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# Testing whether the Nikkei225 best bid/ask price path follows the first order discrete Markov chain – an approach in terms of the total "$\rho$-variation" –

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**Abstract**

This paper empirically shows that in the days near the last trading day of Nikkei 225 Futures the best bid/ask prices follows the highly negatively correlated first order Markov process, and has no trend up to four ticks based on the total $\rho$-variation. This is consistent with the model by Endo et al. and the empirical results therein by different approach. It also derives the theoretical asymptotic formula for the total $\rho$-variation when the process follows the first order random Markov walks, and shows that its fit is satisfactory for $\rho \leq 4$.

**Keywords** length, measurement, Markov random walk

**Research Activity Group** Mathematical Finance

## 1. Introduction

In the liquid market of the Nikkei 225 Futures in the OSE (Osaka Stock Exchange), the bid-ask spread is almost always just one tick. (See [2].) Arrival of a market buy order triggers the settlement at the ask quote, while that of a sell order triggers the settlement at the bid quote. The transaction price oscillates between the highest bid and the lowest ask quotes, which causes seeming strong negative serial correlation in the sequence of consecutive transaction prices. One is interested in the temporal price changes of the best bid/ask quote or their mid-price rather than the transaction price itself.

To analyze the price changes under this situation, Endo et al. [1] proposed a simple double auction model, which predicts that the price changes of the best bid/ask quotes again follows a first order Markov random walk. This prediction is ascertained empirically in the same paper: the null hypothesis of the first order Markov property is not rejected by the Anderson and Goodman test statistics.

In financial markets, one is interested in possible existence of trend in the price path. Short time strong negative correlation caused by intrinsic nature of double auction property does not exclude the possible existence of trend. The target of the Anderson and Goodman test statistics is not to detect the existence of trend. One wishes to test directly the possible existence of longer scale trend. It is just the purpose of this paper.

The Alexander's filter rule [3] is a classical approach to detect the existence of trend in the academic world. This paper works on a variant version of this concept the *total $\rho$-variation* proposed in Kishimoto and Iri [4], which seems to be more natural as a mathematical quantity. (See also Kishimoto [5].)

This paper takes up 72 paths of the best ask price of tick-by-tick data of Nikkei225 futures in 2007 in the OSE, and tests whether we can regard them as paths of first order Markov chains. We also derive the formula for the expectation of the total $\rho$-variation of a locus of the first order Markov random walk to check what extent actual loci of price paths of Nikkei225 Futures follow this formula.

## 2. Markov random walk and the total $\rho$-variation

### *2.1 Markov random walk*

In the model by Endo et al., the $t$th price change $X(t)$ $(t = 1, 2, \dots)$ takes either $+1$ or $-1$. We associate "+1" (resp. "$-1$") with the term "Up" (resp. "Down"). We denoted "Up" (resp. "Down") simply by U (resp. D). Their model explains price changes of securities in terms of the arrival of two types of orders: buy and sell. In conclusion, the $t$th changes $X(t)$ of the best bid/ask price follow the two state Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} p_{\mathrm{UU}} & p_{\mathrm{UD}} \\ p_{\mathrm{DU}} & p_{\mathrm{DD}} \end{pmatrix}. \tag{1}$$

Here, $p_{\mathrm{UU}}$ and $p_{\mathrm{UD}}$ (resp. $p_{\mathrm{DU}}$ and $p_{\mathrm{DD}}$) are the conditional transition probabilities of $U$ and $D$ when the last move was U (resp. D). The security price $S(t)$ at $t$ is given by $S(t) = S(0) + \sum_{\tau=0}^{t} X_\tau$. We denote the path of $S(t)$ $(t \in [t_0, t_1])$ by $C[t_0, t_1]$.

