Galois Connections:
Mathematics, Art, and Archives

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Évariste Galois (1811–1832) has been increasingly recognised as an important mathematician who despite his short life developed mathematical ideas that today have led to applications in computer science (such as Galois connections) and elsewhere. Some of Galois' mathematics can be visualised in interesting and even artistic ways, aided using software. In addition, a significant corpus of the historical documentation on Galois and his family (including his brother Alfred Galois, who was an artist), can now be accessed online as a growing number of institutional archives digitise their collections. This paper introduces some of the mathematics of Galois, ways in which it can be visualised, and also considers the issues and new opportunities with respect to visualising information on Galois and his family (including the connections between them). Although the story of Galois and his close relations can be seen as one of tragedy with lives cut short, from a historical viewpoint Évariste Galois' contribution to humankind has been a triumph.

1. INTRODUCTION

"*Unfortunately what is little recognized is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties.*"

– Évariste Galois (1811–1832)

In this paper, we explore various aspects of the French mathematician Évariste Galois (1811–1832), who died when only aged 20 but made very important contributions to his field, producing an eponymous theory only properly recognised and developed further after his death (Neumann 2011). Rigatelli (1996) provides an extended biography of Galois and there is every a fictionalised version of his life where Galois is represented by a mathematical "*i*" (Petsinis 1995). We consider his life and work with respect to both visualising his mathematical ideas as artistically interesting structures and discovering archival material relating to Galois and his family that is increasingly accessible online in digital form (Bowen & Giannini 2014), although still difficult to find in practice.

In Section 2, Galois' mathematics is considered in overview, including the concept of a *Galois connection*. Section 3 demonstrates how some of Galois' mathematics can be visualised as pleasing and sometimes artistic patterns. In Section 4, archival investigations on historical documents relating to Galois and his close relatives, together with the connections between them, are presented, some of which are available in digitised form. Finally a brief conclusion is presented, noting that there are opportunities for further visualisation and archival research.

2. MATHEMATICS

The contribution of Évariste Galois to mathematics is extraordinary considering his early death at the age of 20 (Neumann 2011, Daintith & Nelson 1989, p. 141). He is an extreme example of the widely held notion that a mathematician produces their best work when young – typically in their 20s, but Galois had barely entered this decade of his life. That said, creative breakthroughs need years of commitment and complete dedication (Robinson 2010). Galois achieved this in his teenage years, a remarkable feat even for a mathematician.

While still in his teens, Galois determined a necessary and sufficient condition for a polynomial to be solvable using radicals, thus resolving a 350 year-old problem. His mathematical investigations were foundational for two important aspects of abstract algebra: Galois theory and group theory.

One of Galois' most celebrated contributions in computer science is that of what is now known as a *Galois connection* (Melton et al. 1986, Mu & Oliveira 2011). For an abstract specification level $S(a)$ and a more concrete design level $D(c)$, there should be a linking predicate $G(c,a)$ connection the two levels. Formally (Hoare & He 1998, p. 41):

$$[(\exists c \bullet D(c) \wedge G(c,a) \Rightarrow S(a)] \textbf{ iff}$$

$$[D(c) \Rightarrow (\forall a \bullet G(c,a) \Rightarrow S(a))]$$

This is known as a Galois connection and is very useful in linking theories in computer science. Determining a suitable predicate $G(c,a)$ connects the two levels or theories in a mathematically useful manner.

A Galois connection can also be modelled as a pair of functions, L and R, between two *complete lattices*, **S** and **T**, in each direction, with the following two *inequations* holding (Hoare & He 1998, p. 98):

$$R(L(X)) \sqsubseteq X$$

$$Y \sqsubseteq L(R(Y))$$

where X is in **S** and Y is in **T**. Note that X and Y may be predicates in stronger and weaker theories respectively. This model of such a Galois connection can be visualised as in Figure 1 (Hoare & He 1998, p. 100).

A Galois lattice (or concept lattice) is useful in formal concept analysis, for example in the study of ontologies. It is a mathematical structure in which the relation between the sets of concepts and attributes is a Galois connection. Such a lattice can be usefully visualised to find patterns within its structure. Building a Galois lattice can be considered as a method for conceptual clustering since it provides a concept hierarchy (Godin et al. 1995).

