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On the qd-type discrete hungry Lotka-Volterra system and its application to the matrix eigenvalue algorithm

Akiko Fukuda[1], Emiko Ishiwata[2], Masashi Iwasaki[3] and Yoshimasa Nakamura[4]

Graduate School of Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan[1]
Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan[2]
Faculty of Life and Environmental Sciences, Kyoto Prefectural University, 1-5 Nagaragi-cho, Shimogamo, Sakyo-ku , Kyoto 606-8522, Japan[3]
Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan[4]

E-mail *j1409704@ed.kagu.tus.ac.jp*

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Abstract

The discrete hungry Lotka-Volterra (dhLV) system is already shown to be applied to matrix eigenvalue algorithm. In this paper, we discuss a form of the dhLV system named as the qd-type dhLV system and associate it with a matrix eigenvalue computation. Along a way similar to the dqd algorithm, we also design a new algorithm without cancellation in terms of the qd-type dhLV system.

Keywords discrete hungry Lotka-Volterra system, dqd algorithm, matrix eigenvalue

Research Activity Group Applied Integrable Systems

1. Introduction

Integrable systems have some relationships to numerical algorithms. For example, the continuous time Toda equation corresponds to one step of the QR algorithm [1] for computing eigenvalues of a symmetric tridiagonal matrix. A discretization of the Toda equation is just the quotient difference (qd) algorithm [2]. The discrete Toda (dToda) equation also leads to a new algorithm for the Laplace transformation [3]. The discrete relativistic Toda equation is applicable for continued fraction expansion [4].

Some of the authors designed new algorithms named the dLV algorithm for computing singular values of a bidiagonal matrix in terms of the integrable discrete Lotka-Volterra (dLV) system [5]. For $k = 1, 2, \dots, 2m-1$ and $n = 0, 1, \dots,$

$$u_k^{(n+1)}(1 + \delta^{(n+1)}u_{k-1}^{(n+1)}) = u_k^{(n)}(1 + \delta^{(n)}u_{k+1}^{(n)}), \quad (1)$$
$$u_0^{(n)} \equiv 0, \quad u_{2m}^{(n)} \equiv 0,$$

where $\delta^{(n)}$ is the n-th discrete step-size and $u_k^{(n)}$ denotes the number of k-th species at the discrete time $\sum_{j=0}^{n-1}\delta^{(j)}$. It is shown in [6] that $u_{2k-1}^{(n)}$ and $u_{2k}^{(n)}$ converge to certain positive constant and zero, respectively, as $n \to \infty$. The dLV algorithm is also surveyed in a recent review paper [7].

Now we introduce new variables

$$q_k^{(n)} := \frac{1}{\delta^{(n)}}(1 + \delta^{(n)}u_{2k-2}^{(n)})(1 + \delta^{(n)}u_{2k-1}^{(n)}), \quad (2)$$
$$e_k^{(n)} := \delta^{(n)}u_{2k-1}^{(n)}u_{2k}^{(n)}. \quad (3)$$

Then, the dLV system (1) yields the recursion formula of the qd algorithm

$$\begin{cases} q_{k+1}^{(n+1)} = q_{k+1}^{(n)} - e_k^{(n+1)} + e_{k+1}^{(n)}, \\ e_k^{(n+1)} = e_k^{(n)}\dfrac{q_{k+1}^{(n)}}{q_k^{(n+1)}}. \end{cases} \quad (4)$$

As mentioned above, this recursion formula is equivalent to the dToda equation. Namely, the dLV system has a relationship to the dToda equation. Rutishauser introduced a modified version, named the dqd (differential qd) algorithm [2], for the purpose of avoiding numerical instability of qd algorithm.

