continuous ω -semiring-semimodule morphism and $g_0^\sharp(f_M)$ and $g_0^\sharp(f_\omega)$ are formed component-wise. It is a routine matter to verify that composition is associative. The identity morphism $X \rightarrow X$ is determined by the corresponding identity morphisms in $\mathbf{Ser}_{S,\Sigma}$ and $\mathbf{Matr}_{(S\langle\langle(X \uplus \Sigma)^*\rangle\rangle, S\langle\langle(X \uplus \Sigma)^\omega\rangle\rangle)}$.

Each hom-set of morphisms $X \rightarrow Y$ is partially ordered by the component-wise order inherited from $\mathbf{Ser}_{(S,\Sigma)}$ and $\mathbf{Matr}_{(S\langle\langle(Y \uplus \Sigma)^*\rangle\rangle, S\langle\langle(Y \uplus \Sigma)^\omega\rangle\rangle)}$. Also, each hom-set

has the structure of a commutative monoid. For any morphisms $f = (f_0, f_M, f_\omega)$ and $g = (g_0, g_M, g_\omega) : X \rightarrow Y$, we define

$$f + g = (f_0 + g_0, f_M + g_M, f_\omega + g_\omega).$$

The components of the morphism $0_{X,Y}$ are the respective zero morphisms.

The category $\mathbf{Ser}_{S,\Sigma}^\omega$ has finite coproducts. On objects, coproduct is again given by disjoint sum.

We now define a dagger operation. Let $f : X \rightarrow X \oplus Y, f = (f_0, f_M, f_\omega)$. Then we define the components of $f^\dagger : X \rightarrow Y$ as $f_0^\dagger : X \rightarrow Y$ in $\mathbf{Ser}_{S,\Sigma}$ and

$$((f^\dagger)_M, (f^\dagger)_\omega) = \langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)^\dagger) = (\langle f_0, \mathbf{id}_Y \rangle^\#(f_M, f_\omega)^\dagger) : X \rightarrow Y$$

in $\mathbf{Matr}_{(S\langle\langle(Y \oplus \Sigma)^*\rangle\rangle, S\langle\langle(Y \oplus \Sigma)^\omega\rangle\rangle)}$.

We prove that the fixed point identity holds. To this end, let $f : X \rightarrow X \oplus Y$ as above. Then

$$f \circ \langle f^\dagger, \mathbf{id}_Y \rangle = (\langle f_0 \circ \langle f_0^\dagger, \mathbf{id}_Y \rangle, \langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)) \circ \langle \langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)^\dagger), \mathbf{id}_Y \rangle).$$

Hence, the first component of f^\dagger is f_0^\dagger . The second and third are given by

$$\begin{aligned} & \langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)) \circ \langle \langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)^\dagger), \mathbf{id}_Y \rangle \\ &= \langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega) \circ \langle (f_M, f_\omega)^\dagger, \mathbf{id}_Y \rangle) \\ &= \langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)^\dagger). \end{aligned}$$

Hence $f \circ \langle f^\dagger, \mathbf{id}_Y \rangle = f^\dagger$.

Theorem 7.1. $\mathbf{Ser}_{(S,\Sigma)}^\omega$ is an iteration category.

Proof. We have already proved that the fixed point identity holds. In order to complete the proof, we establish the parameter, double dagger and group identities.

First we consider the parameter identity. Let $f = (f_0, f_M, f_\omega) : X \rightarrow X \oplus Y$ and $g = (g_0, g_M, g_\omega) : Y \rightarrow Z$. We want to prove that $(f \circ (\mathbf{id}_X \oplus g))^\dagger = f^\dagger \circ g$.

It is clear that the first components of $(f \circ (\mathbf{id}_X \oplus g))^\dagger$ and $f^\dagger \circ g$ are $(f_0 \circ (\mathbf{id}_X \oplus g_0))^\dagger$ and $f_0^\dagger \circ g$, respectively. Since the parameter identity holds in $\mathbf{Ser}_{S,\Sigma}$, we conclude that the first components are equal.

The second and third components of $f \circ (\mathbf{id}_X \oplus g)$ are given by

$$(\mathbf{id}_X \oplus g_0)^\#((f_M, f_\omega)) \circ (\mathbf{id}_X \oplus (g_M, g_\omega)),$$

hence the corresponding components of $(f \circ (\mathbf{id}_X \oplus g))^\dagger$ are given by

$$\langle (f_0 \circ (\mathbf{id}_X \oplus g_0))^\dagger, \mathbf{id}_Z \rangle^\#(((\mathbf{id}_X \oplus g_0)^\#((f_M, f_\omega)) \circ (\mathbf{id}_X \oplus (g_M, g_\omega)))^\dagger)$$

which is

$$\begin{aligned} & \langle f_0^\dagger \circ g_0, \mathbf{id}_Z \rangle^\#(((\mathbf{id}_X \oplus g_0)^\#((f_M, f_\omega)))^\dagger \circ (g_M, g_\omega)) \\ &= ((\mathbf{id}_X \oplus g_0) \circ \langle f_0^\dagger \circ g_0, \mathbf{id}_Z \rangle)^\#((f_M, f_\omega)^\dagger) \circ \langle f_0^\dagger \circ g_0, \mathbf{id}_Z \rangle^\#(g_M, g_\omega) \\ &= ((\mathbf{id}_X \oplus g_0) \circ \langle f_0^\dagger \circ g_0, \mathbf{id}_Z \rangle)^\#((f_M, f_\omega)^\dagger) \circ (g_M, g_\omega). \end{aligned}$$

