JSIAM Letters Vol.11 (2019) pp.5–7 ©2019 Japan Society for Industrial and Applied Mathematics

Modified SE-Sinc approximation with boundary treatment over the semi-infinite interval and its error bound

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Received October 3, 2018, Accepted December 10, 2018

Abstract

The Sinc approximation is known as an efficient function approximation formula for functions that decay exponentially and are defined over the entire infinite interval. Even for functions that do not satisfy such conditions, Stenger constructed an approximation formula based on the Sinc approximation combining with the Single-Exponential (SE) transformation and introducing auxiliary basis functions. In this study, we improve the approximation formula by replacing the SE transformation and the auxiliary basis functions. Two kinds of error bounds for the modified formula are also given.

Keywords Sinc approximation, SE transformation, error bound, function approximation

Research Activity Group Quality of Computations

1. Introduction and summary

The Sinc approximation is a function approximation formula expressed as

$$F(x) \approx \sum_{k=-M}^{N} F(kh)S(k,h)(x), \quad x \in \mathbb{R}, \tag{1}$$

where $S(k,h)$ is the so-called Sinc function defined by

$$S(k,h)(x) = \frac{\sin\left(\pi\left(\frac{x}{h}-j\right)\right)}{\pi\left(\frac{x}{h}-j\right)}.$$

For the Sinc approximation to work accurately, F should satisfy the following conditions:

a) $F(x)$ is defined on the infinite interval $(-\infty,\infty)$,

b) $F(x)$ decays exponentially as $x \to \pm\infty$.

Even for functions that do not satisfy condition a), the Sinc approximation can be applied by combining with a suitable variable transformation. For analytic and exponentially decaying functions $f(t)$ that are defined on the semi-infinite interval $(0,\infty)$ and satisfy $f(0)=0$, e.g., $f(t)=\sqrt{t}\,\mathrm{e}^{-t}$, Stenger [1] proposed the variable transformation

$$t = \psi(x) = \operatorname{arcsinh}(\mathrm{e}^x),$$

by which the function $F(x)=f(\psi(x))$ satisfies both conditions a) and b) and one can apply the formula (1). The transformation $t=\psi(x)$ is sometimes called the Single-Exponential (SE) transformation, because the transformed function $f(\psi(x))$ decays single-exponentially as $x \to \pm\infty$. Hence, a combination of the SE transformation and the Sinc approximation is referred to as the SE-Sinc approximation. Such a transformation is not unique; in fact, it was reported that by replacing the transformation with

$$t = \phi(x) = \log(1+\mathrm{e}^x),$$

the convergence rate was improved [2].

However, in cases such as $g(t)=1+\sqrt{t}\,\mathrm{e}^{-t}$, the transformed function $g(\psi(x))$ satisfies the condition a) but does not satisfy the condition b). This is because the boundary values $g(0)$ and $\lim_{t\to\infty} g(t)$ are not zero. To remedy the issue, Stenger [1] considered the function

$$f(t) = g(t) - \frac{q+p\sinh t}{1+\sinh t}, \tag{2}$$

where

$$p = \lim_{t\to\infty} g(t), \quad q = g(0), \tag{3}$$

so that the function $f(\psi(x))$ satisfies both conditions a) and b), and thus, one can apply (1). This is the SE-Sinc approximation with boundary treatment. The error analysis of the approximation was given [1].

In this study, we aim to improve the approximation by replacing the transformation $t=\psi(x)$ with $t=\phi(x)$. In addition, instead of (2), we consider the function

$$f(t) = g(t) - \frac{q+p(\mathrm{e}^t-1)}{\mathrm{e}^t}, \tag{4}$$

where p and q are defined by (3), and we apply (1) to the function $f(\phi(x))$. Then, we show that the convergence rate may be improved by the modification. We also give a computable error bound for the modified approximation formula.

The remainder of this paper is organized as follows. Error analysis of Stenger's approximation formula is stated in Section 2. A computable error bound for the modified approximation formula is shown in Section 3.

The result is further improved from a practical viewpoint in Section 4. Numerical results are provided in Section 5. Proofs of the presented theoretical results are given in Section 6.