### *2.2 Definition of the total $\rho$-variation*

Let $\Delta = \{t_0 < t_1 < \cdots < t_n\}$ be a subset of $\{0, 1, \dots, T\}$. In our case of the path of piecewise linear function, the total $\rho$-variation $V(\rho; C([0, t]))$ $(\rho > 0, t \in [0, T])$ is

defined by

$$V(\rho; C([0,t])) = \sup_{\{\Delta|(\forall k)\ |S(t_k)-S(T_{k-1})|>\rho\}} \sum_{k=1}^{m} |S(t_k)) - S(T_{k-1})|\,. \tag{2}$$

We notice that $V(0; C[0,t])$ is equal to the total variation in the ordinary sense. For $\rho > 0$, the supremum is attained for some $\Delta^*$. We say that $\rho$-extremum is attained at $S(t^*)$ if $t^* \in \Delta^*$. $S(t^*)$ is called $\rho$-maximum (resp. $\rho$-minimum) if it is a maximum (resp. minimum) point in the ordinary sense. Let us fix $\rho$ at a constant. We understand the $X(t)$ is positively (resp. negatively) correlated if $V(\rho; C([0,t]))$ is large (resp. small).

## 3. Theoretical value of $\mathrm{E}[V(\rho; C([0,t]))]$

Let us derive theoretical value of $\lim_{t\to\infty} \mathrm{E}[V(\rho; C[0, t])]$ as a function of transition matrix $P$. We consider the case where $\mathrm{E}[X(t)] = 0$, because our preliminary empirical calculations show that it produce little difference in our results during the period under consideration. Thus, we put

$$P = \begin{pmatrix} p_{\mathrm{UU}} & p_{\mathrm{UD}} \\ p_{\mathrm{DU}} & p_{\mathrm{DD}} \end{pmatrix} = \begin{pmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{pmatrix}.$$

### *3.1 Construction of an auxiliary Markov chain*

In our case where $V(\rho; C([0,t]))$ is a piecewise constant right continuous function, we need $V(\rho; C([0,t]))$ only for $\rho = 0, 1, 2, \ldots$. Without loss of generality, we assume that $S(0) = 0$ holds.

Suppose that we are at time $t$. For any $\epsilon > 0$ we can find $T$, such that for any $t > T$, there are $\rho$-extremum points between $[0,t]$ with probability more than $1-\epsilon$. Thus, we assume that $S(t)$ takes its $\rho$-extremum points at $t_0 < t_1 < \cdots < t_{n-1} < t^*$ in $[0,t]$. We remark that, in interval $[0,t'] \supset [0,t]$, $S(t)$ takes $\rho$-extremum value at $t_k$ $(0 \le k \le n-1)$ and not necessarily at $t^*$.

Let us introduce a new random variable $U(t) \equiv |S(t) - S(t^*)|$ $(t \ge t^*)$. A pair of random variables $(U(t), X(t))$ define a new Markov chain whose state space is $\{0, 1, \ldots, \rho\} \times \{-1, 1\}$.

We have the following six cases: (Fig. 1)

1) $(U(t), X(t)) = (0, +1)$: This will never happen.
2) $(U(t), X(t)) = (0, -1)$: We have two cases:
   a. With probability $p_{\mathrm{DU}}$, $X(t+1) = +1$ takes place, and we have a new state $(1, +1)$.
   b. With probability $p_{\mathrm{DD}}$, $X(t+1) = -1$ takes place. $S(t)$ becomes the last $\rho$-extremum point in $[0, t+1]$, while $S(t^*)$ is no more a $\rho$-extremum point. We have new state $(0, -1)$ with side effect
$$V(\rho; S(t+1)) = V(\rho; C([0,t])) + 1.$$
3) $1 \le U(t) \le \rho - 1$ and $X(t) = +1$:
   a. With probability $p_{\mathrm{UU}}$, we have a new state $(U(t)+1, +1)$ with no side effect.
   b. With probability $p_{\mathrm{UD}}$, we have a new state $(U(t)-1, -1)$ with no side effect.
4) $1 \le U(t) \le \rho - 1$ and $X(t) = -1$:
   a. With probability $p_{\mathrm{DU}}$, we have a new state $(U(t)+1, +1)$ with no side effect.
   b. With probability $p_{\mathrm{DD}}$, we have a new state $(U(t)-1, -1)$ with no side effect.
5) $(U(t), X(t)) = (\rho, +1)$:
   a. With probability $p_{\mathrm{UU}}$, $X(t+1) = +1$ takes place. $S(t+1)$ is the last $\rho$-extremum point in $[0, t+1]$, and we rename $S(t^*)$ as $S(t_{\rho+1})$. We have new state $(0, -1)$ with side effect $V(\rho; S(t+1)) = V(\rho; C([0,t])) + \rho + 1$.
   b. With probability $p_{\mathrm{UD}}$, $X(t+1) = -1$ takes place, and we have a new state $(1, +1)$ with no side effect.