Recently, in [8,9], we designed a new algorithm named the dhLV algorithm for computing complex eigenvalues of a certain band matrix. The dhLV algorithm is derived from the integrable discrete hungry Lotka-Volterra (dhLV) system [10]. For $k = 1, 2, \dots, M_m$ and $n = 0, 1, \dots,$

$$u_k^{(n+1)}\prod_{j=1}^{M}(1+\delta^{(n+1)}u_{k-j}^{(n+1)}) = u_k^{(n)}\prod_{j=1}^{M}(1+\delta^{(n)}u_{k+j}^{(n)}), \quad (5)$$
$$u_{1-M}^{(n)} \equiv 0, \dots, u_0^{(n)} \equiv 0, \quad u_{M_m+1}^{(n)} \equiv 0, \dots, u_{M_m+M}^{(n)} \equiv 0,$$

where $M_k := (k-1)M+k$, and the meaning of $u_k^{(n)}$ is the same as that of the dLV system. The dLV system (1) is a prey-predator model that the k-th species is predator of the $(k+1)$-th species. On the other hand, the dhLV system (5) is derived by considering the case where the k-th species is predator of the $(k+1)$-th, $(k+2)$-th, $\dots$, $(k+M)$-th species. Of course, if $M = 1$ then (5) coincides with (1).

In this paper, we discuss a new algorithm for computing matrix eigenvalues from a viewpoint of the qd-type dhLV system based on (5). See Section 3 for the qd-type dhLV system. Along a way similar to the dqd algorithm, we derive a recursion formula without subtraction.

This paper is organized as follows. In Section 2, we describe some properties of the dhLV system. In Section 3, we show two invariants of the qd-type dhLV system. We clarify a relationship between the qd-type dhLV system and the matrix eigenvalue algorithm in Section 4. We design an algorithm for computing eigenvalues without cancellation and demonstrate a numerical example. In the final section, we give concluding remarks.

2. Some properties for the dhLV system

In this section, we explain some properties for the dhLV system briefly. The matrix representation of (5) is given as

$$R^{(n)}L^{(n+1)} = L^{(n)}R^{(n)}, \tag{6}$$

$$\begin{aligned} L^{(n)} := (&\boldsymbol{e}_2, \dots, \ \boldsymbol{e}_{M+1}, U_1^{(n)}\boldsymbol{e}_1 + \boldsymbol{e}_{M+2}, \dots, \\ &U_{M_m-1}^{(n)}\boldsymbol{e}_{M_m-1} + \boldsymbol{e}_{M_m+M}, U_{M_m}^{(n)}\boldsymbol{e}_{M_m}), \end{aligned} \tag{7}$$

$$\begin{aligned} R^{(n)} := (&V_1^{(n)}\boldsymbol{e}_1 + \delta^{(n)}\boldsymbol{e}_{M+2}, \dots, \\ &V_{M_m-1}^{(n)}\boldsymbol{e}_{M_m-1} + \delta^{(n)}\boldsymbol{e}_{M_m+M}, \dots, \\ &V_{M_m}^{(n)}\boldsymbol{e}_{M_m}, \dots, V_{M_m+M}^{(n)}\boldsymbol{e}_{M_m+M}), \end{aligned} \tag{8}$$

$$\boldsymbol{e}_k := (\underbrace{0, \dots, 0, 1}_{k}, 0, \dots, 0)^\top, \tag{9}$$

$$U_k^{(n)} := u_k^{(n)} \prod_{j=1}^{M} (1 + \delta^{(n)} u_{k-j}^{(n)}), \tag{10}$$

$$V_k^{(n)} := \prod_{j=0}^{M} (1 + \delta^{(n)} u_{k-j}^{(n)}). \tag{11}$$

Eq. (6) is called Lax form of the dhLV system (5), cf. [11, 12]. Assume that $0 < u_k^{(0)} < K_0$ for $k = 1, 2, \dots, M_m$, then we have $0 < u_k^{(n)} < K$ as is shown in [8, 9], where K_0 is an arbitrary and K is a related positive constant. In (11), if $\delta^{(n)} > 0$ holds for $n = 0, 1, \dots$, then $V_k^{(n)} \geq 1$ holds for $k = 1, 2, \dots, M_m + M$ in the Lax matrix (8). Hence, there exists the inverse matrix of $R^{(n)}$, and (6) can be rewritten as

$$L^{(n+1)} = (R^{(n)})^{-1} L^{(n)} R^{(n)}. \tag{12}$$

This is a similarity transformation from $L^{(n)}$ to $L^{(n+1)}$. Namely, the eigenvalues of $L^{(n)}$ are invariant under the evolution from n to $n + 1$. Moreover, the eigenvalues of $L^{(n)} + dI$ are invariant for any n, where I is a unit matrix and d is an arbitrary constant.

