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Numerical identification of nonhyperbolicity of the Lorenz system through Lyapunov vectors

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Abstract

Understanding nonhyperbolicity in dynamical systems is important, yet, it is usually difficult to see whether a system is hyperbolic or not. In this letter, angles between stable and unstable directions on a point of a chaotic attractor of the Lorenz system with some sets of various parameter values are calculated through identifying Lyapunov vectors numerically. Then we estimate the parameter value where the system becomes nonhyperbolic in one parameter family.

Keywords Lyapunov vectors, nonhyperbolicity, Lorenz system

Research Activity Group Scientific Computation and Numerical Analysis (Role of Unstable Solutions in Pattern Dynamics)

1. Introduction

1.1 Basic background

A dynamical system is said to be hyperbolic if the stable and unstable manifolds are everywhere transversal to each other; otherwise a system is nonhyperbolic. Theory of hyperbolic dynamics have been developed by Smale [1] and many other researchers. In hyperbolic systems approaches by symbolic dynamics and cycle expansion theory always work. It is usually difficult to characterize nonhyperbolic dynamics [2,3], because those approaches are sometimes useless in analyzing a nonhyperbolic system [4, 5]. Therefore to distinguish between hyperbolic and nonhyperbolic dynamical systems is important.

However, it is difficult to see whether a given system is hyperbolic or not, because manifold structures are usually very complicated in chaotic systems. There are, however, some studies to prove hyperbolicity from both rigorous and non-rigorous approaches. Davis et al. [6] conjectured that the real Hénon family with some parameter region is hyperbolic. Later, Arai [7] proposed a rigorous computational method to prove hyperbolicity of discrete dynamical systems. He applied the method to the real Hénon family and proved the existence of many regions of hyperbolic parameters in the parameter plane of the family. Kuptsov and Kuznetsov [8] studied a coupled Ginzburg Landau equation from the viewpoint of hyperbolicity and nonhyperbolicity through Lyapunov vectors calculated by the numerical algorithm proposed by Ginelli et al. [9]. In this letter, we try to investigate the validity of Lyapunov vectors and the occurrence of nonhyperbolicity in the well known Lorenz system.

1.2 Lyapunov vectors

Lyapunov vectors are the vectors invariant under both forward and backward time iterations, and the expansion and the contraction rates of the vectors correspond to Lyapunov spectra [10–12]. They indicate stable and unstable directions of the tangent space at each point of an invariant set. The local (un)stable manifold at each point is spanned by the (un)stable directions.

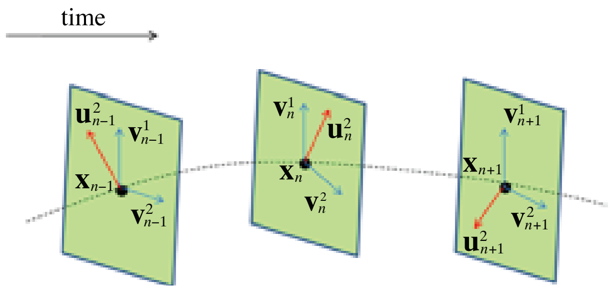

Fig. 1. Conceptual figure of calculating Lyapunov vectors ((i) Identifying "orthogonal Lyapunov vector" from the calculation of positive time direction (ii) Identifying Lyapunov vectors from the calculation of inverse time direction).

Ginelli and co-workers recently proposed a nice algorithm to compute Lyapunov vectors and named them covariant Lyapunov vectors (CLVs) [9]. The algorithm enables us to study local manifold structures for various systems including high dimensional systems. The Lyapunov vectors are computed in the following way (see Fig. 1). Let's consider an N-dimensional map $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$. For the forward procedure, we compute a set of orthogonal vectors $\mathbf{v}^k_n$ $(k = 1, 2, \ldots, N)$ at time n accompanied by the QR procedure [13]. $\mathbf{v}^k_n$ corresponds to the k-th column of the Q_n at the phase space point $\mathbf{x}_n$. To calculate Lyapunov vectors, we basically use $\mathbf{v}_n$ and R_n which are stored for the forward procedure. Let $\mathbf{u}^j_n$ be a generic vector inside the subspace spanned by $\mathbf{v}^k_n$, $k = 1, 2, \ldots, j$. We iterate this vector backward in time by inverting the matrix R_n: one has

