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Investigation of the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps

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Abstract

Lyapunov exponent is commonly used as a measure of chaos. Chaos Degree is propsed as another measure of chaos based on information theory. The advantage of Chaos Degree is that it is calculable from data. However, there is a difference between Chaos Degree and Lyapunov exponent and determination of chaos requires caution by using Chaos Degree. In this paper, we find out information-theoretic interpretation of the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps.

Keywords chaos, Chaos Degree, Lyapunov exponent

Research Activity Group Applied Chaos

1. Introduction

Lyapunov exponent is commonly used as a measure of chaos. However, It is difficult to calculate Lyapunov exponents if equations of dynamical systems are not given. On the other hand, Entropic Chaos Degree (Chaos Degree) [1] is proposed as another measure of chaos which can be directly calculated from data. In order to apply Chaos Degree to analyze actual data, however, the relationship between Chaos Degree and other measures of chaos such as Lyapunov exponent or Kolmogorov-Sinai entropy (KS entropy) shoud be clarified.

Relations among Chaos Degree, Lyapunov exponent and KS entropy are discussed in [2]. Previous research [3] analytically shows that Chaos Degree in the limit that partition number approaches to infinity is *greater* than Lyapunov exponent, and suggests that the difference between Chaos Degree and Lyapunov exponent depends on the choice of partition. In many cases, Chaos Degree is greater than Lyapunov exponent and we should be careful to determine chaos by using Chaos Degree. For example, preceding studies [4, 5] show that Chaos Degree is positive for quasi-periodic orbits and it is hard to distinguish chaotic orbits from quasi-periodic orbits.

This paper reports on an analytical investigation of the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps.

2. Definition of Chaos Degree

The definition of Chaos Degree in difference equations is as below.

We assume that the difference equation is determined by a map $f : I \to I$ $(\equiv [a,b]^d \subset \mathbf{R}^d, a, b \in \mathbf{R}, d \in \mathbf{N})$, i.e., $x_{n+1} = f(x_n)$ $(n = 0, 1, \dots)$. Let x_0 be an initial value, and $A = \{A_i\}$ be a finit partition of I such that

$$I = \bigcup_{k=1}^{N} A_k, \quad A_i \cap A_j = \emptyset \quad (i \neq j).$$

The probability distribution $p_{i,A}^{(m)}(M)$ at the time m is given as

$$p_{i,A}^{(m)}(M) = \frac{1}{M} \sum_{k=m}^{m+M-1} 1_{A_i}(x_k),$$

and the joint probability distribution $p_{i,j,A}^{(m,m+1)}(M)$ between the time m and $m+1$ is given as

$$p_{i,j,A}^{(m,m+1)}(M) = \frac{1}{M} \sum_{k=m}^{m+M-1} 1_{A_i}(x_k) 1_{A_j}(x_{k+1}).$$

Then Chaos Degree $D^{(M,m)}(A, f)$ for the orbit $\{x_k\}$ is defined by

$$D^{(M,m)}(A, f) = \sum_{i=1}^{N} \sum_{j=1}^{N} p_{i,j,A}^{(m,m+1)}(M) \log \frac{p_{i,A}^{(m)}(M)}{p_{i,j,A}^{(m,m+1)}(M)}.$$

In this paper, we simplify $p_{i,A}^{(m)}(M) = p(i)$ and $p_{i,j,A}^{(m,m+1)}(M) = p(i,j)$, then Chaos Degree H_{CD} is calculated as:

$$\begin{aligned} H_{CD} &= \sum_{i=1}^{N} \sum_{j=1}^{N} p(i,j) \log \frac{p(i)}{p(i,j)} \\ &= -\sum_{i=1}^{N} p(i) \sum_{j=1}^{N} p(j|i) \log p(j|i), \qquad (1) \end{aligned}$$

where the conditional probability $p(j|i) = p(i,j)/p(i)$.