Using a Galois lattice helps to display an order structure, in which dependencies among row and column objects, together with dependencies between rows and columns, are both revealed; for example, in visualisation for social network analysis (Freeman 2000). Marghoubi et al. (2006) have investigated the use of a Galois lattice for the extraction and visualisation of association rules, applied to spatial data mining, helping to discover hidden relationships. First the spatial context is determined and then pattern mining is undertaken in an efficient manner.
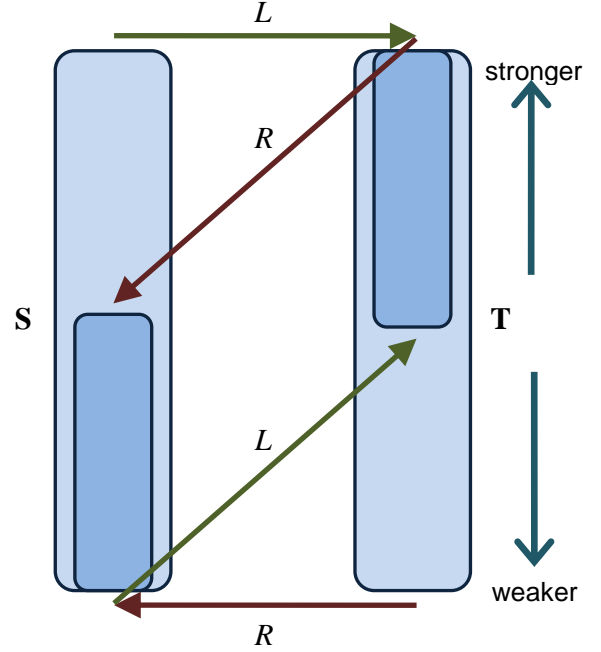

Figure 1: *Visualisation of a Galois connection (Hoare & He 1998, p. 100)*

3. ART

The Galois family had an artistic streak running in it and Évariste's brother Alfred was an artist. Alfred Galois produced one of the few images of Évariste posthumously (see Figure 2).

Mathematical structures can often be beautiful and complex (Bakshee 1999, Beddard & Dodds 2009, Wolfram 2002). Some of Évariste Galois' mathematics can be visualised as interesting and even artistic patterns. We explore some examples in this section.

Galois fields have been visualised (Stein 2012). It is convenient to have Galois fields of size 2^n for various n for software applications to allow them to fit into computer bytes or words. The Galois field $GF(2^n)$ has 2^n elements, each represented as polynomials of degree less than n with all the coefficients having values of either 0 or 1. Then to encode an element of $GF(2^n)$ as a number, an n-bit binary number is required. For example, consider the Galois field $GF(2^3)$ with 2^3 or 8 elements. These are 3-bit binary numbers. For example, the binary element 011 represents the polynomial $x + 1$ and the element 110 stands for $x^2 + x$. Using 8-bit representations of elements of $GF(2^8)$ for example, it is possible to create an image where the pixel in the ith row and jth column is the sum in the Galois field of i and j (as binary numbers). See Figure 3 for a visualisation of this.

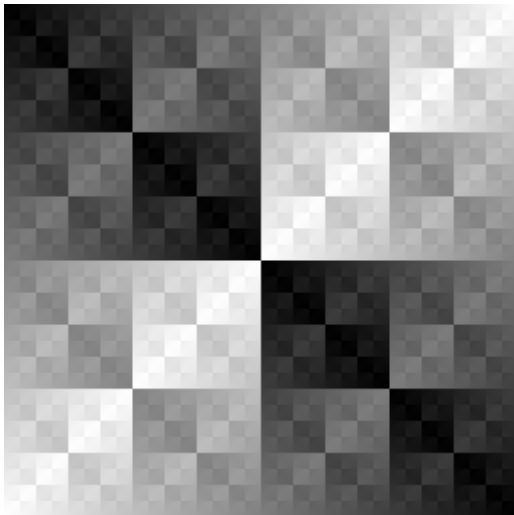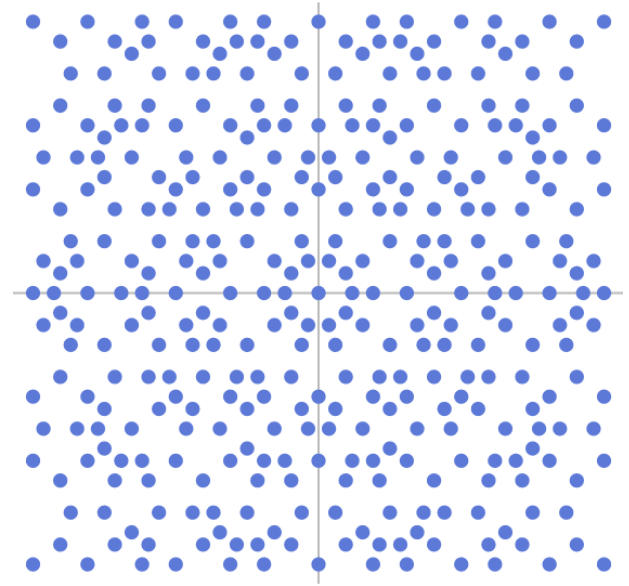**Figure 2:** *Posthumous portrait of Évariste Galois by his brother Alfred Galois (1848)*

