JSIAM Letters Vol.1 (2009) pp.25–27 ©2009 Japan Society for Industrial and Applied Mathematics

On oscillatory solutions of the ultradiscrete Sine-Gordon equation

Shin Isojima[1] and Junkichi Satsuma[1]

Department of Physics and Mathematics, College of Science and Engineering, Aoyama Gakuin University, 5-10-1 Fuchinobe, Sagamihara, Kanagawa, 229-8558 Japan[1]

E-mail *isojima@gem.aoyama.ac.jp, satsuma@gem.aoyama.ac.jp*

Received December 6, 2008, Accepted February 21, 2009

Abstract

Exact solutions of the ultradiscrete Sine-Gordon equation which have oscillating structure are constructed. They are considered to be a counterpart of the breather solution of the Sine-Gordon equation. They are given by setting specific parameters in the discrete soliton solutions and ultradiscretizing the resulting solutions.

Keywords soliton, cellular automaton, Sine-Gordon equation, ultradiscrete system, breather solution

Research Activity Group Applied Integrable Systems

1. Introduction

Cellular automaton (CA) is a discrete dynamical system which consists of a regular array of cells. Each cell takes a finite number of states updated by a given rule in discrete time steps. Although the updating rule is usually simple, CAs may give very complex evolution patterns (see for example [1]). Moreover, CAs are suitable for computer experiments since all variables take discrete values. Hence CAs may be good models to capture the essential mechanisms for physical, social or biological phenomena by simple rules.

Ultradiscretization [2] is a procedure transforming a given difference equation into a CA or an ultradiscrete system. In general, to apply this procedure, we first replace a dependent variable in a given equation x_n with a new variable X_n by

$$x_n = e^{X_n/\varepsilon} \tag{1}$$

upon introduction of a parameter $\varepsilon > 0$. Then in the limit $\varepsilon \downarrow 0$, addition, multiplication and division of the original variables are replaced with max, addition and subtraction for the new ones, respectively. Note that x_n should be positive definite for (1) and that no general way to cover subtraction in a discrete equation. In addition to overcoming these difficulty, it is also an open problem how to capture oscillatory phenomena in ultradiscrete systems. A partial answer is given in [3] and [4], in which ultradiscretization of the elliptic functions is discussed. The authors and coworkers reported an ultradiscrete analogue of the Airy function as the solution of an initial value problem in [5].

It has already been reported that some ultradiscrete systems constructed from discrete soliton equations possess soliton solutions similar to those of the discrete or corresponding continuous systems (see for example [2, 6, 7]). However, an ultradiscrete solution propagating with oscillation, as the breather solution of the Sine-Gordon (SG) equation, has not been reported. In this letter, we propose solutions of an ultradiscrete analogue of the SG (udSG) equation [8] which have oscillating structure. They are considered to be a counterpart of the breather solution. They are constructed by proper setting of parameters in the known discrete soliton solutions and ultradiscretizing the resulting solutions.

2. Ultradiscrete Sine-Gordon Equation

The SG equation, one of the well-known soliton equations,

$$\frac{\partial^2 \varphi}{\partial x \partial t} = \sin \varphi \tag{2}$$

is famous for possessing the breather solution, which describes oscillatory phenomena and is given as the special case of the 2-soliton solution. Hirota proposed an integrable discrete analogue of the SG equation [9]

$$\sin\left(\frac{\phi_{n+1}^{m+1} + \phi_{n-1}^{m-1} - \phi_{n+1}^{m-1} - \phi_{n-1}^{m+1}}{4}\right) = \delta^2 \sin\left(\frac{\phi_{n+1}^{m+1} + \phi_{n-1}^{m-1} + \phi_{n+1}^{m-1} + \phi_{n-1}^{m+1}}{4}\right) \tag{3}$$

through the bilinearizing technique. Note that this equation also has the breather solution.

For the purpose of constructing an udSG equation, the authors and coworkers proposed another discrete SG equation [8]

$$\begin{vmatrix} \left(1-\delta^2\right) u_{n-1}^{m-1} - 1 & \dfrac{1+\delta^2}{u_{n-1}^{m+1}} - 1 \\ \dfrac{1+\delta^2}{u_{n+1}^{m-1}} - 1 & \left(1-\delta^2\right) u_{n+1}^{m+1} - 1 \end{vmatrix} = 0. \tag{4}$$