3. Asymmetric tent map

Let $T_k(x)$ be a tent map with the peak at $x = 1/k$ ($k \in \mathbf{N}$, $k \geq 2$) such that

$$T_k(x) = \begin{cases} kx & (0 \leq x \leq \frac{1}{k}) \\ \frac{k}{k-1}(1-x) & (\frac{1}{k} \leq x \leq 1) \end{cases}. \tag{2}$$

If $k = 2$ then $T_k(x)$ is a *symmetric* tent map, else $T_k(x)$ is an *asymmetric* tent map.

We try to find out how partitions affect the difference between Chaos Degree and Lyapunov exponent, focusing on asymmetric tent map $T_k(x)$ because its shape is simple so that it is easy to calculate both theoritical values of Chaos Degree and Lyapunov exponents.

4. Investigation prosedure

First, we calculate Chaos Degree of an asymmetric tent map $T_k(x)$. Next, we claculate a difference between Chaos Degree and Lyapunov Exponent ($H_{CD} - \lambda$) and try to interpret what the difference means.

5. Calculation of Chaos Degree

Let $\{A_i\}$ be an equipartition with partition number N such that $N = nk$ ($n \in \mathbf{N}$). For example, the case $k = 4$ and $n = 2$ is shown in Fig. 1.

There are two kinds of repeating patterns in $0 \leq x \leq 1/k$ and $1/k \leq x \leq 1$. In Fig. 1, each pattern is colored red or green. Each of these patterns appears n times.

If (A_i, A_j) such that $x_m \in A_i$ and $x_{m+1} \in A_j$ does not belong to any patterns, then $p(j|i) = 0$ and Chaos Degree is not affected. Therefore, we consider the two patterns in calculation of Chaos Degree below.

In case $0 \leq x \leq 1/k$, as shown in Fig. 2 (a), $T_k(A_{i_1})$ intersects with just k components, therefore conditional probability is

$$p(j|i_1) = \begin{cases} \frac{1}{k} & (j = j_1, j_2, \dots j_k) \\ 0 & (otherwise) \end{cases}. \tag{3}$$

In case $1/k \leq x \leq 1$, as shown in Fig. 2 (b), each of $T_k(A_{i_u})$ ($u = 1, 2, \dots k-1$) intersects with just two components. Suppose the index numbers of the two components are v_1 and v_2, then

$$p(j|i_u) = \begin{cases} \frac{u}{k} & (j = j_{v_1}) \\ \frac{k-u}{k} & (j = j_{v_2}) \\ 0 & (otherwise) \end{cases}. \tag{4}$$

Futher, $T_k(x)$ has the uniform invariant probability density, therefore

$$p(i) = \frac{1}{N} = \frac{1}{nk}. \tag{5}$$

From equations (1), (3), (4) and (5), we obtain

$$\begin{aligned} H_{CD} &= \frac{1}{nk} nk \left(-\frac{1}{k} \log \frac{1}{k} \right) \\ &\quad + \frac{1}{nk} n \sum_{u=1}^{k-1} \left(-\frac{u}{k} \log \frac{u}{k} - \frac{k-u}{k} \log \frac{k-u}{k} \right) \\ &= -\frac{1}{k} \log \frac{1}{k} \\ &\quad + \frac{1}{k} \sum_{u=1}^{k-1} \left(-\frac{u}{k} \log \frac{u}{k} - \frac{k-u}{k} \log \frac{k-u}{k} \right). \end{aligned} \tag{6}$$

Note that H_{CD} does not depend on n.

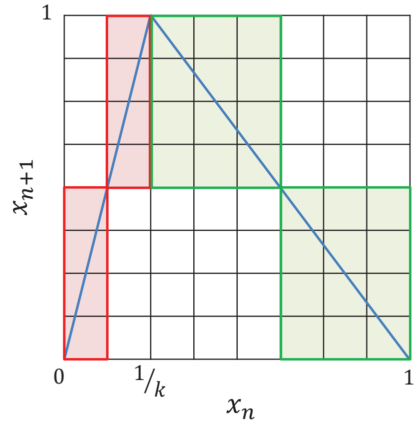

Fig. 1. Example of an asymmetric tent map $T_k(x)$ and an nk equipartition for $k = 4$ and $n = 2$.