The asymptotic behavior of the dhLV system is as follows.

$$\lim_{n\to\infty} u_{M_k}^{(n)} = c_k, \quad k = 1, 2, \dots, m, \tag{13}$$

$$\lim_{n\to\infty} u_{M_k+p}^{(n)} = 0, \quad p = 1, 2, \dots, M. \tag{14}$$

See [9] for the proof of (13) and (14). By combining (10) and (11) with (13) and (14), it is obvious that the limits of $U_k^{(n)}$ and $V_k^{(n)}$ also exist. As $n \to \infty$, the Lax matrix $L^{(n)} + dI$ converges to

$$L(d) := \lim_{n\to\infty} (L^{(n)} + dI) = \begin{pmatrix} L_1(d) & & & \mathbf{0} \\ E_M & L_2(d) & & \\ & \ddots & \ddots & \\ \mathbf{0} & & E_M & L_m(d) \end{pmatrix}, \tag{15}$$

where $L_k(d)$ and E_M are $(M+1)\times(M+1)$ block matrices defined by

$$L_k(d) := \begin{pmatrix} d & & & c_k \\ 1 & d & & \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & d \end{pmatrix}, \quad E_M := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ & \mathbf{0} & & \vdots \\ & & & 0 \end{pmatrix}.$$

It is of significance to note that $L^{(n)} + dI$ can be divided into several block matrices. The characteristic polynomial of $L(d)$ is given as

$$\det(\lambda I - L(d)) = \prod_{k=1}^{m} \left\{ (\lambda - d)^{M+1} - c_k \right\}.$$

Therefore, we obtain the eigenvalues $\lambda_{k,l}$ of $L^{(0)} + dI$ as follows.

$$\lambda_{k,l} = \sqrt[M+1]{c_k} \left\{ \cos\left(\frac{2l\pi}{M+1}\right) + i \sin\left(\frac{2l\pi}{M+1}\right) \right\} + d,$$

$$l = 1, 2, \dots, M+1, \quad k = 1, 2, \dots, m,$$

where $i = \sqrt{-1}$. For a sufficiently large n, $\lambda_{k,l}$ becomes the approximate value of the eigenvalues of $L^{(0)} + dI$. As a result, the dhLV algorithm is designed for computing eigenvalues of $L^{(0)} + dI$ in [8, 9].

3. Invariants of the qd-type dhLV system

In this section, we investigate some properties of a recursion formula derived from the Lax form (6).

By comparing the both sides of (6), the variables $U_k^{(n)}$ in (7) and $V_k^{(n)}$ in (8) satisfy the following relations

$$\delta^{(n)} U_k^{(n+1)} + V_{k+M+1}^{(n)} = \delta^{(n)} U_{k+M+1}^{(n)} + V_{k+M}^{(n)}, \tag{16}$$

$$V_k^{(n)} U_k^{(n+1)} = U_k^{(n)} V_{k+M}^{(n)}, \quad k = 1, 2, \dots, M_m. \tag{17}$$

We call (16) and (17) the qd-type dhLV system. Let us here impose the boundary condition

$$V_{k-M}^{(n)} \equiv 1, \quad U_{k-M}^{(n)} \equiv 0, \quad k = 0, 1, \dots, M,$$

$$V_{M_m+M+k}^{(n)} \equiv 1, \quad U_{M_m+k}^{(n)} \equiv 0, \quad k = 1, 2, \dots, M.$$

The existence of invariants is one of characteristic properties in integrable systems. Now we give two propositions concerning invariants independent of the discrete variable n.

Proposition 1 *Variable $U_k^{(n)}$ satisfies*

$$\sum_{k=1}^{M_m} U_k^{(n+1)} = \sum_{k=1}^{M_m} U_k^{(n)}. \tag{18}$$

Proof Taking a sum on both sides of (16) for $k = -M, -M+1, \ldots, M_m$, as

$$\sum_{k=-M}^{M_m} (\delta^{(n)} U_k^{(n+1)} + V_{k+M+1}^{(n)}) = \sum_{k=-M}^{M_m} \left(\delta^{(n)} U_{k+M+1}^{(n)} + V_{k+M}^{(n)}\right).$$

Let us expand the above equation and substitute the boundary condition, then we have (18).