Figure 4: *Simple visualisation of arithmetic symmetry of roots using mathematical ideas by Galois (Balakrishnan & Venkatachalam 2014)*
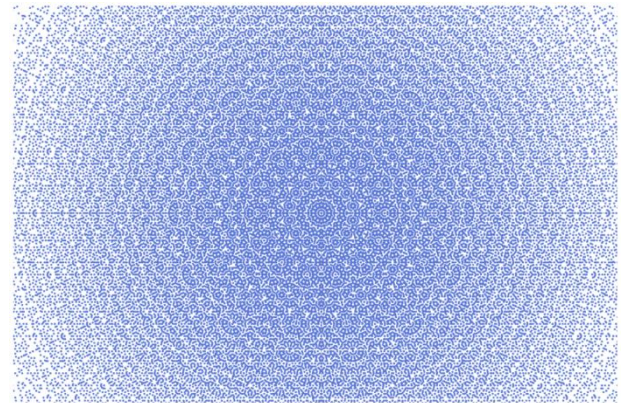

Figure 5: *More complex visualisation of arithmetic symmetry (Balakrishnan & Venkatachalam 2014)*

4. ARCHIVES

Archival documents can visualise history and as seen for example in documentary films since they provide the necessary detail to create vivid pictures of points or moments in time. Much archival material has yet to be discovered, due to the large amount of it and the difficulty of searching it effectively.

With new archival documentation from the Archives Nationales, Archives de Paris, and État Civil of Bourg-la-Reine, we visualise the family life of Évariste in Paris, focusing on the Galois residences at the rue Jean de Beauvais, where his father committed suicide in 1829, and the rue d'Enfer Saint Michel 57, where his brother, Alfred Galois, a painter, died after a prolonged illness on 4 June 1849.

Figure 3: *Visualisation of a Galois field (Stein 2012)*

An online Galois visualisation tool illustrating arithmetic symmetries using roots of unity is available for use by anyone (Balakrishnan & Venkatachalam 2014). Figures 4 and 5 provide examples of simple and more complex visual-isations, generated using this interactive tool (http://people.maths.ox.ac.uk/balakrishnan/galois/).

Documents relating to the life of Évariste Galois' mother, Adélaïde Marie Demante (born Paris 1788; died Paris 1871, aged 83), reveal the social milieu of Évariste's short life and his mother's connections to academic circles in Paris, where her father, Thomas François Demante, was docteur agrégé à la Faculté de droit de l'ancienne université de Paris (see Figure 6), while her brother, Antoine Marie Demante, and his son, Auguste Gabriel, were members of the Faculté de droit and also Chevalier de la Légion d'honneur. His mother's classical education and enthusiasm for learning no doubt influenced Évariste's passionate approach to life. Évariste's father, Nicolas Gabriel, the mayor of Borg-la-Reine, Évariste's place of birth, was the son of Théodore Michel Galois (born 1774, died 1831), who was an Officer of the Royal Order of the Légion d'honneur (see Figure 7).
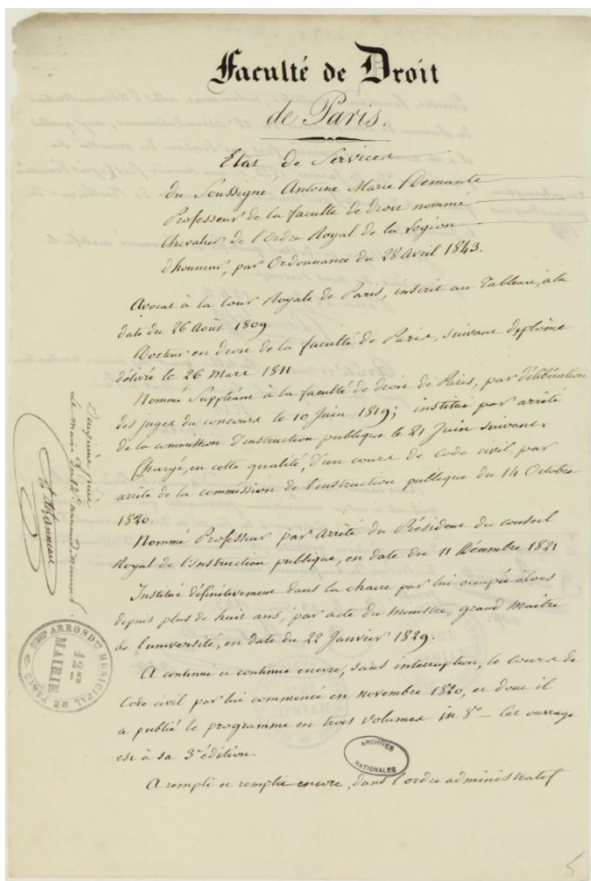