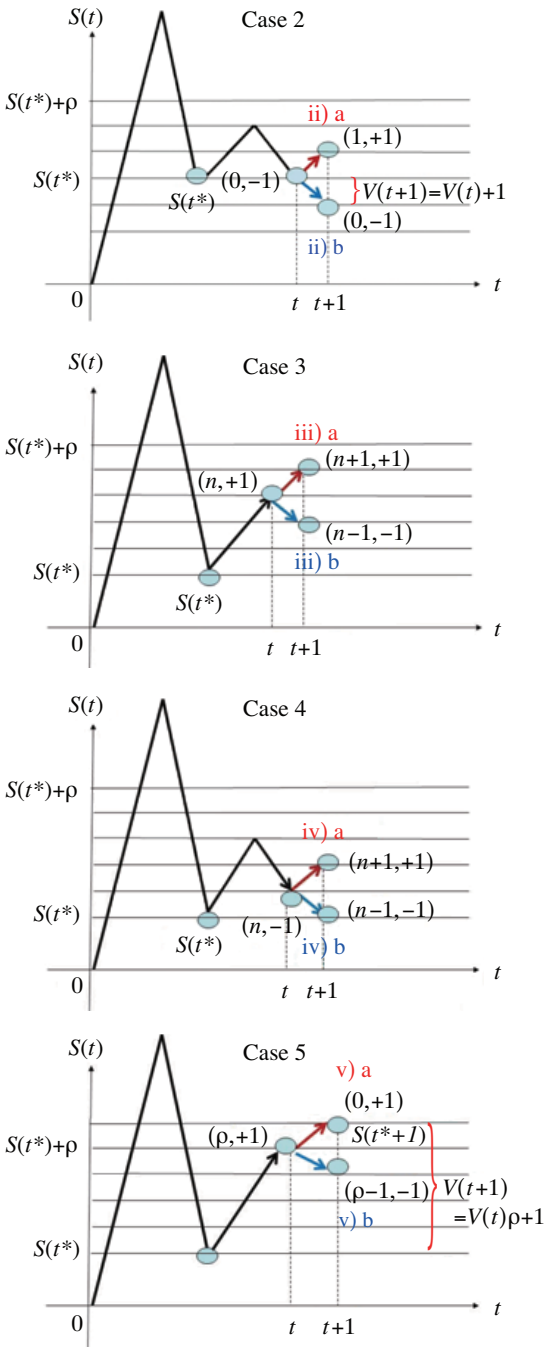


Fig. 1. Four cases 2)–5) out of six cases 1)–6), when $t$ is incremented by 1.

6) $(U(t), X(t)) = (\rho, -1)$: This will never happen.