(QED)

Proposition 2 *Variable $U_k^{(n)}$ satisfies*

$$\prod_{k=1}^{m} U_{M_k}^{(n+1)} = \prod_{k=1}^{m} U_{M_k}^{(n)}. \tag{19}$$

Proof Let us recall that, in (12), $L^{(n+1)}$ has the same eigenvalues as $L^{(n)}$ for $n = 0, 1, \ldots$. Then it is obvious that

$$\det(L^{(n+1)}) = \det(L^{(n)}), \quad n = 0, 1, \ldots. \tag{20}$$

By cofactor expansion, the determinants of $L^{(n)}$ and $L^{(n+1)}$ are given as

$$\det(L^{(n)}) = (-1)^{mM} U_{M_1}^{(n)} U_{M_2}^{(n)} \ldots U_{M_m}^{(n)},$$
$$\det(L^{(n+1)}) = (-1)^{mM} U_{M_1}^{(n+1)} U_{M_2}^{(n+1)} \ldots U_{M_m}^{(n+1)},$$

respectively. Substituting the above expression into (20), we have

$$(-1)^{mM} U_{M_1}^{(n+1)} U_{M_2}^{(n+1)} \ldots U_{M_m}^{(n+1)} = (-1)^{mM} U_{M_1}^{(n)} U_{M_2}^{(n)} \ldots U_{M_m}^{(n)}.$$

This leads to (19).

(QED)

Let us assume that $0 < U_k^{(0)} < \hat{K}_0$ for $k = 1, 2, \ldots, M_m$, where $\hat{K}_0$ is an arbitrary positive constant. Then $0 < \sum_{k=1}^{M_m} U_k^{(0)} < \hat{K}_1$ and $0 < \prod_{k=1}^{m} U_{M_k}^{(0)} < \hat{K}_2$, where $\hat{K}_1$, $\hat{K}_2$ are positive constants related to $\hat{K}_0$. Propositions 1 and 2 also imply that, $0 < \sum_{k=1}^{M_m} U_k^{(n)} < \hat{K}_1$ and $0 < \prod_{k=1}^{m} U_{M_k}^{(n)} < \hat{K}_2$. Under the assumption $0 < u_k^{(0)} < \hat{K}_0$, it is concluded that $0 < U_k^{(n)} < \hat{K}_3$ for $n = 0, 1, \ldots$, where $\hat{K}_3$ is a positive constant related to $\hat{K}_0$. Note that the time evolution is performed in the arithmetic such that positivity of variables is assured. This property is important for designing numerical algorithms.

4. The qd-type dhLV system and matrix eigenvalue

In this section, we propose an application of the qd-type dhLV system to matrix eigenvalue computation. Assume that there exists the limit of $\delta^{(n)}$ as $n \to \infty$, and let $\delta^* := \lim_{n\to\infty} \delta^{(n)}$. By taking account of (10) and (11), the limits of $U_k^{(n)}$ and $V_k^{(n)}$ also exist as $n \to \infty$. Namely,

$$\lim_{n\to\infty} U_{M_k}^{(n)} = c_k, \quad k = 1, 2, \ldots, m,$$
$$\lim_{n\to\infty} U_{M_k+p}^{(n)} = 0, \quad p = 1, 2, \ldots, M,$$
$$\lim_{n\to\infty} V_{M_k+p}^{(n)} = \delta^* c_k + 1, \quad p = 0, 1, \ldots, M.$$

We simply rewrite the qd-type dhLV system (16) and (17) as the following recursion formula.

$$U_k^{(n+1)} = \frac{U_k^{(n)} V_{k+M}^{(n)}}{V_k^{(n)}}, \tag{21}$$

$$V_k^{(n)} = \delta^{(n)} U_k^{(n)} + V_{k-1}^{(n)} - \delta^{(n)} \frac{U_{k-M-1}^{(n)} V_{k-1}^{(n)}}{V_{k-M-1}^{(n)}}. \tag{22}$$

The time evolution from n to $n+1$ in (21) with (22) is applicable for computing eigenvalues of $L^{(0)} + dI$. For $U_k^{(0)} > 0$ the time evolution in (21) with (22) generates the same matrix as (15), where $V_k^{(0)}$ is calculated if $U_k^{(0)}$ is given. In other words, computed eigenvalues by (21) with (22) are theoretically equal to those by the dhLV algorithm.