Figure 6: Faculté de Droit de Paris document

4.1 Évariste's family life in Paris at 16 rue Jean de Beauvais, 1829–1832: four fateful years

Focusing on the critical years of the Galois and Demante families in Paris from the death of Nicolas Gabriel Galois to that of Évariste, we present an archival narrative, beginning with the inventory after the death of Évariste Galois' father in 1829, to the marriage contract of Évariste's mother, Adélaïde Marie Demante with Jean Christophe Loyer in 1832, which took place shortly after Évariste's

death. As an epilogue, we consider Loyer's 1837 inventory after his death, the marriage contract and inventory of Alfred Galois, Évariste's brother, and the 1851 sale by Mme Demante of the Galois house in Bourg la Reine, and finally her death there in 1871.
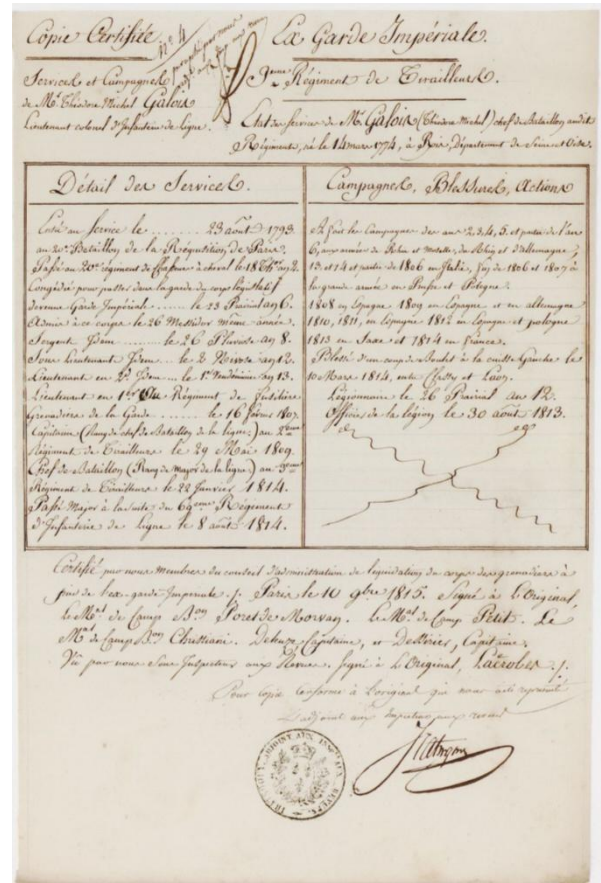

Figure 7: Copie Certifiée La Garde Impériale (Certified copy of the Imperial Guard) for Théodore Michel Galois (http://www.culture.gouv.fr/LH/LH078/PG/FRDAFAN83_OL1064022v004.htm)

Central to this *l'histoire de famille*, is the Demante family house on rue Jean de Beauvais no.16 (formerly, St. Jean de Beauvais), where the dialogue between Évariste and his parents and grandparents no doubt took centre stage, and where the republican views of Évariste Galois' father held sway, influencing the young mind of Évariste, and drawing him into his ill-fated political struggles. Set in the fifth arrondissement in the *quartier* of the Sorbonne at the heart of academic life in Paris, the home on the rue Jean de Beauvais belonged to Évariste's maternal grandparents, Thomas François Demante (lawyer to Parliament, doctor of the faculty of law of Paris, and former President of the civil tribunal of Louviers), and his wife, Marie Elisabeth Thérèse Durand.