Let us define the state probability vector $\boldsymbol{w}_t = (w_{t,1}, w_{t,2}, \ldots, w_{t,2(\rho+1)})$ at $t$ by

$$w_{t,2k-1} = \Pr[U(t) = k-1, X(t) = -1], \quad (k = 1, 2, \ldots, \rho+1),$$

$$w_{t,2k} = \Pr[U(t) = k-1, X(t) = +1], \quad (k = 1, 2, \ldots, \rho+1).$$

The transition matrix $A = (a_{ij})$ is given by

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & 0 & A_{23} & 0 & 0 \\ 0 & A_{32} & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 0 & A_{\rho,\rho+1} \\ A_{\rho+1,1} & 0 & \cdots & A_{\rho+1,\rho} & 0 \end{pmatrix},$$

$$A_{11} = \begin{pmatrix} p_{\mathrm{DD}} & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & p_{\mathrm{DU}} \\ 0 & 0 \end{pmatrix},$$

$$A_{23} = A_{\rho,\rho+1} = \begin{pmatrix} 0 & p_{\mathrm{DU}} \\ 0 & p_{\mathrm{UU}} \end{pmatrix},$$

$$A_{21} = A_{32} = \begin{pmatrix} p_{\mathrm{DD}} & 0 \\ p_{\mathrm{UD}} & 0 \end{pmatrix},$$

$$A_{\rho+1,1} = \begin{pmatrix} 0 & 0 \\ p_{\mathrm{UD}} & 0 \end{pmatrix}, \quad A_{\rho+1,\rho} = \begin{pmatrix} 0 & 0 \\ 0 & p_{\mathrm{DD}} \end{pmatrix}.$$

### *3.2 Asymptotic probability*

From the standard theory on the finite state Markov chain, $A$ has one and only one characteristic value 1 whose characteristic vector gives the asymptotic probability of $\boldsymbol{w}_t$ at $t \to \infty$ if normalized as

$$\sum_{k=1}^{2(\rho+1)} w_{t,k} = 1.$$

Let "T" denote the transposition of a vector. Asymptotic probability is given by the solution of

$$\boldsymbol{w}^{\mathrm{T}}(I - A) = 0.$$

Thus, we have

$$\begin{aligned} &( w_{t,2\rho+1}, \quad w_{t,2\rho+2} ) \\ &\quad = \left( 0, \quad \frac{\pi}{\rho(\rho-1)\pi^2 + (2\rho-1)\pi + 1} \right) \end{aligned}$$

and

$$\begin{aligned} &( w_{t,2n-1}, \quad w_{t,2n} ) \\ &\quad = ( w_{t,2\rho+1}, \quad w_{t,2\rho+2} ) \begin{pmatrix} \pi & \pi-1 \\ 1-\pi & 2-\pi \end{pmatrix}^{\rho+1-n} \\ &\qquad (n = 1, 2, \ldots, \rho+1). \end{aligned}$$

The proportional relationship among asymptotic probabilities are shown in Figs. 2 and 3.

### *3.3 Formula for* $\mathrm{E}[V(\rho; C([0,t]))]$

As $t \to \infty$, we have

$$\mathrm{E}[V(\rho; S(t+1)) - V(\rho; C([0,t]))]$$

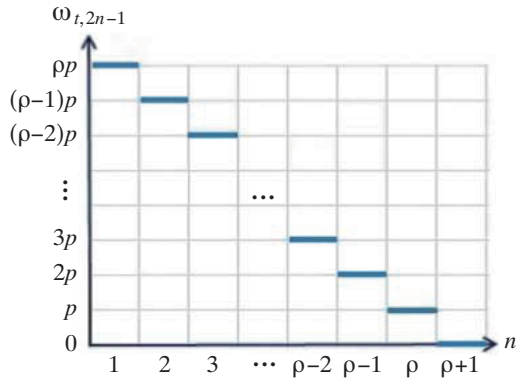


Fig. 2. Scaling relation among the points of $w_{t,2n-1}$.

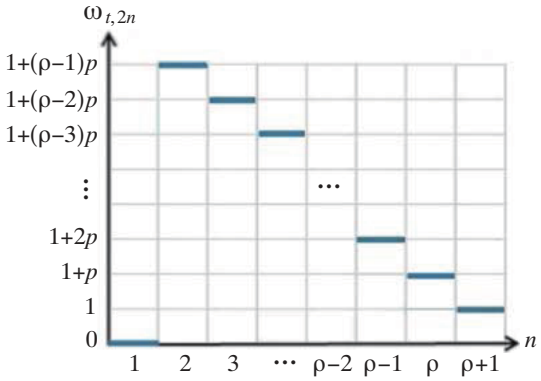


Fig. 3. Scaling relation among the points of $w_{t,2n}$.