In finite arithmetic, it is doubtful whether the time evolution in (21) with (22) is performed with high accuracy. This is because that cancellation by subtraction may occur. Subtraction also appears in the recursion formula of the qd algorithm.

Rutishauser [2] recognized some numerical instability of the qd algorithm (4) where variables $q_k^{(n)}$ and $e_k^{(n)}$ are not related to the dLV variables $u_k^{(n)}$. So he introduced an modified version, named the dqd (differential qd) algorithm [2], for the purpose of avoiding numerical instability. Along a way similar to the dqd algorithm, we derive a recursion formula without subtraction. Let us introduce a new variable

$$P_k^{(n)} := V_{k-1}^{(n)} - \delta^{(n)} \frac{U_{k-M-1}^{(n)} V_{k-1}^{(n)}}{V_{k-M-1}^{(n)}}. \tag{23}$$

Then $P_k^{(n)}$ satisfies the recursion formula

$$P_k^{(n)} = \frac{V_{k-1}^{(n)}}{V_{k-M-1}^{(n)}} P_{k-M-1}^{(n)}, \tag{24}$$

where we set $P_k^{(n)} = 1$ for $k = -M, -M+1, \ldots, 0$. By using $P_k^{(n)}$, (22) is rewritten as

$$V_k^{(n)} = \delta^{(n)} U_k^{(n)} + P_k^{(n)}. \tag{25}$$

Obviously, (24) and (25) have no subtraction, and the cancellation does not occur. The recursion formula (21) with (24) and (25) is essentially equivalent to the qd-type dhLV system (16) and (17). Note that the ratio $V_k^{(n)}/V_{k-M}^{(n)}$ appears in both (21) and (24). Let $Q_k^{(n)} := V_k^{(n)}/V_{k-M}^{(n)}$, and set $Q_0^{(n)} = 1$ for $n = 0, 1, \ldots$. Then the time evolution of the qd-type dhLV system is performed by the following Procedure 1. In Procedure 1, $U_k^{(0)}$ is given by the entry of $L^{(0)} + dI$ and $\delta^{(n)}$ for $n = 0, 1, \ldots$ is an optional parameter. The time evolution requires less operations in Procedure 1 than in original (21) with (22). As shown in [8, 9], $c_k = \lim_{n\to\infty} U_{M_k}^{(n)}$ is equal to the eigenvalue of $L^{(0)} + dI$. We call this algorithm the

Table 1. Computed eigenvalues of the Toeplitz matrix T by the dhLV and the qd-type dhLV algorithms

by the dhLV algorithm	by the qd-type dhLV algorithm
2.00000000000000 + i 1.788617417884120	2.00000000000000 + i 1.788617417884119
0.211382582115879	0.211382582115880
2.00000000000000 − i 1.788617417884120	2.00000000000000 − i 1.788617417884119
3.78861741788412	3.78861741788412
2.00000000000000 + i 1.333397829783662	2.00000000000000 + i 1.333397829783662
0.666602170216338	0.666602170216338
2.00000000000000 − i 1.333397829783662	2.00000000000000 − i 1.333397829783662
3.33339782978366	3.33339782978366
2.00000000000000 + i 0.5683177818055106	2.00000000000000 + i 0.5683177818055107
1.43168221819449	1.43168221819448
2.00000000000000 − i 0.5683177818055106	2.00000000000000 − i 0.5683177818055107
2.56831778180551	2.56831778180551

qd-type dhLV algorithm.