2. Stenger's approximation formula

First, let us introduce the following function space.

Definition 1 *Let K, α, and β be positive constants, and let $\mathscr{D}$ be a simply connected complex domain that contains $(0,\infty)$. Then, $\mathbf{L}_{K,\alpha,\beta}(\mathscr{D})$ denotes a family of functions f that are analytic on $\mathscr{D}$ and satisfy for all $z \in \mathscr{D}$ that*

$$|f(z)| \le K \left| \frac{z}{1+z} \right|^{\alpha} |\,\mathrm{e}^{-z}\,|^{\beta}.$$

The error analysis of Stenger's approximation formula was given as follows. Here, $\mathscr{D}_d$ denotes a strip complex domain defined by $\mathscr{D}_d = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\}$ for $d > 0$.

Theorem 2 (Stenger [1, Example 4.2.11]) *Let d be a constant with $0 < d < \pi/2$. For a given function g, let p and q be given by (3), and let f defined by (2) belong to $\mathbf{L}_{K,\alpha,\beta}(\psi(\mathscr{D}_d))$. Let $\mu = \min\{\alpha,\beta\}$, let h be set as*

$$h = \sqrt{\frac{\pi d}{\mu n}}, \tag{5}$$

and let M and N be set as

$$\begin{cases} M = n, \quad N = \left\lceil \frac{\alpha n}{\beta} \right\rceil & (\text{if } \mu = \alpha), \\ N = n, \quad M = \left\lceil \frac{\beta n}{\alpha} \right\rceil & (\text{if } \mu = \beta). \end{cases} \tag{6}$$

Then, there exists a constant C independent of n such that

$$\sup_{t\in(0,\infty)} \left| g(t) - \left[\frac{q + p\sinh t}{1+\sinh t} + \sum_{k=-M}^{N} \left(g(\psi(kh)) - \frac{q + p\,\mathrm{e}^{kh}}{1+\mathrm{e}^{kh}} \right) S(k,h)(\psi^{-1}(t)) \right] \right| \le C\sqrt{n}\,\mathrm{e}^{-\sqrt{\pi d \mu n}}. \tag{7}$$

In this theorem, the constant C in (7) is not explicitly given. We note that if the constant C is given in a computable form, we can use the right-hand side of (7) as a mathematically rigorous error bound. Based on the idea, we give the error bound for the new formula as described in the next section.

3. Modified approximation formula

This paper gives the following error bound for the modified approximation formula. The proof is given in Section 6.1.

Theorem 3 *Let d be a constant with $0 < d < \pi$. For a given function g, let p and q be given by (3), and let f defined by (4) belong to $\mathbf{L}_{K,\alpha,\beta}(\phi(\mathscr{D}_d))$. Let $\mu = \min\{\alpha,\beta\}$, let h be set as (5), and let M and N be set as (6). Then, it holds that*

$$\sup_{t\in(0,\infty)} \left| g(t) - \left[\frac{q + p(\mathrm{e}^t - 1)}{\mathrm{e}^t} + \sum_{k=-N}^{N} \left(g(\phi(kh)) - \frac{q + p\,\mathrm{e}^{kh}}{1+\mathrm{e}^{kh}} \right) S(k,h)(\phi^{-1}(t)) \right] \right| \le C\sqrt{n}\,\mathrm{e}^{-\sqrt{\pi d \mu n}}, \tag{8}$$

where C is a constant defined by

$$C = \frac{2K}{\sqrt{\pi d \mu}} \left\{ \frac{2\left(\frac{\mathrm{e}}{\mathrm{e}-1}\right)^{\frac{\mu}{2}}}{\sqrt{\pi d \mu}\,(1-\mathrm{e}^{-2\sqrt{\pi d \mu}})\cos^{\alpha+\beta}\left(\frac{d}{2}\right)} + 1 \right\}. \tag{9}$$

In view of (7) and (8), the convergence rates of the two formulas seem to be the same: $\mathrm{O}(\mathrm{e}^{-\sqrt{\pi d \mu n}})$. However, there is a difference in the condition of d, that is, $0 < d < \pi/2$ in Theorem 2 and $0 < d < \pi$ in Theorem 3. This means that d in our formula may be larger than that in Stenger's formula. This difference originates from the variable transformations ψ and ϕ; Okayama–Machida [3] gave a detailed explanation on this point.