$c_{n-1}^{i,j} = \sum_k [R_n]^{-1}_{i,k} c_n^{k,j}$, where $[R]_{i,j}$ is a matrix element of R and $c_n^{i,j} = (\mathbf{v}_n^i, \mathbf{u}_n^j)$ are the expansion coefficients. After iterating $\mathbf{u}_n^j$ backward for a long time, the vector eventually gives the most expanding direction within the subspace spanned by $\mathbf{v}_n^k$, $k = 1, 2, \ldots, j$. In fact $\mathbf{u}_n^j$ gives j-th expanding direction for the forward time iteration and thus $\mathbf{u}_n^j$ are j-th Lyapunov vectors at the phase space point $\mathbf{x}_n$. The knowledge of the Lyapunov vectors allows testing hyperbolicity by determining angles between subspaces E^s spanned by contracting CLVs and ones E^u spanned by expanding CLVs. The angle is defined as follows [8]:

$$\angle(E^s, E^u) = \cos^{-1} \max_{\substack{|\mathbf{u}^s|=|\mathbf{u}^u|=1 \\ \mathbf{u}^s \in E^s, \mathbf{u}^u \in E^u}} (|\mathbf{u}^s, \mathbf{u}^u|).$$

To see the validity of the Lyapunov vectors we first apply them to the Hénon map. Hénon map is a two-dimensional map on $\mathbf{R}^2$, which is described by

$$x_{n+1} = a - x_n^2 + by_n, \quad y_{n+1} = x_n$$

where the parameters a, b ($\in \mathbf{R}$) are constants. This is a diffeomorphism if $b \neq 0$, and the Jacobian of the system is $-b$. Hénon map is the only one diffeomorphism on $\mathbf{R}^2$ described by a polynomial of order 2 and the inverse of which is also written by a polynomial.

Fig. 2 shows the distribution of the angle between stable and unstable directions at each point of a chaotic attractor of the Hénon map with two parameter values, which is calculated from Lyapunov vectors. This shows that both parameters give nonhyperbolic structures. It is already known that if the Hénon map is hyperbolic the system cannot have a chaotic attractor. So our results are consistent with this known result and the result by Arai [7] in which both of these parameters are outside the parameter regions in $\mathbf{R}^2$ at which the system is proved to be hyperbolic.

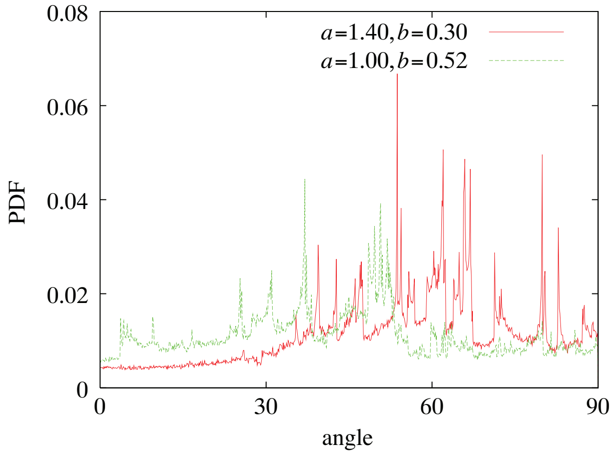

Fig. 2. Distribution of the angle(degree) between stable and unstable directions at each point of a chaotic attractor of the Hénon map (30000 iterations, bin size=0.1).

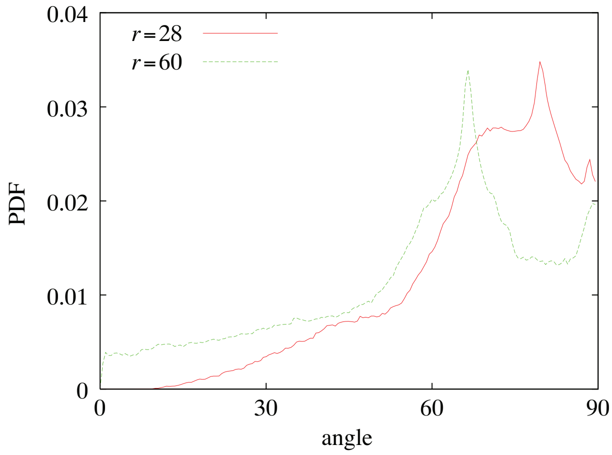

Fig. 3. Distribution of the angle (degree) between stable and unstable directions at each point of a chaotic attractor of the Lorenz system ($r = 28, 60$).

2. Lyapunov vectors of the Lorenz system

The Lorenz system

$$\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = rx - y - xz, \quad \frac{dz}{dt} = xy - bz$$

is one of the most famous chaotic systems. The system with the classical parameter values ($\sigma = 10, b = 8/3$, $r = 28$) have been extensively studied [14]. It is known that the system with the classical parameter values is (singular) hyperbolic and has a chaotic attractor which includes an infinite number of unstable periodic orbits [2, 15, 16]. It is also an interesting problem to see the structure change by varying some parameter values. Here, we only change the parameter r from the classical parameter value and investigate the change of manifold structures by calculating Lyapunov vectors.