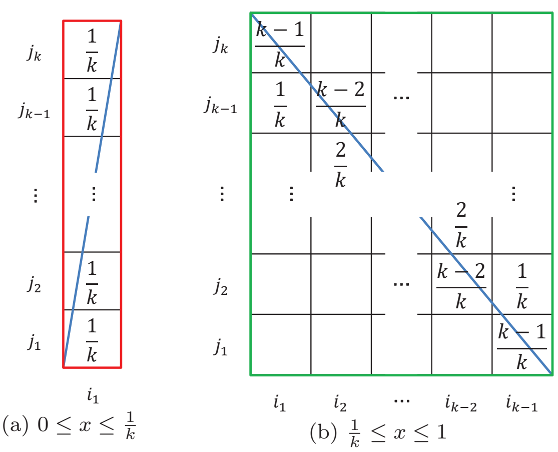

Fig. 2. Schematic picture of two kinds of repeating patterns and conditional probability $p(j|i)$. In pattern (b), $p(j|i) = 0$ is omitted.

6. Calculation of the difference between Chaos Degree and Lyapunov Exponent

Lyapunov exponent λ is given as

$$\lambda = -\frac{1}{k} \log \frac{1}{k} - \frac{k-1}{k} \log \frac{k-1}{k}. \tag{7}$$

Chaos Degree H_{CD} for $k \geq 2$ and Lyapunov exponent λ is shown in Fig. 3. The horizontal axis a is the peak of $T_k(x)$ i.e. $a = 1/k$. As can be seen from the figure, when $k = 2$ ($a = 1/2$; $T_k(x)$ is a symmetric tent map), Chaos Degree equals Lyapunov exponent, and in other cases, Chaos Degree is *greater* than Lyapunov exponent.

Theorem 1 *Let $T_k(x)$ be an asymmetric tent map which is given by equation (2) and $\{A_i\}$ be an nk equipartition of $I = [0, 1]$, then we have*

$$H_{CD} \geq \lambda,$$

where H_{CD} is Chaos Degree and λ is Lyapunov expo-

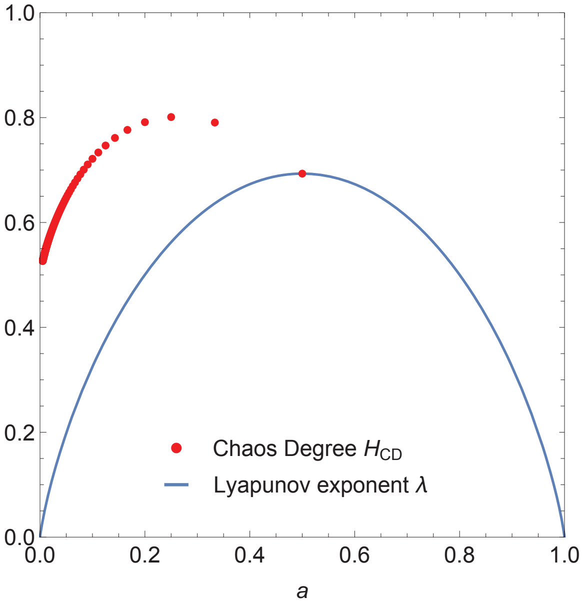

Fig. 3. Chaos Degree and Lyapunov exponent of asymmetric tent map $T_k(x)$.

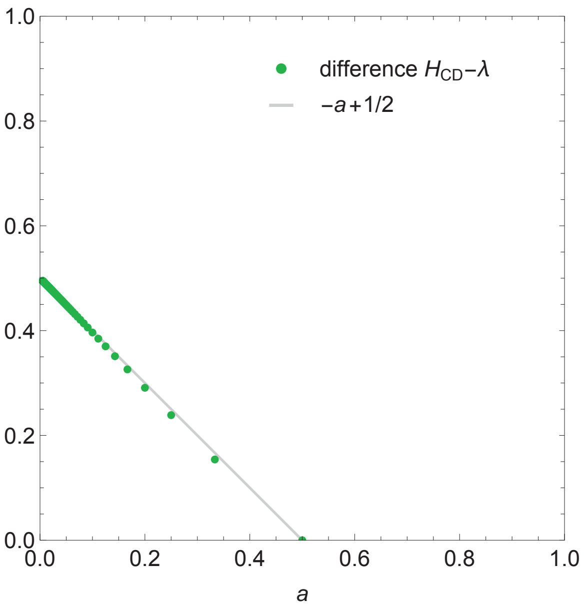

Fig. 4. The difference between Chaos Degree and Lyapunov exponent $(H_{CD}-\lambda)$ with the line $-a+1/2$.

nent.