On the other hand, the second and third components of f^\dagger and $f^\dagger \circ g$ are respectively given by $\langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)^\dagger)$ and

$$g_0^\#(\langle f_0^\dagger, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)^\dagger)) \circ (g_M, g_\omega) = (\langle f_0^\dagger, \mathbf{id}_Y \rangle \circ g_0)^\#((f_M, f_\omega)^\dagger) \circ (g_M, g_\omega).$$

But $(\mathbf{id}_X \oplus g_0) \circ \langle f_0^\dagger \circ g_0, \mathbf{id}_Z \rangle = \langle f_0^\dagger \circ g_0, g_0 \rangle = \langle f_0^\dagger, \mathbf{id}_Y \rangle \circ g_0$, so that the second and third components are also equal.

We prove that the double dagger identity holds. To this end, let $f = (f_0, f_M, f_\omega) : X \rightarrow X \oplus X \oplus Y$ and $\tau = \langle \mathbf{id}_X, \mathbf{id}_X \rangle \oplus \mathbf{id}_Y$. Then the first component of $f^{\dagger\dagger}$ is $f_0^{\dagger\dagger}$, and the first component of $(f \circ \tau)^\dagger$ is $(f_0 \circ \tau)^\dagger$. These are equal since the double dagger identity holds in $\mathbf{Ser}_{S, \Sigma}$.

The second and third components of f^\dagger and $f^{\dagger\dagger}$ are

$$\langle f_0^\dagger, \mathbf{id}_{X \oplus Y} \rangle^\#((f_M, f_\omega)^\dagger)$$

and

$$\begin{aligned} & \langle f_0^{\dagger\dagger}, \mathbf{id}_Y \rangle^\#(\langle f_0^\dagger, \mathbf{id}_{X \oplus Y} \rangle^\#((f_M, f_\omega)^\dagger)) \\ &= (\langle f_0^\dagger, \mathbf{id}_{X \oplus Y} \rangle \circ \langle f_0^{\dagger\dagger}, \mathbf{id}_Y \rangle)^\#((f_M, f_\omega)^{\dagger\dagger}) \\ &= (\langle f_0^\dagger \circ \langle f_0^{\dagger\dagger}, \mathbf{id}_Y \rangle, f_0^{\dagger\dagger}, \mathbf{id}_Y)^\#((f_M, f_\omega)^{\dagger\dagger}) \\ &= \langle f_0^{\dagger\dagger}, f_0^{\dagger\dagger}, \mathbf{id}_Y \rangle^\#((f_M, f_\omega)^{\dagger\dagger}) \\ &= (\tau \circ \langle f^{\dagger\dagger}, \mathbf{id}_Y \rangle)^\#((f_M, f_\omega)^{\dagger\dagger}), \end{aligned}$$

respectively. Now the second and third components of $f \circ \tau$ are given by $\tau^\#((f_M, f_\omega) \circ \tau)$ and thus the second and third components of $(f \circ \tau)^\dagger$ are given by

$$\begin{aligned} & (\langle f \circ \tau \rangle^\dagger, \mathbf{id}_Y)^\#((\tau^\#((f_M, f_\omega) \circ \tau))^\dagger) \\ &= \langle f^{\dagger\dagger}, \mathbf{id}_Y \rangle^\#((\tau^\#((f_M, f_\omega) \circ \tau))^\dagger) \\ &= (\langle f^{\dagger\dagger}, \mathbf{id}_Y \rangle)^\#((\tau^\#((f_M, f_\omega)))^{\dagger\dagger}) \\ &= (\langle f^{\dagger\dagger}, \mathbf{id}_Y \rangle)^\#(\tau^\#(((f_M, f_\omega))^{\dagger\dagger})) \\ &= (\tau \circ \langle f^{\dagger\dagger}, \mathbf{id}_Y \rangle)^\#((f_M, f_\omega)^{\dagger\dagger}). \end{aligned}$$

Hence the double dagger identity holds.