2.1 Hyperbolic and nonhyperbolic structure

Fig. 3 shows the distribution of the angle between stable and unstable directions at each point of a chaotic attractor of the Lorenz system. Practically points of a chaotic attractor are replaced by points on a chaotic orbit with time length T=30000 calculated by the fourth order Runge-Kutta method with time step width 0.001. In the case of $r = 28$ the PDF does not seem to take positive around zero angle, whereas the PDF for $r = 60$ seems to take positive around zero angle. Fig. 4 shows the minimum angle between stable and unstable directions along a segment of a chaotic orbit (time length T) of the Lorenz system ($r = 28, 60$) for three initial conditions. In the case of the Lorenz system with classical parameter values ($r = 28$) the minimum angle seems to converge to some positive value. However, in the case of the Lorenz system with $r = 60$, the minimum angle seems to decrease toward 0. This implies that the system is hyperbolic in $r = 28$ and nonhyperbolic in $r = 60$. It is known that the Lorenz system with the classical parameter values is (singular) hyperbolic [2, 15, 16], whereas the system with $r = 60$ is thought to be nonhyperbolic where the cycle expansion theory [17] does not work [4, 5]. Sparrow [14] conjectured that the system generates a homoclinic tangency as r increases from 28. The results obtained in Figs. 3 and 4. which are calculated from the Lyapunov vectors are consistent with these facts.

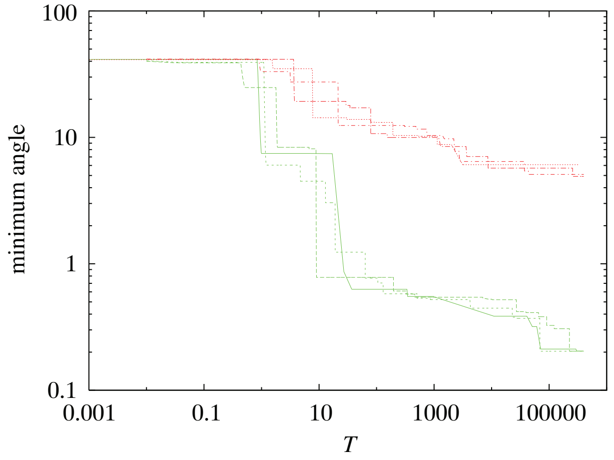

Fig. 4. Minimum angle (degree) between stable and unstable directions at each point along a segment of a chaotic orbit (time length T) of the Lorenz system from three initial conditions ($r = 28$(red), 60(green)).

2.2 First tangency

The occurrence of the nonhyperbolicity by changing some parameter values is an important phenomenon of a structure change. Especially the determination of the first tangency point is one of the most important but difficult problems [3,18]. In this subsection we try to approach the first tangency problem that appears when r is increased from 28 in the Lorenz system by numerically calculating the Lyapunov vectors at points of a chaotic attractor. Remark that in the usual case the term first tangency refers to a first bifurcation on the boundary of uniformly hyperbolic parameter region, but the problem here is not the case.

Fig. 5 is the minimum angle between stable and unstable directions at points of a chaotic attractor approximated by a chaotic orbit with time length $T = 30000$ for the Lorenz system for various r ($24.5 < r < 124.5$). In the range from $r = 40$ to 70 minimum angles seem not to be small enough, but if we use a longer orbit for representing points of a chaotic attractor, the minimum angles tend to become smaller as we have seen in Fig. 4 ($r = 60$). It seems that the structure change is monotonic by increasing r in the range $24.5 < r < 124.5$ and that after the occurrence of nonhyperbolicity the system keeps nonhyperbolicity for r which realizes a chaotic attractor. Fig. 6 is the detailed figure of Fig. 5 for $28 < r < 33$ which is calculated by a chaotic orbit with time length $T = 10^6$. As r increases the minimum angle decreases, and the system seems to become nonhyperbolic around 32. In fact from Fig. 7 PDF of the angle between stable and unstable directions of the Lorenz system at $r = 32$ takes positive around zero angle, whereas PDFs at $r = 28, 30$ do not take positive around zero angle. This means that the Lorenz system becomes nonhyperbolic between $r = 30$ and 32. The result is consistent with the estimation from the observation of the Poincaré section without calculating manifolds [14].

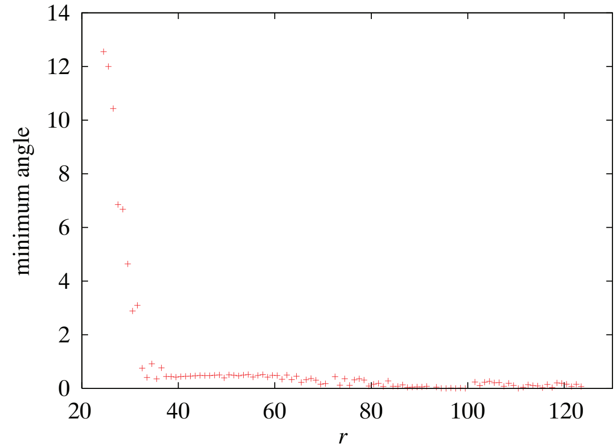

Fig. 5. Minimum angle (degree) between stable and unstable directions at each point on a chaotic attractor for various r.