This equation is reduced to the trilinear form

$$\begin{vmatrix} \left(1-\delta^2\right)\tau_{n-2}^{m-2} & \tau_{n-2}^{m} & \left(1+\delta^2\right)\tau_{n-2}^{m+2} \\ \tau_{n}^{m-2} & \tau_{n}^{m} & \tau_{n}^{m+2} \\ \left(1+\delta^2\right)\tau_{n+2}^{m-2} & \tau_{n+2}^{m} & \left(1-\delta^2\right)\tau_{n+2}^{m+2} \end{vmatrix} = 0 \qquad (5)$$

through the variable transformation

$$u_n^m = \frac{\tau_{n+1}^{m+1}\tau_{n-1}^{m-1}}{\tau_{n+1}^{m-1}\tau_{n-1}^{m+1}}. \qquad (6)$$

If we set

$$\delta = \tanh\left(\frac{L}{2\varepsilon}\right), \quad \tau_n^m = e^{T_n^m/\varepsilon}, \quad u_n^m = e^{U_n^m/\varepsilon} \qquad (7)$$

and take the limit $\varepsilon \downarrow 0$, we have the udSG equation for U_n^m

$$\begin{aligned} &\max\Big[-|L| + U_{n+1}^{m+1} + U_{n-1}^{m-1},\, |L| - U_{n+1}^{m-1},\, |L| - U_{n-1}^{m+1}\Big] \\ &\quad = \max\Big[\,|L| - U_{n+1}^{m-1} - U_{n-1}^{m+1},\ U_{n+1}^{m+1},\, U_{n-1}^{m-1}\Big] \end{aligned} \qquad (8)$$

from (4) and for T_n^m

$$\begin{aligned} \max\Big[&-|L| + T_{n+2}^{m+2} + T_n^m + T_{n-2}^{m-2}, \\ &|L| + T_{n+2}^{m-2} + T_n^{m+2} + T_{n-2}^{m}, \\ &|L| + T_{n+2}^{m} + T_n^{m-2} + T_{n-2}^{m+2}\Big] \\ = \max\Big[\, &|L| + T_{n+2}^{m-2} + T_n^m + T_{n-2}^{m+2}, \\ &T_{n+2}^{m+2} + T_n^{m-2} + T_{n-2}^{m}, \\ &T_{n+2}^{m} + T_n^{m+2} + T_{n-2}^{m-2}\Big] \end{aligned} \qquad (9)$$

from (5) and the relation between T_n^m and U_n^m

$$U_n^m = T_{n+1}^{m+1} + T_{n-1}^{m-1} - T_{n+1}^{m-1} - T_{n-1}^{m+1} \qquad (10)$$

from (6). Refer to [8] for more detail about the udSG equation and its soliton solutions.

3. Oscillatory Solution

For the purpose of our discussion, we give the 2-soliton solution of (5). Let p_j, q_j be parameters satisfying the dispersion relation

$$\delta^2({p_j}^2+1)({q_j}^2+1) = ({p_j}^2-1)({q_j}^2-1) \qquad (11)$$

and a_j be arbitrary phase constants. Phases x_j and interaction factors b_{jk} are defined by

$$x_j = {p_j}^n {q_j}^m, \qquad (12)$$

$$b_{jk} = \frac{({p_j}^2 - {p_k}^2)^2}{((p_j p_k)^2 - 1)^2}, \qquad (13)$$

respectively. In terms of these notations, the 2-soliton solution is written as

$$\tau_n^m = 1 + a_1 x_1 + a_2 x_2 + a_1 a_2 b_{12} x_1 x_2. \qquad (14)$$

Now, we construct the 2-periodic solution by specific setting of parameters in (14). Let us set

$$p_2 = -p_1,\ q_2 = q_1,\ a_1 = \alpha_1 + \alpha_2,\ a_2 = \alpha_2. \qquad (15)$$

Then (14) is reduced to

$$\tau_n^m = \begin{cases} 1 + (\alpha_1 + 2\alpha_2)x_1 & (n : \text{even}), \\ 1 + \alpha_1 x_1 & (n : \text{odd}). \end{cases} \qquad (16)$$

The phase constant in (16) depends on whether n is an even number or an odd number. This structure plays a crucial role for 2-periodic behaviour of the solution.