4. Conditions in terms of given function

In both Theorems 2 and 3, the conditions are not directly imposed on the given function g, but on the translated function f defined by (2) or (4). From a practical viewpoint, however, it is preferable to write the conditions on the given function g. Therefore, this paper shows the following lemma. The proof is given in Section 6.2.

Lemma 4 *Let d be a constant with $0 < d < \pi$. For a given function g, let p and q be given by (3). Assume that g is analytic on $\phi(\mathscr{D}_d)$, and there exist positive constants L_1 and L_2 such that*

$$|g(z) - q| \le L_1 \left| \frac{z}{1+z} \right|,$$
$$|g(z) - p| \le L_2 \left|\mathrm{e}^{-z}\right|$$

hold for all $z \in \phi(\mathscr{D}_d)$. Then, the function f defined by (4) belongs to $\mathbf{L}_{K,\alpha,\beta}(\phi(\mathscr{D}_d))$ with $K = L_1 + L_2 c_d$ and $\alpha = \beta = 1$, where c_d is a constant defined by

$$c_d = \frac{1 + \log\left(1 + \left(1 + \frac{1}{\cos\left(\frac{d}{2}\right)}\right)\right)}{\log\left(1 + \left(1 + \frac{1}{\cos\left(\frac{d}{2}\right)}\right)\right)} \left(1 + \frac{1}{\cos\left(\frac{d}{2}\right)}\right). \tag{10}$$

Thanks to this lemma, we can rewrite Theorem 3 as follows, where the conditions are imposed only on g.

Theorem 5 *Suppose that the assumptions in Lemma 4 are fulfilled. Then, by setting h as $h = \sqrt{\pi d/n}$, we have*

$$\sup_{t\in(0,\infty)} \left| g(t) - \left[\frac{q + p(\mathrm{e}^t - 1)}{\mathrm{e}^t} + \sum_{k=-n}^{n} \left(g(\phi(kh)) - \frac{q + p\,\mathrm{e}^{kh}}{1+\mathrm{e}^{kh}} \right) S(k,h)(\phi^{-1}(t)) \right] \right| \le C\sqrt{n}\,\mathrm{e}^{-\sqrt{\pi d n}},$$

where C is a constant defined by

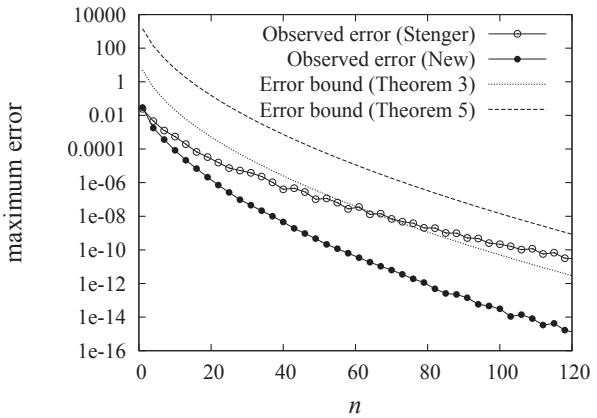

Fig. 1. Errors of Stenger's and the new SE-Sinc approximation formulas for the function $g(t)$ in (11) and error bounds for the new formula by Theorems 3 and 5.

$$C = \frac{2(L_1 + L_2 c_d)}{\sqrt{\pi d}} \left\{ \frac{2\left(\frac{\mathrm{e}}{\mathrm{e}-1}\right)^{\frac{1}{2}}}{\sqrt{\pi d}\,(1-\mathrm{e}^{-2\sqrt{\pi d}})\cos^2\left(\frac{d}{2}\right)} + 1 \right\},$$

where c_d is a constant defined by (10).