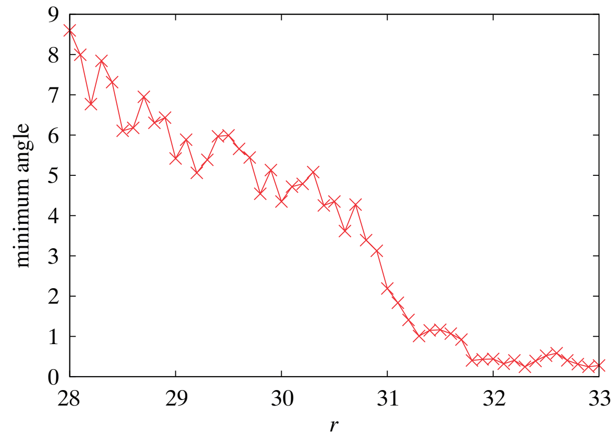

Fig. 6. Minimum angle (degree) between stable and unstable directions at each point on a chaotic attractor for various r.

3. Concluding remarks

3.1 Conclusion

In this letter, the validity of the Lyapunov vectors is confirmed from the Hénon map and the Lorenz system. At first we confirmed the hyperbolicity and nonhyperbolicity of the systems for well known parameter values. Hyperbolicity and nonhyperbolicity are identified from the angles between stable and unstable directions on a point of a chaotic attractor which are determined by the numerically calculated Lyapunov vectors.

Next the ranges of hyperbolic and nonhyperbolic parameter values of the Lorenz system are studied in detail. It is conjectured, from the calculation of Lyapunov vectors, that the first tangency parameter of r is between 30 and 32, which was estimated from the observation of the Poincaré section without calculating manifolds.

3.2 Recent works

Yang et al. [19] obtained Lyapunov vectors of the Kuramoto-Sivashinsky equation numerically and discussed physical modes in relation to the nonhyperbolicity of the system. This seems to be one of the interesting ways to use Lyapunov vectors. Identification of global manifolds are very difficult, but Doedel et al. [20] numerically identified the global stable manifold of the

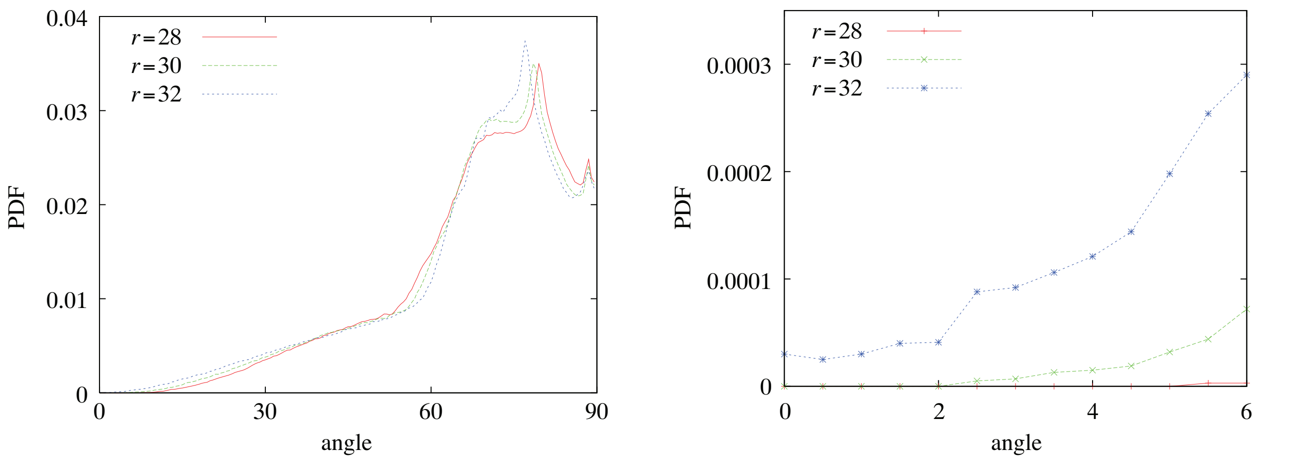

Fig. 7. Distributions of the angle (degree) between stable and unstable directions at each point on a chaotic attractor ($r = 28, 30, 32$) (left) and its detailed figure (right).

origin of the Lorenz system. It will also give us some interesting features in relation to nonhyperbolicity. In addition, from our recent study, it is found that the generation of nonhyperbolicity in the Lorenz system can be understood well by employing periodic orbits. Results will be reported in our papers in preparation [21, 22].

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