Let us ultradiscretize (16). First, we put

$$\begin{aligned} &p_1 = e^{P_1/\varepsilon}, \quad q_1 = e^{Q_1/\varepsilon}, \\ &\alpha_1 = e^{A_1/\varepsilon}, \quad \alpha_2 = e^{A_2/\varepsilon} \quad (A_1 < A_2), \end{aligned} \qquad (17)$$

and take the limit $\varepsilon \downarrow 0$. Then we have the ultradiscrete analogue of (16),

$$T_n^m = \begin{cases} \max(0,\, P_1 n + Q_1 m + A_2) & (n : \text{even}), \\ \max(0,\, P_1 n + Q_1 m + A_1) & (n : \text{odd}). \end{cases} \qquad (18)$$

Note that P_1 and Q_1 should satisfy the dispersion relation

$$|P_1 + Q_1| = |L| + |P_1 - Q_1|, \qquad (19)$$

which is obtained by ultradiscretizing (11). Substituting (18) into (10), we have U_n^m solving (8). For general parameters, the solution describes travelling pulse with oscillation. In order to emphasize its periodic behaviour, we set $P = Q = |L|/2$, which satisfy (19), and introduce new independent variables (k, l) by

$$n = k - l, \quad m = k + l. \qquad (20)$$

Fig. 1–3 show behaviour of U_n^m for various values of parameters A_1, A_2. In all cases, the solution gives localized pulse for fixed *time* l. Each pulse is almost stable and its shape changes for l in period 2. Hence, this solution clearly describes oscillatory phenomena. Furthermore, its behaviour is similar to that of the breather solution.

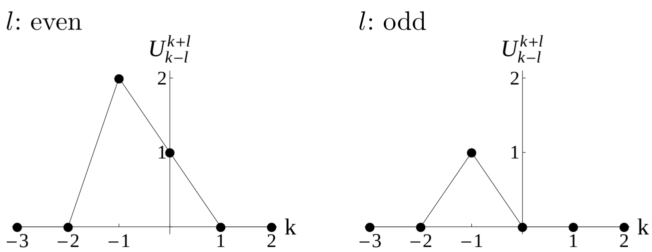

Fig. 1. An example of oscillatory solution. $L = 2$, $P_1 = Q_1 = 1$, $A_1 = 1$, $A_2 = 2$.

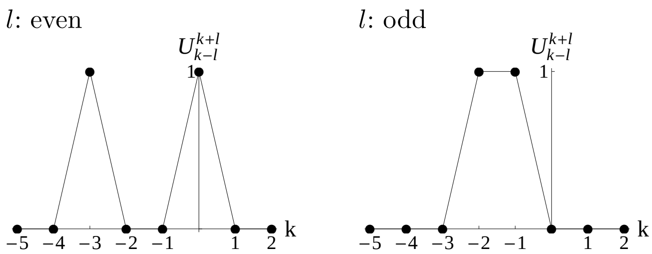

Fig. 2. An example of oscillatory solution. $L = 2$, $P_1 = Q_1 = 1$, $A_1 = 1$, $A_2 = 5$.

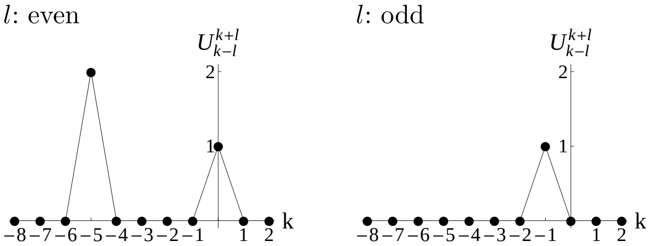

Fig. 3. An example of oscillatory solution. $L = 2$, $P_1 = Q_1 = 1$, $A_1 = 1$, $A_2 = 10$.

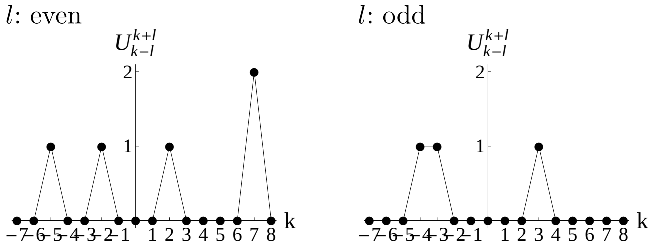

Fig. 4. An example of oscillatory solution with richer structure. $L = 2$, $P_1 = Q_1 = 1$, $P_3 = Q_3 = -1$, $A_1 = 1$, $A_2 = 5$, $A_3 = 1$, $A_4 = 10$.