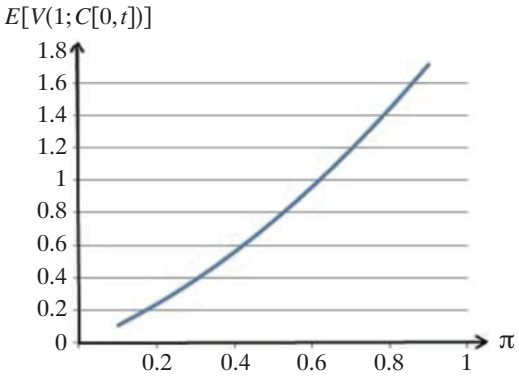


Fig. 4. $\mathrm{E}[V(1; C[0,t])]$ for a Markov random walk.

$$= \pi(w_{t,2} + \rho w_{t,2\rho+1}),$$

where the first term on the right-hand side is due to Case 2 in Fig. 1 and the second term is due to Case 5. The expectation of total $\rho$-variation is

$$\mathrm{E}[V(\rho; C([0,t]))] = \frac{\pi^2 + \pi\rho}{\rho(\rho-1)\pi^2 + 2(\rho-1)\pi + 1} t \quad (3)$$

as $t \to \infty$. A graph $\mathrm{E}[V(1; C[0,t])]$ as a function of $\pi$ is given in Fig. 4.

## 4. Empirical test on Nikkei 225 futures trade

### *4.1 The data set*

We used tick-by-tick data of Nikkei 225 Futures on the OSE in 2007 provided by Nikkei Media Marketing Inc.

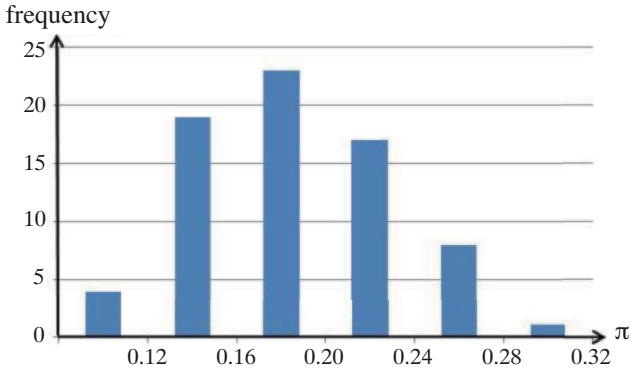


Fig. 5. Histogram of the observed value of $\pi$.

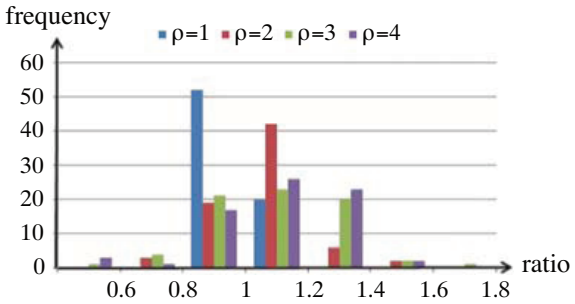


Fig. 6. Histogram for the ratio of calculated value to real value as $\rho = 1, \rho = 2, \rho = 3, \rho = 4$.

The contract months of Nikkei 225 Futures are March, June, September and December. The last trading day is the business day preceding the second Friday of each contract month. We deal only with the data of 10 days before the last trading day for 4 delivery months in 2007. Due to incompleteness of data, we only calculate from March 1st to 8th for the delivery month of March 2007.

Trading hours are composed of two sessions: 9:00–11:00 (morning session) and 12:30–15:10 (afternoon session). Their opening prices and the closing prices are determined by the periodic double auction called "itayose", while the other transactions prices are determined by continuous auctions. We only work on the transactions by the continuous auctions, and discard the transactions at opening and closing of the morning and the afternoon sessions. We analyze morning and afternoon sessions separately. Thus we have 72 samples.