Évariste's sister, Nathalie Théodore Galois, married Benoit Chantelot in Paris on 28 January 1829, with

a second ceremony on 5 February 1829 in Bourg La Reine. Her marriage took place just five months before her father's suicide by asphyxiation on 2 July 1829 at the house on rue Jean de Beauvais, where his inventory after death was taken on 11 November 1829. The inventory establishes that his three children from his marriage with Mme Demante, namely Nathalie Théodore, Évariste, and Alfred (see Figure 8), were the unique inheritors of his estate, each receiving one third. Significantly, Évariste Galois' father's inventory details the 1825 estate settlement of his father-in-law, Thomas François Demante, by which Adélaïde Marie Demante received a one-third share of the house on the rue Jean de Beauvais. In October 1829, she moved from Bourg la Reine to Paris, taking up residence there, a move that coincided with Évariste's acceptance at the l'Ecole préparatoire. Consequently, Évariste and his mother were living at the family house in Paris just months after his

father's suicide there and shortly before he received his baccalauréat des sciences on 14 December 1829, so that mother and son were living there through the tumultuous years of the political upheaval of 1830 and 1832. Thus, the house on the rue Jean de Beauvais became in some respects an arbiter of history as it brought together the Galois and Demante families at critical moments of personal crisis. For a family tree of the Galois and Demante families, see Figure 8 below.

The July Revolution of 1830 proved highly disruptive to Évariste's studies and the intricate deep thinking of his mathematical work seems to belie his impulsive and passionate politics. This was a turning point in his life that distracted him from his true mathematical vocation. Without the political upheavals, he could have made even more interesting mathematical discoveries.
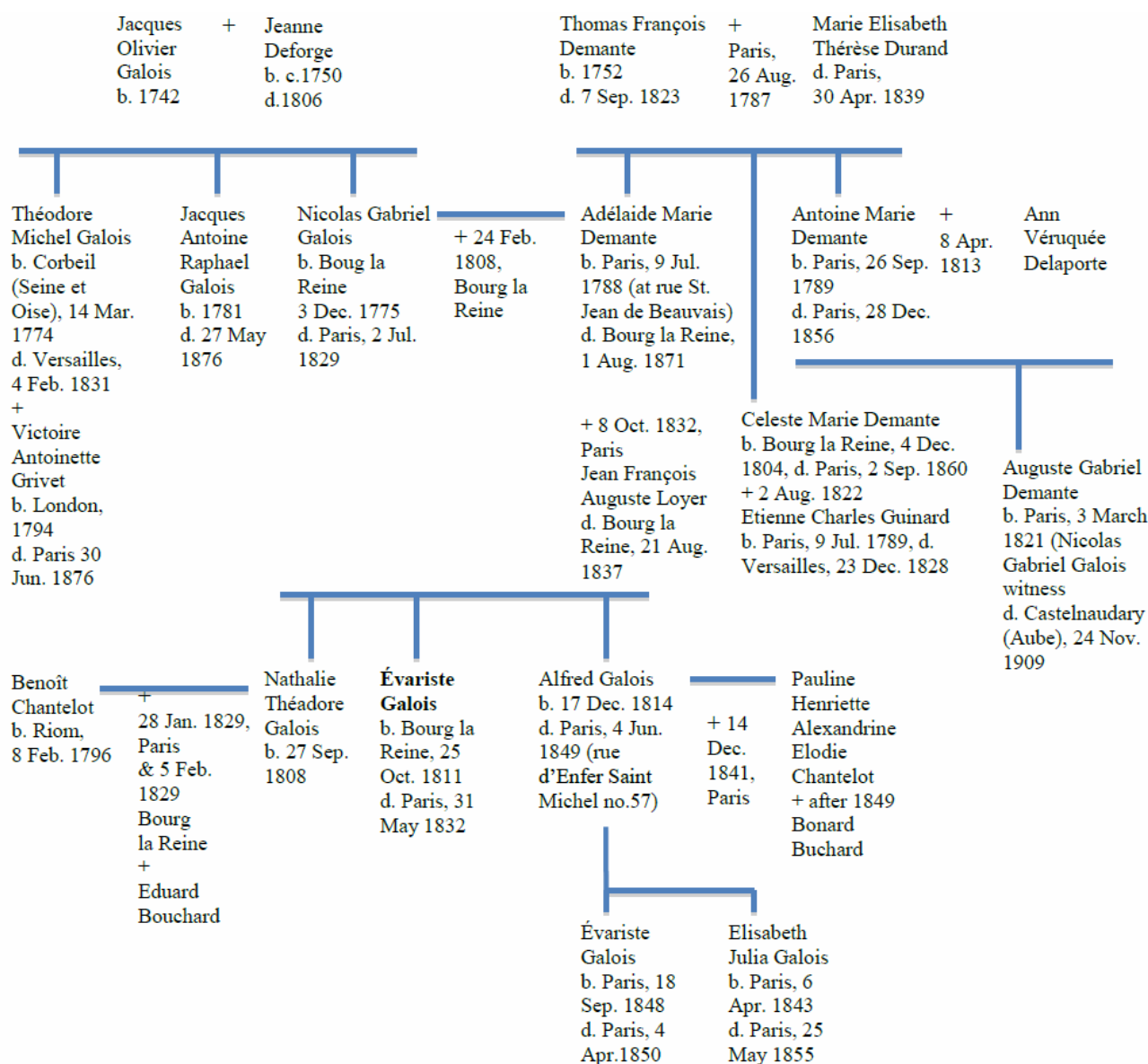