Proof From equations (6) and (7), the difference between Chaos Degree and Lyapunov exponent can be calculated as follows:

$$H_{CD}-\lambda = \frac{1}{k}\sum_{u=1}^{k-1}\left(-\frac{u}{k}\log\frac{u}{k}-\frac{k-u}{k}\log\frac{k-u}{k}\right) - \left(-\frac{k-1}{k}\log\frac{k-1}{k}\right)$$
$$= \frac{1}{k}\sum_{u=1}^{k-1}\left(-\frac{u}{k}\log\frac{u}{k}-\frac{k-u}{k}\log\frac{k-u}{k}\right) + \frac{1}{k}\sum_{u=1}^{k-1}\log\frac{k-1}{k}$$
$$= \frac{1}{k}\sum_{u=1}^{k-1}\left(-\frac{u}{k}\log\frac{u}{k-1}-\frac{k-u}{k}\log\frac{k-u}{k-1}\right). \tag{8}$$

Thus, we have $H_{CD}-\lambda \geq 0$, i.e. $H_{CD}\geq\lambda$. The equality holds for $k=2$ (symmetric tent map case).

(QED)

The difference between Chaos Degree and Lyapunov Exponent is shown in Fig. 4. The horizontal axis a is the peak of $T_k(x)$ i.e. $a=1/k$. When $a=0$, although Lyapunov exponent $\lambda=0$, the difference $H_{CD}-\lambda$ is maximum. This shows that it is difficult to determine chaos for weak chaos by using Chaos Degree H_{CD}.

Fig. 4 also shows that values of the difference $H_{CD}-\lambda$ are close to the line $-a+1/2$. We can understand this behavior as below. From equation (8),

$$H_{CD}-\lambda = \frac{1}{k}\sum_{u=1}^{k-1}\left(-\frac{u}{k}\log\frac{u}{k-1}-\frac{k-u}{k}\log\frac{k-u}{k-1}\right)$$
$$= 2\left(1-\frac{1}{k}\right)^2\sum_{u=1}^{k-1}\left(-\frac{u}{k-1}\log\frac{u}{k-1}\right)\frac{1}{k-1}.$$

For sufficiently large k,

$$H_{CD}-\lambda \sim 2\left(1-\frac{2}{k}\right)\int_0^1(-x\log x)dx$$
$$= -a+\frac{1}{2}.$$

7. Interpretation of the difference

Theorem 2 *Suppose that $T_k(x)$ is an asymmetric tent map and $\{A_i\}$ is an nk equipartition of $I=[0,1]$. The difference between Chaos Degree and Lyapunov exponent $H_{CD}-\lambda$ can be calculated as an average of some kind of information $-\log q(i,j)$, where $q(i,j)$ is defined as*

$$q(i,j) \stackrel{\text{def}}{=} \frac{\|T_k(A_i)\cap A_j\|}{\|A_j\|}.$$

Proof Since $\{A_i\}$ is an nk equipartition,

$$\|A_i\| = \Delta = \frac{1}{nk} \quad (i=1,2,\ldots,N).$$

In case $0\leq x\leq 1/k$,

$$q(i_1,j) = \frac{\|A_j\|}{\|A_j\|} = 1 \quad (j=j_1,j_2,\ldots j_k).$$

In case $1/k\leq x\leq 1$, as shown in Fig. 5,

$$q(i_u,j) = \begin{cases} \frac{u}{k-1} & (j=j_{v_1}) \\ \frac{k-u}{k-1} & (j=j_{v_2}) \\ 0 & (\textit{otherwise}) \end{cases}.$$