Our next task is to prove that the (simplified) composition identity holds:

$$(f \circ g)^\dagger = f \circ (g \circ (f \oplus \mathbf{id}_Z))^\dagger,$$

where $f : X \rightarrow Y$ and $g : Y \rightarrow X \oplus Z$. We will establish this identity only in the case when Z is the initial object \emptyset . In this case, the identity takes the following form:

$$(f \circ g)^\dagger = f \circ (g \circ f)^\dagger,$$

where $f : X \rightarrow Y$ and $g : Y \rightarrow X$.

It is again clear that the first components of the two sides are equal, since the identity holds in $\mathbf{Ser}_{S,\Sigma}$.

Now the second and third components of $(f \circ g)^\dagger$ are

$$\begin{aligned} & ((f_0 \circ g_0)^\dagger)^\#((g_0^\#((f_M, f_\omega)) \circ (g_M, g_\omega))^\dagger) \\ &= ((g_0 \circ (f_0 \circ g_0)^\dagger)^\#((f_M, f_\omega)) \circ (f_0 \circ g_0)^\dagger)^\#((g_M, g_\omega))^\dagger \\ &= (((g_0 \circ f_0)^\dagger)^\#((f_M, f_\omega)) \circ ((f_0 \circ g_0)^\dagger)^\#((g_M, g_\omega))^\dagger)^\dagger. \end{aligned}$$

The corresponding components of $f \circ (g \circ f)^\dagger$ are

$$\begin{aligned} & ((g_0 \circ f_0)^\dagger)^\#((f_M, f_\omega)) \circ ((g_0 \circ f_0)^\dagger)^\#((f_0^\#((g_M, g_\omega)) \circ (f_M, f_\omega))^\dagger) \\ &= ((g_0 \circ f_0)^\dagger)^\#((f_M, f_\omega)) \circ (((f_0 \circ g_0)^\dagger)^\#((g_M, g_\omega)) \circ ((g_0 \circ f_0)^\dagger)^\#((f_M, f_\omega))^\dagger) \\ &= (((g_0 \circ f_0)^\dagger)^\#((f_M, f_\omega)) \circ ((f_0 \circ g_0)^\dagger)^\#((g_M, g_\omega))^\dagger)^\dagger. \end{aligned}$$

Hence, the second and third components are also equal.

Our last task is to prove that the group identities hold. Suppose that G is a finite group of order n whose elements are the integers $\{1, \dots, n\}$, say. Let $g : X \rightarrow X \oplus \dots \oplus X \oplus Y$, where there are n occurrences of X in the target. For each $i \in \{1, \dots, n\}$, let

$$\rho_i = \langle \mathbf{id}_{i_1}, \dots, \mathbf{id}_{i_n} \rangle \oplus \mathbf{id}_Y : X \oplus \dots \oplus X \oplus Y \rightarrow X \oplus \dots \oplus X \oplus Y.$$

Moreover, let

$$\tau = \langle \mathbf{id}_X, \dots, \mathbf{id}_X \rangle : X \oplus \dots \oplus X \rightarrow X.$$

The group identity associated with G is

$$\langle g \circ \rho_1, \dots, g \circ \rho_n \rangle^\dagger = \tau \circ (g \circ (\tau \oplus \mathbf{id}_Y))^\dagger.$$

We will prove this only in the case when $Y = \emptyset$. The first components of the two sides are again equal. The second and third components of the morphism on the left hand side are given by

$$\begin{aligned} & \langle (g_0 \circ \rho_1, \dots, g_0 \circ \rho_n)^\dagger \rangle^\#(\langle (g_M, g_\omega) \circ \rho_1, \dots, (g_M, g_\omega) \circ \rho_n \rangle^\dagger) \\ &= \langle (\tau \circ (g_0 \circ \tau)^\dagger)^\#(\langle (g_M, g_\omega) \circ \rho_1, \dots, (g_M, g_\omega) \circ \rho_n \rangle^\dagger) \rangle \\ &= \langle (\tau \circ (g_0 \circ \tau)^\dagger)^\#(\langle (g_M, g_\omega) \circ \rho_1, \dots, (\tau \circ (g_0 \circ \tau)^\dagger)^\#(\langle (g_M, g_\omega) \circ \rho_n \rangle^\dagger) \rangle) \rangle \\ &= \tau \circ \langle (\tau \circ (g_0 \circ \tau)^\dagger)^\#(\langle (g_M, g_\omega) \circ \tau \rangle^\dagger) \rangle. \end{aligned}$$

But this is exactly the morphism determined by the second and third components of the morphism on the right hand side of the group identity associated with G . This completes the proof of the theorem. \square

Remark of the second author.

From May 17 to May 20, 2016, Zoltan stayed and worked in my home and almost finished his presumably final paper as coauthor. He wrote this paper in his typical style of working: After extensive discussions he typed the tex file directly into the computer without using a concept or notes. This paper is complete except for the announced “Conclusion”. His planned content of the “Conclusion” is unknown to me and I have not tried to reconstruct it. Therefore, this paper appears according to Zoltan’s conceptions.

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