We worked on the changes of the best ask price. When the best ask price moves $k$ticks ($k > 1$) instantaneously, we regard it as consecutive $k$ times transitions in the same direction.

### *4.2 Basic statistics*

For each of 72 sample paths, we can directly estimate $\pi$. We give the histogram of the estimated values of $\pi$ in Fig. 5, which suggests that $X(t)$ has strong negative serial correlation.

One is also interested whether the theoretical asymptotic formula (3) well predicts the observed total $\rho$-variations, or not. In Fig. 6, we give the histogram of the ratios of observed total $\rho$-variations ($\rho = 1, 2, 3, 4$) to their asymptotic predictions. To remove the *initial point effect*, we used $(\pi^2+\pi\rho)(T-t^*)/[\rho(\rho-1)\pi^2+2(\rho-1)\pi+1]$ as theoretical prediction, where $t^*$ is the location of the first extremum point. We give its average and variance in Table 1.

Table 1. The time on finding the first $\rho$-extremum point. The time is divided by the total time of a path.

| | $\rho = 1$ | $\rho = 2$ | $\rho = 3$ | $\rho = 4$ |
|---|---|---|---|---|
| Mean | 1.67E−03 | 8.79E−03 | 1.71E−02 | 3.16E−02 |
| Var | 5.92E−06 | 2.05E−04 | 1.06E−03 | 3.91E−03 |

Table 2. Numbers of rejections at the significance level 5 percent.

| | $\rho = 1$ | $\rho = 2$ | $\rho = 3$ | $\rho = 4$ |
|---|---|---|---|---|
| ratios of rejection | 5/72 | 2/72 | 1/72 | 1/72 |

We see that the best fit is given for $\rho = 1$, and the asymptotic formula seems to predicts the reality for all $\rho$'s, though its variance must be further investigated.

### *4.3 Method for testing and its results*

Suppose that 72 original paths of the best ask price are given. For each path, we randomly generated 999 paths of the same path length based on the estimated $\pi$. We calculated 1000 total $\rho$-variation. If the rank $r$ of $V(\rho; C[0, m])$ of the original path satisfies $25 < r < 975$, we judges that the null hypothesis is accepted for this path at the significance level 5 percent. The null hypothesis that the best ask path follows the first order Markov process seems to be accepted for $\rho = 1, 2, 3, 4$. For $\rho$ more than 5, we do not have enough $\rho$-extremum points for testing.

We give the numbers of rejections in Table 2. It seems that the null hypothesis is accepted.

## 5. Conclusion

We tested whether the best ask price path of the Nikkei 225 Futures in the OSE follows the first order Markov chain or not, based on the total $\rho$-variation for $\rho = 1, 2, 3, 4$. The null hypothesis is not rejected. We also calculated the asymptotic theoretical expectation of the total $\rho$-variation. Its fit seems to be satisfactory though its variance must be further investigated.

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## References

[1] M. Endo, S. Zuo and K. Kishimoto, Modeling intra-day stock price changes in terms of a continuous double auction system (in Japanese), Trans. JSIAM, **16** (2006), 305–316.

[2] M. Li, M. Endo, S. Zuo and K. Kishimoto, Order imbalances explain 90% of returns of Nikkei 225 futures, Applied Economics Lett., **17** (2010), 1241–1245.

[3] S. S. Alexander, Price movement in speculative markets: Trends or random walk, Industrial Management Review, **2** (1961), 7–26.

[4] K. Kishimoto and M. Iri, A practical approach to the definition and measurement of "Length", Jpn J. Indust. Appl. Math., **6** (1989), 179–207.

[5] K. Kishimoto, Analytical expression of the expectation of a scale dependent "Length" of a path of Brownian motion with drift, Prog. Theor. Phys., **82** (1989), 465–470.