For the sake of constructing the solution with richer structure, we consider the 4-soliton solution

$$\begin{aligned}\tau_n^m = 1 + \sum_{j=1}^{4} a_j x_j + \sum_{j<k,\ j,k=1}^{4} a_j a_k b_{jk} x_j x_k \\ + \sum_{j<k<l,\ j,k,l=1}^{4} a_j a_k a_l b_{jk} b_{jl} b_{kl} x_j x_k x_l \\ + a_1 a_2 a_3 a_4 b_{12} b_{13} b_{14} b_{23} b_{24} b_{34} x_1 x_2 x_3 x_4. \end{aligned} \tag{21}$$

If we put (15) and

$$p_4 = -p_3,\ q_4 = q_3,\ a_3 = \alpha_3 + \alpha_4,\ a_4 = \alpha_4, \tag{22}$$

then we have

$$\tau_n^m = \begin{cases} 1 + (\alpha_1 + 2\alpha_2)x_1 + (\alpha_3 + 2\alpha_4)x_3 \\ \quad + (\alpha_1 + 2\alpha_2)(\alpha_3 + 2\alpha_4)b_{13}x_1x_3 & (n: \text{even}), \\ 1 + \alpha_1 x_1 + \alpha_3 x_3 + \alpha_1 \alpha_3 b_{13} x_1 x_3 & (n: \text{odd}). \end{cases} \tag{23}$$

Moreover, setting

$$\begin{aligned} p_j = e^{P_j/\varepsilon},\ q_j = e^{Q_j/\varepsilon}, \\ \alpha_j = e^{A_j/\varepsilon} \quad (A_1 < A_2,\ A_3 < A_4) \end{aligned} \tag{24}$$

and taking the limit $\varepsilon \downarrow 0$, we have

$$T_n^m = \begin{cases} \max[\ 0,\ X_1 + A_2,\ X_3 + A_4, \\ \qquad X_1 + X_3 + A_2 + A_4 \\ \qquad\quad + 2(|P_1 - P_3| - |P_1 + P_3|)\ ] & (n: \text{even}), \\ \max[\ 0,\ X_1 + A_1,\ X_3 + A_3, \\ \qquad X_1 + X_3 + A_1 + A_3 \\ \qquad\quad + 2(|P_1 - P_3| - |P_1 + P_3|)\ ] & (n: \text{odd}). \end{cases} \tag{25}$$

The solution U_n^m constructed from (25) and (10) describes interaction among oscillating pulses. We consider the specific case $P_1 = Q_1 = |L|/2$, $P_3 = Q_3 = -|L|/2$ and introducing independent variables (k, l) defined by (20). We observe pulses which are almost stable and change their shape in period 2 (see Fig. 4).

We can obtain a solution which describes larger numbers of oscillating pulse by starting from the $(2N)$-soliton solution. We would, however, comment that we have only two choices of P, Q such that $P = Q$ and (19) holds, namely $P = Q = \pm|L|/2$. Hence, the oscillatory solution constructed from the $(2N)$-soliton solution may be understood as nonlinear superposition of the solutions given in this section.

4. Concluding Remarks

We have given exact solutions of the udSG equation which describe oscillatory phenomena. They are considered to be a counterpart of the breather solution. It is an interesting problem to construct oscillatory solutions for other ultradiscrete systems by applying the procedure developed in Section 3. We also comment that the period of oscillation of our solution is essentially 2 by its construction. It is a future problem to find the ultradiscrete system having solutions with arbitrary periods.

References

[1] S. Wolfram, A New Kind of Science, Wolfram Media, Inc., Champaign, 2002.

[2] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, From soliton equations to integrable cellular automata through a limiting procedure, Phys. Rev. Lett., **76** (1996), 3247–3250.

[3] D. Takahashi, T. Tokihiro, B. Grammaticos, Y. Ohta and A. Ramani, Constructing solutions to the ultradiscrete Painlevé equations, J. Phys. A: Math. Gen., **30** (1997), 7953–7966.

[4] A. Nobe, Ultradiscretization of elliptic functions and its applications to integrable systems, J. Phys. A: Math. Gen., **39** (2006), L335–L342.

[5] S. Isojima, B. Grammaticos, A. Ramani and J. Satsuma, Ultradiscretization without positivity, J. Phys. A: Math. Gen., **39** (2006), 3663–3672.

[6] J. Matsukidaira, J. Satsuma, D. Takahashi, T. Tokihiro and M. Torii, Toda-type cellular automaton and its N-soliton solution, Phys. Lett. A, **225** (1997), 287–295.

[7] M. Murata, S. Isojima, A. Nobe and J. Satsuma, Exact solutions for discrete and ultradiscrete modified KdV equations and their relation to box-ball systems, J. Phys. A: Math. Gen., **39** (2006), L27–L34.

[8] S. Isojima, M. Murata, A. Nobe and J. Satsuma, An ultradiscretization of the sine-Gordon equation, Phys. Lett. A, **331** (2004), 378–386.

[9] R. Hirota, Nonlinear partial difference equations III; discrete sine-Gordon equation, J. Phys. Soc. Japan, **43** (1977), 2079–2086.