Therefore, the difference $H_{CD}-\lambda$ is calculated as:

$$H_{CD}-\lambda = \frac{1}{k}\sum_{u=1}^{k-1}\left(-\frac{u}{k}\log\frac{u}{k-1}-\frac{k-u}{k}\log\frac{k-u}{k-1}\right)$$
$$= \frac{1}{nk}n\sum_{u=1}^{k-1}(-p(j_{v_1}|i_u)\log q(i_u,j_{v_1}) -p(j_{v_2}|i_u)\log q(i_u,j_{v_2}))$$

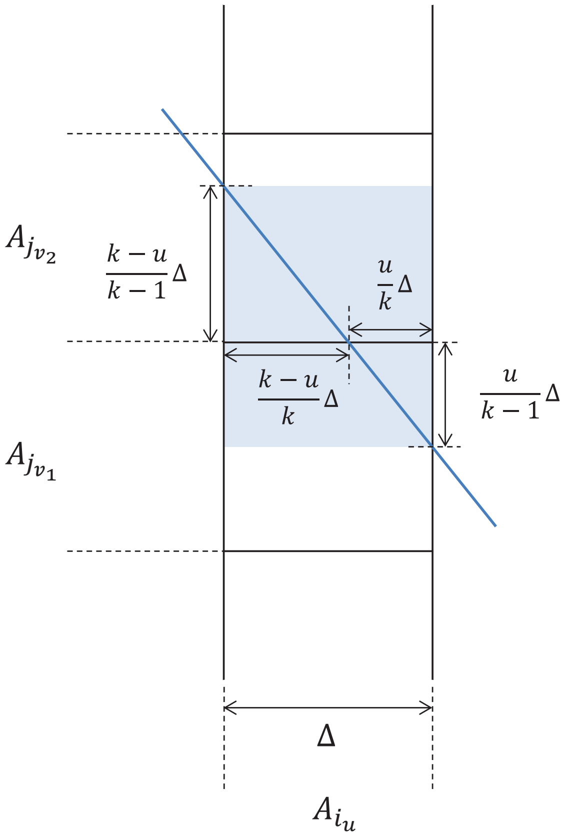

Fig. 5. Figure of intersection between $T_k(A_{i_u})$ and two components $A_{j_{v_1}}$, $A_{j_{v_2}}$.

$$
\begin{aligned}
&= \frac{1}{nk} n \sum_{u=1}^{k-1} \left(\sum_{v=1}^{k} p(j_v|i_u)\{-\log q(i_u, j_v)\} \right) \\
&= \frac{1}{nk} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} p(j|i)\{-\log q(i, j)\} \right) \\
&= \sum_{i=1}^{N} p(i) \sum_{j=1}^{N} p(j|i)\{-\log q(i, j)\}. \qquad (9)
\end{aligned}
$$

(QED)

If we regard $-\log q(i,j)$ as an amount of information, equation (9) is intuitively interpreted as below. When we assume that T_k maps A_i, Chaos Degree is calculated by entropy of conditional probability $p(j|i)$, then Chaos Degree is the same value in the case that the output is *uniformly* distributed in the whole of A_j and is greater than Lyapunov exponent. This difference is owing to lack of consideration about how output is distributed in each A_j. An actual distribution of output is limited in $T_k(A_i) \cap A_j$ and entropy should be *less* than the value in the case that output is distributed in whole of A_j. We can assume that this is because the entropy is added with $-\log q(i,j)$, which is an amount of information necessary to know that output is in $T_k(A_i) \cap A_j$.

8. Conclusion

We try to interpret the difference between Chaos Degree and Lyapunov exponent for asymmetric tent maps and find out that the difference $H_{CD} - \lambda$ can be calculated as an average of some kind of information $-\log q(i,j)$ and thus be non-negative. It is assumed that information $-\log q(i,j)$ is related to intersection of mapping $T_k(x)$ and components of partition $\{A_i\}$.

It is left for a future work to generalize the interpretation to other one-dimensional maps.

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