Procedure 1

set boundary conditions of $U_k^{(n)},\ V_k^{(n)},\ P_k^{(n)},\ Q_k^{(n)}$
for $n := 0, 1, \ldots, n_{\max}$ **do**
  for $k := 1, 2, \ldots, M_m + M$ **do**
    $P_k^{(n)} = Q_{k-1}^{(n)} P_{k-M-1}^{(n)}$
    $V_k^{(n)} = \delta^{(n)} U_k^{(n)} + P_k^{(n)}$
    $Q_k^{(n)} = V_k^{(n)} / V_{k-M}^{(n)}$
  end for
  for $k := 1, 2, \ldots, M_m$ **do**
    $U_k^{(n+1)} = Q_{k+M}^{(n)} U_k^{(n)}$
  end for
end for

Now we present a numerical experiment carried out on our computer with OS: Windows XP, CPU: Genuine Intel (R) 1.66GHz, RAM: 1.99GB. We also use Wolfram Mathematica 6.0 with double-precision floating point arithmetic. As a numerical example, we consider a 12×12 Toeplitz matrix T as $L^{(0)} + dI$ with $M = 3$, $m = 3$, $d = 2$ and $U_k^{(0)} = 1.5$ for $k = 1, 2, \ldots, 9$. Let $\delta^{(n)} = 1.0$ for $n = 0, 1, \ldots$.

Table 1 shows computed eigenvalues by the dhLV algorithm [8, 9] and the qd-type dhLV algorithm. We see from Table 1 that both algorithms can compute the same eigenvalues with almost the same accuracy. The operation number of the dhLV algorithm and of the qd-type dhLV algorithm are $6M$ and 5 times, respectively, for the evolution from n to $n+1$ of one variable. From the viewpoint of the operation number, the qd-type dhLV algorithm is better than the dhLV algorithm.

5. Concluding remarks

In this paper, we discuss some properties of the qd-type dhLV system. Based on the qd-type dhLV system and its properties, we design a new algorithm for computing complex eigenvalues of a certain band matrix, similar to the dhLV algorithm. Along the way similar to the dqd algorithm, we design the qd-type dhLV algorithm without subtraction. We also confirm that the new algorithm can compute same eigenvalues as the dhLV algorithm through a numerical example. In order to compare numerical accuracy and running time of the qd-type dhLV algorithm with or without subtraction and the dhLV algorithm, it is necessary to perform more numerical experiments.

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References

[1] W. W. Symes, The QR algorithm and scattering for the finite nonperiodic Toda lattice, Physica D, **4** (1982), 275–280.
[2] H. Rutishauser, Lectures on Numerical Mathematics, Birkhauser, Boston, 1990.
[3] Y. Nakamura, Calculating Laplace transforms in terms of the Toda molecule, SIAM J. Sci. Comput., **20** (1999), 306–317.
[4] Y. Minesaki and Y. Nakamura, The discrete relativistic Toda molecule equation and a Padé approximation algorithm, Numer. Algorithms, **27** (2001), 219–235.
[5] R. Hirota, Conserved quantities of a "random-time Toda equation", J. Phys. Soc. Japan, **66** (1997), 283–284.
[6] M. Iwasaki and Y. Nakamura, An application of the discrete Lotka-Volterra system with variable step-size to singular value computation, Inverse Problems, **20** (2004), 553–563.
[7] M. T. Chu, Linear algebra algorithm as dynamical systems, Acta Numerica, **17** (2008), 1–86.
[8] A. Fukuda, E. Ishiwata, M. Iwasaki and Y. Nakamura, A numerical factorization of characteristic polynomial by the discrete hungry Lotka-Volterra system (in Japanese), Trans. Japan. Soc. Indust. Appl. Math., **18** (2008), 409–425.
[9] A. Fukuda, E. Ishiwata, M. Iwasaki and Y. Nakamura, The discrete hungry Lotka-Volterra system and a new algorithm for computing matrix eigenvalues, Inverse Problems, **25** (2009), 015007.
[10] Y. Nakamura (Eds.), Applied Integrable Systems (in Japanese), Shokabo, Tokyo, 2000.
[11] S. Tsujimoto, R. Hirota and S. Oishi, An extension and discretization of Volterra equation I (in Japanese), IEICE Tech. Rep. NLP **92–90** (1993), 1–3.
[12] S. Tsujimoto, On the discrete Toda lattice hierarchy and orthogonal polynomials (in Japanese), RIMS Kôkyûroku, **1280** (2002), 11–18.