Figure 8: *Galois/Demante family tree*

The year of 1831 saw the death of Évariste's paternal uncle, Théodore Michel Galois, on 4 February 1831, a member of the Royal Order of the Legion of Honour as lieutenant colonel of the 6th regiment of infantryman. Évariste enjoyed a close relationship with him, so that his death added to his loss of family support. 1831 was also the year of Évariste's arrest and imprisonment for political activities at a time when his mother was dividing her time between Paris and Bourg la Reine. We see that at most critical moment in his life, Évariste lacked parental guidance; one can imagine the anguish of Mme Demante as she observed his intensifying anti-monarchy sentiments whilst fearing that she was on the verge of losing her cherished son.

31 May 1832, the day after Évariste's ill-fated pistol duel, marks the tragic end of the life of a great mathematician, ironically just a few days before the Paris uprising of 1832, on 5–6 June. Perhaps surprisingly, only four months after her son's death, on 8 October 1832, Mme Demante married Jean Christophe Loyer, described in the marriage contract as "Maître d'Hôtel Garni à Elbeuf" and a widower with two children from his first marriage to Victoire Françoise Rouaux. Significantly, Loyer owned a house in Paris located at 6 rue Jean de Beauvais, which he brought to the marriage, providing Mme Demante with a quarter share. Thus, it seems likely that the newlyweds met on or around rue Jean de Beauvais. After the marriage Mme Demante moved from no. 16 to no. 6.

4.2 Epilogue

When Loyer died on 21 August 1837, he and Mme Demante were still residing at 6 rue Jean de Beauvais, where his inventory after death was taken, even though he died in Bourg la Reine where Mme Demante maintained property. A few years later, Alfred Galois, born 17 December 1814 in Bourg la Reine, married Pauline Henriette Alexandrine Elodie Chantelot on 14 December 1841. By that date, Mme Demante was living with her son, Alfred, on the rue d'Enfer Saint Michel, at no. 61.

Alfred had two children with Mlle Chatelot: Elisabeth Julia Galois, born 6 April 1843 and died 25 May 1855, at age 12, and a son named after his brother, Évariste Galois, born 8 September 1848 and died on 4 April 1850 at only 18 months old. Sadly Alfred's death in 1849 at his apartment on rue d'Enfer Saint Martin 57, which was due to illness, ended his seemingly happy family life. His marriage contract and inventory after his death describe him as a "peintre, artiste," although little is known of his painting. These documents further establish that the life of the Galois family was marked by turbulent times and *tristesse* for which

the historical backdrop was the French revolution of 1830.