5. Numerical example

In this section, we show numerical approximation results for the function

$$g(t) = 1 + \frac{\mathrm{e}^{-t}}{1+t}, \tag{11}$$

by using Stenger's and the new SE-Sinc approximation formulas with boundary treatment. This function g fulfills the conditions in Theorem 2 with $d = 1.5$, $\alpha = \beta = 1$, and some positive constant K. It also fulfills the conditions in Theorem 3 with $d = 3$, $\alpha = \beta = 1$, and $K = 1$. Furthermore, it fulfills the conditions in Theorem 5 with $d = 3$, $\alpha = \beta = 1$, $L_1 = 1 + c_d/\sin d$, and $L_2 = 1/\sin d$, where c_d is a constant defined by (10). All programs were written in C with double-precision floating-point arithmetic. We observed the error $|g(t) - g_n(t)|$, where $g_n(t)$ denotes an approximation formula, on the following 101 points:

$$t = 2^{-50}, 2^{-49}, \ldots, 2^{49}, 2^{50},$$

and the maximum error among those points is plotted with a solid line in Fig. 1. From Fig. 1, we see that the new formula converges faster than Stenger's formula. In addition, error bounds for the new approximation formula by Theorem 3 and by Theorem 5 are shown with a dotted line and a dashed line, respectively. We see that both error bounds include the observed error, and converge with the same rate as the observed error. It makes sense that the error bound by Theorem 3 is sharper than that by Theorem 5, because $K < L_1 + L_2 c_d$ holds. Therefore, if users can get the value of the constant K, the error bound in Theorem 3 should be used. However, generally, it is not easy to get the value of K from the given function g. Theorem 5 is quite useful in such a case.

6. Proofs

6.1 Proof of Theorem 3

Theorem 3 is derived by using the following theorem.

Theorem 6 (Okayama et al. [2, Theorem 2.2]) *Let d be a constant with $0 < d < \pi$, and let $f \in \mathbf{L}_{K,\alpha,\beta}(\phi(\mathscr{D}_d))$. Let $\mu = \min\{\alpha, \beta\}$, let h be set as* (5)*, and let M and N be set as* (6)*. Then, it holds that*

$$\sup_{t\in(0,\infty)} \left| f(t) - \sum_{k=-M}^{N} f(\phi(kh)) S(k,h)(\phi^{-1}(t)) \right| \le C\sqrt{n}\,\mathrm{e}^{-\sqrt{\pi d \mu n}},$$

where C is a constant defined by (9).

Putting f as (4) in this theorem, we obtain Theorem 3.

6.2 Proof of Lemma 4

To prove Lemma 4, the following lemma is useful.

Lemma 7 (Okayama–Machida [3, Lemma 7]) *Let d be a constant with $0 < d < \pi$. Then, it holds for all $\zeta \in \mathscr{D}_d$ that*

$$\left| \frac{1 + \log(1+\mathrm{e}^{\zeta})}{\log(1+\mathrm{e}^{\zeta})} \cdot \frac{1}{1+\mathrm{e}^{-\zeta}} \right| \le c_d,$$

where c_d is a constant defined by (10).

Putting $z = \phi(\zeta)$ in this lemma, we immediately have the following result.

Corollary 8 *Let d be a constant with $0 < d < \pi$. Then, it holds for all $z \in \phi(\mathscr{D}_d)$ that*

$$|1 - \mathrm{e}^{-z}| \le c_d \left| \frac{z}{1+z} \right|,$$

where c_d is a constant defined by (10).

By using this result, Lemma 4 is shown as follows.

Proof We can rewrite the function f in (4) as

$$f(z) = (1 - \mathrm{e}^{-z})(g(z) - p) + \mathrm{e}^{-z}(g(z) - q),$$

from which we have

$$\begin{aligned} |f(z)| &\le |1 - \mathrm{e}^{-z}|\,|g(z) - p| + |\mathrm{e}^{-z}|\,|g(z) - q| \\ &\le c_d \left| \frac{z}{1+z} \right| L_2 |\mathrm{e}^{-z}| + |\mathrm{e}^{-z}| L_1 \left| \frac{z}{1+z} \right| \\ &= (L_1 + L_2 c_d) \left| \frac{z}{1+z} \right| |\mathrm{e}^{-z}|. \end{aligned}$$

This completes the proof.

(QED)

Acknowledgments

This work was partially supported by JSPS Grant-in-Aid for Young Scientists (B) Number JP17K14147.

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