Mme Demante, now twice a widower, was still living on the rue d'Enfer when she sold the Galois family house with "jardins Anglais" in Bourg la Reine at the Grande Rue no. 20, on 26 March 1851 to Pierre Ravon, mayor of Bourg la Reine, and Dame Amadine Zoe Deriquebourcq. According to the contract of sale, the house was sold for 35,000 francs together with an annual *rente viagère* for Mme Demante, by which she would receive 850 francs annually. Page 9 of the contract shows that by 1851 her only surviving heirs were Nathalie Théodore and Alfred's daughter, Julie Elisabeth; it notes the deaths of Évariste, Alfred, and Alfred's son Évariste (see Figure 9).
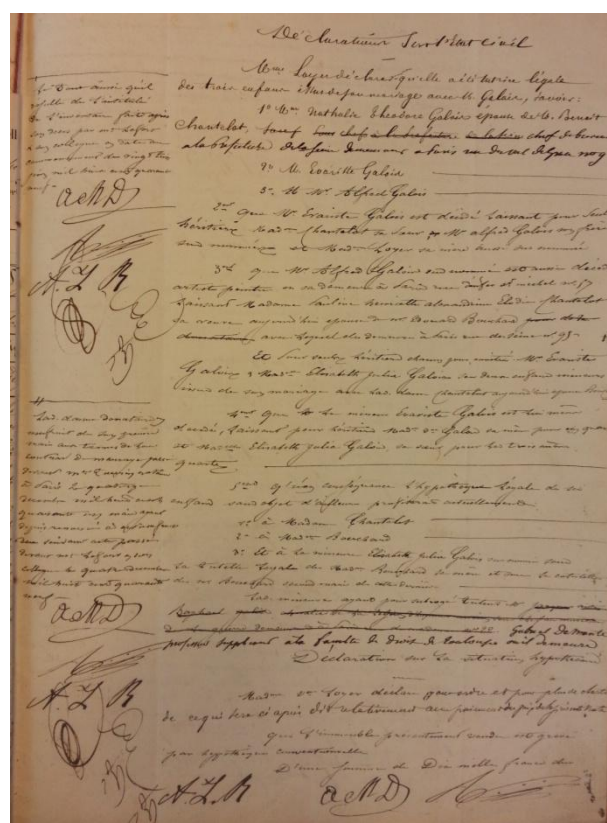

Figure 9: Archives Nationales, Minutier Central, Sale by Mme Loyer (Mme Demante) to M. and Mme Ravon, 26 March 1851 – Déclaration sur l'état civil

Mme Demante, retired, aged 83, died on 1 August 1871 in Bourg la Reine. No family witnesses were present, as was also the case when Évariste died, aged 20, in Paris at the Chochin Hospital in the fifth arrondissement, 39 years earlier. The declaration of his death, which incorrectly states his age to be 21 (a mistake that has been perpetuated by others since then), was made by two hospital employees. Documenting here the series of emotionally charged family interactions at the Demante residence in Paris at 16 rue Jean de Beauvais, it becomes clear that Évariste's family life profoundly

influenced his thoughts and actions, leading to his premature death in 1832.

7. CONCLUSION

"No mathematician should ever allow him to forget that mathematics, more than any other art or science, is a young man's game. … Galois died at twenty..."
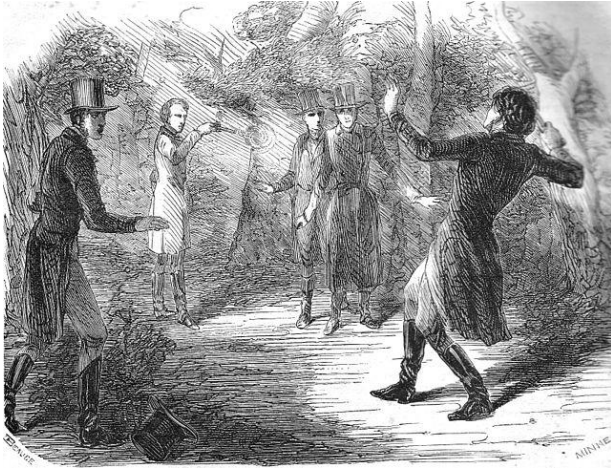
– G. H. Hardy (1877–1947)

Figure 10: *The duel of Évariste Galois in 1832*

Évariste died in Paris on 31 May 1832 from the gunshot wounds that he suffered in a duel (see Figure 10). Even his last letter to a friend, Auguste Chevalieron, on the eve of the duel, was on the subject of mathematics, including some that can be visualised as regular solids and a Fano plane (see Figure 11), an order 2 finite projective plane with the smallest possible number of points and lines, namely seven of each, where there are three lines passing through each point and three points on every line (Kostant 1995, Le Bruyn 2008). The portrait of Évariste by his younger brother, Alfred, remains an iconic image of a great mathematician (see Figure 2 earlier), celebrated by later mathematicians to this day (Neumann 2011) and also theoretical computer scientists (Hoare & He 1998) for his important foundational achievements.
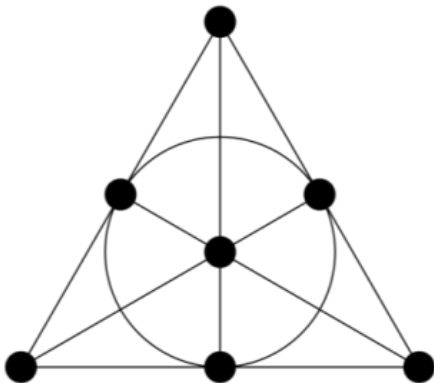

Figure 11: *Visualisation of a Fano plane, relating to Évariste Galois' last letter in 1832 (Wikimedia Commons)*

The death record of Évariste Galois is shown in Figure 12. Bringing archival documents together as a visualisation, we have created an archival map of place (Paris and Borg-la-Reine), time (1811–1871), family connections, and documents, illustrating the cascading events of Évariste's life, leading to his death in 1832, followed his mother's remarriage shortly afterwards in that same year, and beyond that to her death much later in 1871.
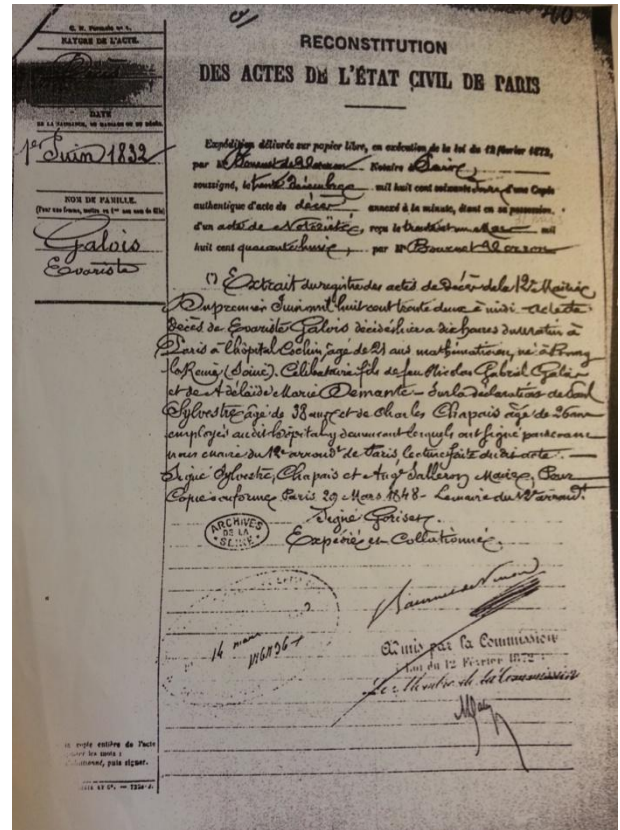

Figure 12: *Archives de Paris, Etat Civil, death record, Évariste Galois, 31 May 1832*

This paper has attempted to provide a view of the French mathematician Évariste Galois (1811–1832) and his close relatives (including his brother, the painter Alfred Galois) from several aspects, including an overview of his mathematical contribution, the possibilities of visualising his mathematics ideas, and the issues of searching historical archives to help understand his family life and background better. There are still many possibilities of visualising Galois' mathematics in more interesting and artistic ways and discovering further historic documents with information on his wider family.
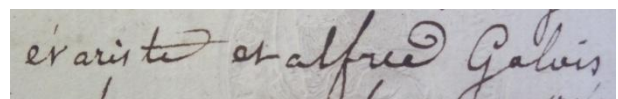

Figure 13: *Mention of Évariste and Alfred Galois in the 1829 inventory of Galois' father (page 2)*
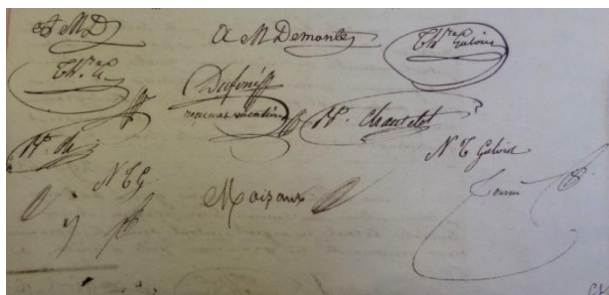
8. ACKNOWLEDGEMENTS

Figure 14: *Galois and Demante family signatures in the 1829 inventory of Nicolas Gabriel*

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